Thai Journal of Mathematics
Volume 16 (2018) Number 2 : 489–502

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

New Forward Backward Splitting Methods for Solving Pseudomonotone Variational Inequalities

Kan Buranakorn, Somyot Plubtieng[1] and Tadchai Yuying

Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
e-mail : kanburanakorn@hotmail.com (K. Buranakorn)
somyotp@nu.ac.th (S. Plubtieng)
tadchai_99@hotmail.com (T. Yuying)

Abstract : In this paper, we propose the new forward backward splitting method for solving variational inequality problem. The proposed method can be viewed as an extension of the extragradient method by additional projection step at each iteration under the relaxed condition that the mapping is pseudomonotone. The convergence of the proposed method has been proved. Moreover, to show the effectiveness of the proposed method the numerical experiments are performed.

Keywords : forward backward splitting method; projection method; pseudomonotone operator; variational inequality.
2010 Mathematics Subject Classification : 47H05; 65K05; 90C25.

1 Introduction

Over the past decades, researchers have developed a variety of efficient algorithms for solving variational inequality problem (VI):

$$\text{Find a vector } x^* \in \Omega \text{ such that } (x - x^*)^T F(x^*) \geq 0,\ \forall x \in \Omega, \tag{1.1}$$

where Ω is assumed to be a nonempty closed convex subset of $\mathbb{R}^n$ and F is assumed to be a mapping from $\mathbb{R}^n$ into itself. Denote $\mathrm{VI}(F, \Omega)$ is the solution set of (1.1).

[1]Corresponding author.

The projection method, proposed by Goldstein-Levitin-Polyak [1, 2], is the basic solution algorithm for solving VI due to its simple implementation. This method begin with any starting point, then generates a new point as follows:

$$x^{k+1} = P_\Omega[x^k - \beta_k F(x^k)],$$

where $P_\Omega(\cdot)$ is the projection from $\mathbb{R}^n$ onto Ω. Under the assumptions that the mapping F is strongly monotone and Lipschitz continuous, the convergence of the projection method can be guaranteed. In fact, it may not be easy to estimate the strongly monotone modulus and the Lipschitz constant of the mapping F. This stringent condition causes researchers to develop methods by reducing these conditions. Soon thereafter, Korpelevich [3] proposed the extra gradient method, which get rid of the strong monotonicity of the mapping. The iteration are generated by the following:

$$\begin{aligned} \bar{x} &= P_\Omega[x^k - \beta_k F(x^k)], \\ x^{k+1} &= P_\Omega[x^k - \beta_k F(\overline{x})]. \end{aligned}$$

The sequence $\{x^k\}$ converges to a solution of VI, when the mapping F is monotone and Lipschitz continuous, and $0 < \beta_k < 1/L$.

Recently, many researchers have proposed various new approaches [4–9] for solving VI. Most methods were invented to improve the efficiency by control the step size parameters depend on some suitable principles. Furthermore, the Lipschitz constant were removed by using the line search technique.

The prediction correction method developed by He et al. [8] adopted the following iterative scheme:

$$x^{k+1} = P_\Omega[x^k - \gamma\alpha_k g(x^k, \beta_k)],\ 0 < \gamma < 2$$

where

$$\begin{aligned} e(x^k, \beta_k) &= x^k - P_\Omega[x^k - \beta_k F(x^k)], \\ g(x^k, \beta_k) &= e(x^k, \beta_k) - \beta_k[F(x^k) - F(\bar{x}^k)], \\ \alpha_k &= \frac{e(x^k, \beta_k)^T g(x^k, \beta_k)}{\|g(x^k, \beta_k)\|^2}. \end{aligned}$$

The convergence of this method can be guaranteed under the conditions that F is monotone and β_k is chosen suitably. On the other hand, Bnouhachem et al. [18] introduced a new kind of extra gradient method by additional projection step at each iteration which the convergence of this method can be guaranteed also by the monotonicity of F (see also [17]).

Inspired by the research mentioned above, the new forward backward splitting method are introduced. The convergent theorem for solving pseudomonotone variational inequality are proved and discuss in later section. Finally, some numerical experiments are presented to show the efficiency of the new method.

2 Preliminaries

In this section, some definitions and lemmas from the literature are presented which are used throughout the paper. For convenience, we consider the projection under the Euclidean norm.

Definition 2.1. [10]The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *α-strongly monotone* if there exists $\alpha > 0$ such that

$$(x-y)^T(F(x)-F(y)) \geq \alpha\|x-y\|^2,\ \forall x,y \in \mathbb{R}^n.$$

Definition 2.2. [10]The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* if

$$(x-y)^T(F(x)-F(y)) \geq 0, \forall x,y \in \mathbb{R}^n.$$

Definition 2.3. [11]The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *pseudomonotone* if

$$(x-y)^T F(y) \geq 0 \ \Rightarrow\ (x-y)^T F(x) \geq 0, \forall x,y \in \mathbb{R}^n.$$

Remark 2.4. *The implications strongly monotone implies monotone and monotone implies pseudomonotone are evident.*

Example 2.5. Let K be a nonempty closed convex subset of $\mathbb{R}$ and $F : K \to \mathbb{R}$ be a single-valued mapping.

(1) If a mapping F is defined by $F(x) = 1 - x$ and $K = [0,1]$. Thus, we can check that the mapping F is a pseudomonotone mapping, but not a monotone mapping and a strongly monotone mapping.

(2) If a mapping F is defined by $F(x) = c$, where c is a constant and $K = \mathbb{R}$. We observe that the mapping F is monotone, but not strongly monotone mapping.

Definition 2.6. Let C be a nonempty subset of $\mathbb{R}^n$, the distance from a point $x \in \mathbb{R}^n$ to C is $d_C(x) = \inf_{y\in C}\|x-y\|$. If C is also closed and convex, then for every $x \in \mathbb{R}^n$, there exists a unique point $P_C(x) \in C$ such that $\|x - P_C(x)\| = d_C(x)$. The point $P_C(x)$ is the projection of x onto C.

Lemma 2.7. [10] *Let Ω be a closed convex subset of $\mathbb{R}^n$. Then the following hold:*

(1) $(y - P_\Omega(y))^T(x - P_\Omega(y)) \leq 0$, $\forall y \in \mathbb{R}^n$ *and* $\forall x \in \Omega$,

(2) $\|P_\Omega(y) - x\|^2 \leq \|y-x\|^2 - \|y - P_\Omega(y)\|^2$, $\forall y \in \mathbb{R}^n$ *and* $\forall x \in \Omega$,

(3) $\|P_\Omega(y) - P_\Omega(x)\|^2 \leq (y-x)^T(P_\Omega(y) - P_\Omega(x))$, $\forall y,x \in \mathbb{R}^n$.

Lemma 2.8. [12] *Let Ω be a closed convex subset of $\mathbb{R}^n$. Then x^* is a solution of* VI(F,Ω) *if and only if*

$$x^* = P_\Omega[x^* - \beta F(x^*)], \forall \beta > 0. \tag{2.1}$$

From Lemma 2.8, we note that $x \in \text{VI}(F,\Omega)$ if and only if $e(x,\beta) = 0$ where

$$e(x,\beta) := x - P_\Omega[x - \beta F(x)],\ \forall \beta > 0. \tag{2.2}$$

Generally, the term $\|e(x,1)\|$ is referred to as the error bound of VI(F,Ω).

Lemma 2.9. [13] *For any $x \in \mathbb{R}^n$ and $\tilde{\beta} \geq \beta > 0$, we have*

$$\|e(x,\beta)\| \leq \|e(x,\tilde{\beta})\|, \tag{2.3}$$

and

$$\frac{\|e(x,\beta)\|}{\beta} \geq \frac{\|e(x,\tilde{\beta})\|}{\tilde{\beta}}. \tag{2.4}$$

3 New Forward Backward Splitting Method

In this section, we describe the proposed methods. The proposed methods generate two predictors and evaluates F three times per iteration. We incorporate the algorithm with an Armijo-like line search similar to [14] and [6] in which β_k should satisfy two criteria. We also choose β_k with the same way in He et al. [8] to make it a good starting step size for the next iteration. And then investigate the strategy of how to choose the step size α_k.

Remark 3.1. [8] *The sequence β_k is monotonically nonincreasing. However, this may cause a slow convergence if*

$$r_k := \frac{\beta_k\|(F(\bar{x}_1^k) - F(\bar{x}_2^k))\|}{\|\bar{x}_1^k - \bar{x}_2^k\|}$$

is too small. In order to solve this problem, enlarging the step size β for the next iteration is necessary. Therefore, in $k+1^{th}$ iteration, we take

$$\beta_{k+1} = \begin{cases} 2\beta_k/m_2, & \text{if } 2r_k \leq m_2; \\ \beta_k, & \text{otherwise.} \end{cases}$$

where $m_2 \in (0,\sqrt{2})$ is a constant.

Algorithm3.1

Step 1 : Let $x^0 \in \Omega$, $\varepsilon > 0, \beta_0 = 1, m_1 \in (0,1), m_2 \in (0,\sqrt{2}), \gamma \in (0,2)$ and $k = 0$.
Step 2 : If $\|e(x^k,1)\| \leq \varepsilon$, then stop. Otherwise, go to Step 3.
Step 3 : (1) For a given $x^k \in \Omega$, calculate

$$\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)],$$

$$\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)].$$

(2) If $\|e(\bar{x}_1^k,1)\| \leq \varepsilon$, then stop. Otherwise, continue.
(3) If β_k satisfies both

$$r_k := \frac{\|\beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k))\|}{\|\bar{x}_1^k - \bar{x}_2^k\|} \leq m_1, \tag{3.1}$$

and

$$\frac{\|\beta_k[(\bar{x}_1^k-\bar{x}_2^k)^T(F(x^k)-F(\bar{x}_1^k))-(x^k-\bar{x}_2^k)^T(F(\bar{x}_1^k)-F(\bar{x}_2^k))]\|}{\|\bar{x}_1^k-\bar{x}_2^k\|^2}\leq m_2^2 \quad (3.2)$$

then go to Step 4; otherwise, continue.
(4) Perform an Armijo-like line search via reducing β_k

$$\beta_k := \frac{3}{4}*\beta_k*\min\{1,\frac{m_1}{r_k}\},$$

and go to Step 3.
Step 4 : Take the new iteration $x^{k+1}(\alpha_k)$ by setting

$$x^{k+1}(\alpha_k)=P_\Omega[x^k-\gamma\alpha_k g(\bar{x}_1^k,\bar{x}_2^k)],$$

where

$$0<\gamma<2,\ g(\bar{x}_1^k,\bar{x}_2^k)=(\bar{x}_1^k-\bar{x}_2^k)-\beta_k(F(\bar{x}_1^k)-F(\bar{x}_2^k)),$$

$$\lambda=\frac{\|\bar{x}_1^k-\bar{x}_2^k\|}{\|F(x^k)\|},\ \alpha_k=\frac{(x^k-\bar{x}_2^k)^T g(\bar{x}_1^k,\bar{x}_2^k)}{\|g(\bar{x}_1^k,\bar{x}_2^k)+\lambda F(x^k)\|^2}.$$

Step 5: Choosing a suitable β_{k+1} for the next iteration (same as [8]).

$$\beta_{k+1}=\begin{cases}2\beta_k/m_2, & \text{if } 2r_k\leq m_2;\\ \beta_k, & \text{otherwise.}\end{cases}$$

Return to Step 2, with k replaced by $k+1$.

Lemma 3.2. [17,18] *In the k^{th} iteration, if $\|e(x^k,1)\|\geq\varepsilon$, then the Armijo-like line search procedure with Criterion* (3.1) *and* (3.2) *is finite.*

Remark 3.3. *It is a natural question that how to choose a suitable optimal α_k is an important issue. Criterion* (3.2) *only could ensure $\alpha_k>0$. In order to obtain a lower bound (away from zero) on α_k, we need Criterion* (3.1)*. We will discuss these issues in this section.*

For convenience of later analysis, we use the following notations:

$$\begin{aligned}\rho_1 &= (\bar{x}_1^k-\bar{x}_2^k)^T g(\bar{x}_1^k,\bar{x}_2^k)\\ &= \|\bar{x}_1^k-\bar{x}_2^k\|^2-\beta_k(\bar{x}_1^k-\bar{x}_2^k)^T(F(\bar{x}_1^k)-F(\bar{x}_2^k)),\end{aligned} \quad (3.3)$$

and

$$\begin{aligned}\rho_2 &= (x^k-\bar{x}_1^k)^T g(\bar{x}_1^k,\bar{x}_2^k)\\ &= (x^k-\bar{x}_1^k)^T(\bar{x}_1^k-\bar{x}_2^k)-\beta_k(x^k-\bar{x}_1^k)^T(F(\bar{x}_1^k)-F(\bar{x}_2^k)),\end{aligned} \quad (3.4)$$

then $(x^k-\bar{x}_2^k)^T g(\bar{x}_1^k,\bar{x}_2^k)=\rho_1+\rho_2$.

Now, in order to prove the fact that α_k is bounded away from zero, we need the next lemma.

Lemma 3.4. *Let $x^k \in \Omega$, $\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)]$ and $\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)]$, then*

$$\begin{aligned}\rho_2 \quad \geq \quad & \|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) \\ & -(x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))]. \end{aligned} \tag{3.5}$$

Proof. Since $\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)], \bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)]$, we can apply Lemma 2.7, with $y = x^k - \beta_k F(x^k)$ and $x = \bar{x}_1^k - \beta_k F(\bar{x}_1^k)$, we obtain that

$$(x^k - \beta_k F(x^k) - (\bar{x}_1^k - \beta_k F(\bar{x}_1^k)))^T(\bar{x}_1^k - \bar{x}_2^k) \geq \|\bar{x}_1^k - \bar{x}_2^k\|^2.$$

By some manipulations, we have

$$(x^k - \bar{x}_1^k)^T(\bar{x}_1^k - \bar{x}_2^k) \geq \|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)). \tag{3.6}$$

Using (3.6) and the definition of ρ_2, we obtain that

$$\rho_2 \geq \| \bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) - (x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))]$$

as claimed. □

Lemma 3.5. *Let $\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)]$, $\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)]$ and*

$$\alpha_k = \frac{(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k)}{\|g(\bar{x}_1^k, \bar{x}_2^k) + \lambda F(x^k)\|^2}.$$

Then α_k is bounded away from zero.

Proof. Applying the Lemma 3.4 and Criterion (3.2), we have

$$\begin{aligned}\rho_1 + \rho_2 \quad \geq \quad & \|\bar{x}_1^k - \bar{x}_2^k\|^2 - \beta_k(\bar{x}_1^k - \bar{x}_2^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k)) + \|\bar{x}_1^k - \bar{x}_2^k\|^2 \\ & +\beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) - (x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))] \\ = \quad & 2\|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) \\ & -F(\bar{x}_1^k)) - (x^k - \bar{x}_2^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))] \\ \geq \quad & 2\|\bar{x}_1^k - \bar{x}_2^k\|^2 - m_2^2\|\bar{x}_1^k - \bar{x}_2^k\|^2 = (2 - m_2^2)\|\bar{x}_1^k - \bar{x}_2^k\|^2. \end{aligned} \tag{3.7}$$

Recalling the definition of $g(\bar{x}_1^k, \bar{x}_2^k) = (\bar{x}_1^k - \bar{x}_2^k) - \beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k))$ and applying Criterion (3.1), we get that

$$\begin{aligned}\|g(\bar{x}_1^k, \bar{x}_2^k) + \lambda F(x^k)\|^2 \quad \leq \quad & (\|\bar{x}_1^k - \bar{x}_2^k\| + \|\beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k))\| + \|\lambda F(x^k)\|)^2 \\ \leq \quad & ((1 + m_1)\|\bar{x}_1^k - \bar{x}_2^k\| + \|\bar{x}_1^k - \bar{x}_2^k\|)^2 \\ = \quad & (2 + m_1)^2\|\bar{x}_1^k - \bar{x}_2^k\|^2. \end{aligned} \tag{3.8}$$

Moreover, by using (3.7) together with (3.8), we get that

$$\alpha_k = \frac{\rho_1 + \rho_2}{\|g(\bar{x}_1^k, \bar{x}_2^k) + \lambda F(x^k)\|^2} \geq \frac{2 - m_2^2}{(2 + m_1)^2} > 0 \text{ where } m_2 \in (0, \sqrt{2}). \tag{3.9}$$

The proof is complete. □

4 Main Results

The aim of this section is to show the convergence result of proposed method.

Theorem 4.1. *The sequence* $\{x^{k+1}(\alpha_k)\}$ *generated by Algorithm* 3.1 *is bounded.*

Proof. Since

$$(x - x^*)^T F(x^*) \geq 0, \forall x \in \Omega \text{ where } x^* \in \mathrm{VI}(F, \Omega),$$

and

$$\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)] \in \Omega,$$

we obtain

$$(\bar{x}_2^k - x^*)^T F(x^*) \geq 0.$$

By pseudomonotonicity of F, i.e;

$$(x - y)^T F(y) \geq 0 \text{ implies } (x - y)^T F(x) \geq 0, \ \forall x, y \in \Omega,$$

we get that

$$(\bar{x}_2^k - x^*)^T F(\bar{x}_2^k) \geq 0.$$

Then

$$\beta_k (\bar{x}_2^k - x^*)^T F(\bar{x}_2^k) \geq 0.$$

Since

$$\bar{x}_1^k - \beta_k F(\bar{x}_1^k) \in \mathbb{R}^n \text{ and } x^* \in \Omega.$$

By Lemma 2.7(1), i.e;

$$(y - P_\Omega(y))^T (x - P_\Omega(y)) \leq 0, \ \forall x \in \Omega, y \in \mathbb{R}^n,$$

we get that

$$(\bar{x}_2^k - x^*)^T (\bar{x}_1^k - \bar{x}_2^k - \beta_k F(\bar{x}_1^k)) \geq 0.$$

Thus

$$(\bar{x}_2^k - x^*)^T (\bar{x}_1^k - \bar{x}_2^k - \beta_k F(\bar{x}_1^k) + \beta_k F(\bar{x}_2^k)) \geq 0.$$

And then

$$(\bar{x}_2^k - x^k + x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \geq 0.$$

This implies that

$$(x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \geq (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k). \tag{4.1}$$

By Lemma 2.7(2), i.e;

$$\|x - P_\Omega(y)\|^2 \leq \|x - y\|^2 - \|y - P_\Omega(y)\|^2, \ \forall y \in \mathbb{R}^n, \forall x \in \Omega,$$

and

$$x^{k+1}(\alpha_k) := P_\Omega[x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)].$$

Since $x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k) \in \mathbb{R}^n$ and $x^* \in \Omega$, we obtain

$$\begin{aligned}
\|x^* - x^{k+1}(\alpha_k)\|^2 &\leq \|x^* - (x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \|(x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)) - x^{k+1}(\alpha_k)\|^2 \\
&= \|\alpha_k g(\bar{x}_1^k, \bar{x}_2^k) - (x^k - x^*)\|^2 - \|(x^k - x^{k+1}(\alpha_k)) - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\
&= \|\alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - 2\alpha_k (x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) + \|x^k - x^*\|^2 \\
&\quad - (\|x^k - x^{k+1}(\alpha_k)\|^2 - 2\alpha_k (x^k - x^{k+1}(\alpha_k))^T g(\bar{x}_1^k, \bar{x}_2^k) \\
&\quad + \|\alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2) \\
&= \|x^k - x^*\|^2 - \|x^k - x^{k+1}(\alpha_k)\|^2 - 2\alpha_k (x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \\
&\quad + 2\alpha_k (x^k - x^{k+1}(\alpha_k))^T g(\bar{x}_1^k, \bar{x}_2^k) \\
&\leq \|x^k - x^*\|^2 - \|x^k - x^{k+1}(\alpha_k)\|^2 - 2\alpha_k (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\
&\quad + 2\alpha_k (x^k - x^{k+1}(\alpha_k))^T g(\bar{x}_1^k, \bar{x}_2^k).
\end{aligned} \tag{4.2}$$

The second inequality follow directly from (4.1).

Denote that $\theta(\alpha_k) := \| x^k - x^*\|^2 - \| x^{k+1}(\alpha_k) - x^*\|^2$, we have

$$\begin{aligned}
\theta(\alpha_k) &= \|x^k - x^*\|^2 - \|x^{k+1}(\alpha_k) - x^*\|^2 \\
&\geq \|x^k - x^{k+1}(\alpha_k)\|^2 + 2\alpha_k (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) - 2\alpha_k (x^k - x^{k+1}(\alpha_k))^T g(\bar{x}_1^k, \bar{x}_2^k) \\
&= 2\alpha_k (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) + \|(x^k - x^{k+1}(\alpha_k)) - g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \|\alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\
&\geq 2\alpha_k (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) - \alpha_k^2 \|g(\bar{x}_1^k, \bar{x}_2^k)\|^2.
\end{aligned} \tag{4.3}$$

The right-hand side is a quadratic function of α_k and its maximum can be obtained at

$$\alpha_k^* = \frac{(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k)}{\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2}. \tag{4.4}$$

Let $\gamma \in (0, 2)$ be a relaxation factor and $\alpha_k = \gamma\alpha_k^*$, it follows that

$$\begin{aligned}
\theta(\gamma\alpha_k^*) &\geq 2\gamma\alpha_k^* (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) - \gamma^2 \alpha_k^{*2} \|g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\
&= \gamma\alpha_k^* (2(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) - \gamma\alpha_k^* \|g(\bar{x}_1^k, \bar{x}_2^k)\|^2) \\
&= \gamma\alpha_k^* (2(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) - \gamma\alpha_k^* \frac{(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k)}{\alpha_k^*}) \\
&= \gamma\alpha_k^* (2 - \gamma)(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k).
\end{aligned} \tag{4.5}$$

By (3.7) and (3.9), we obtain

$$\alpha_k^*(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \geq \frac{(2-m_2^2)^2}{(2+m_1)^2}\|\bar{x}_1^k - \bar{x}_2^k\|^2. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\theta(\gamma\alpha_k^*) \geq \gamma(2-\gamma)\frac{(2-m_2^2)^2}{(2+m_1)^2}\|\bar{x}_1^k - \bar{x}_2^k\|^2.$$

This implies that

$$\|x^k - x^*\|^2 - \| x^{k+1}(\gamma\alpha_k^*) - x^*\|^2 \geq \gamma(2-\gamma)\frac{(2-m_2^2)^2}{(2+m_1)^2}\|\bar{x}_1^k - \bar{x}_2^k\|^2.$$

Thus

$$\|x^{k+1}(\gamma\alpha_k^*) - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma(2-\gamma)\frac{(2-m_2^2)^2}{(2+m_1)^2}\|\bar{x}_1^k - \bar{x}_2^k\|^2. \tag{4.7}$$

According to (4.7),it follows that

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \leq \cdots \leq \|x^0 - x^*\|.$$

It is clear that the sequence $\{x^k\} \subset \mathbb{R}^n$ generated by Algorithm 3.1 is bounded. The proof is completed. □

Theorem 4.2. *Suppose that the solution set of* VI(F,Ω) *is nonempty. Then the sequence* $\{\bar{x}_1^k\} \subset \mathbb{R}^n$ *generated by Algorithm* 3.1 *converges to a solution of* VI(F,Ω).

Proof. Let x^* be a solution of VI(F,Ω). First, it follows from (4.7) that

$$\sum_{k=0}^{\infty}\frac{\gamma(2-\gamma)(2-m_2^2)^2}{(2+m_1)^2}\|\bar{x}_1^k - \bar{x}_2^k\|^2 \leq \|x^0 - x^*\|^2 < +\infty,$$

which means that

$$\lim_{k\to\infty}\|\bar{x}_1^k - \bar{x}_2^k\|^2 = 0.$$

According to Theorem 4.1, the sequence $\{x^k\}$ is bounded. Consequently, $\{\bar{x}_1^k\}$ is also bounded. Thus, it has at least one cluster point. Assume that x^* is a cluster point of $\{\bar{x}_1^k\}$. Then there exists a subsequence $\{\bar{x}_1^{k_j}\}$ that converges to x^*. It follows from the continuity of e and (2.3) that

$$\|e(x^*,\beta)\| = \lim_{k_j\to\infty}\|e(\bar{x}_1^{k_j},\beta)\| \leq \lim_{k_j\to\infty}\|e(\bar{x}_1^{k_j},\beta_{k_j})\| = \lim_{k_j\to\infty}\|\bar{x}_1^{k_j} - \bar{x}_2^{k_j}\| = 0.$$

Therefore, x^* is a solution of $\text{VI}(F,\Omega)$. In the following, we prove the sequence $\{\bar{x}_1^k\}$ has exactly one cluster point. Assume that $\bar{x}$ is another cluster point, and denote $\delta := \|\bar{x} - x^*\| > 0$. Since x^* and $\bar{x}$ are cluster point of the sequence $\{\bar{x}_1^k\}$, there is a $k_1 \in \mathbb{N}$ such that

$$\|\bar{x}_1^k - x^*\| \leq \frac{\delta}{2},\ \forall k \geq k_1,$$

and there is a $k_2 \in \mathbb{N}$ such that

$$\|\bar{x}_1^k - \bar{x}\| \leq \frac{\delta}{2},\ \forall k \geq k_2,$$

and choose $k_0 = \max\{k_1, k_2\}$. On the other hand, since $x^* \in \text{VI}(F,\Omega)$, thus

$$\|\bar{x}_1^k - x^*\| \leq \|\bar{x}_1^{k_0} - x^*\|,\ \forall k \geq k_0,$$

It follows that

$$\|\bar{x}_1^k - \bar{x}\| \geq \|\bar{x} - x^*\| - \|\bar{x}_1^{k_0} - x^*\| \geq \frac{\delta}{2},\ \forall k \geq k_0,$$

This contradicts with $\bar{x}$ is a cluster point, thus the sequence $\{\bar{x}_1^k\}$ converges to $x^* \in \text{VI}(F,\Omega)$. □

5 Numercal Experiments

In this section, we use two exanples in [5] and [15] to show the efficiency of the proposed new algorithm. All codes are written in Matlab 7.12 and run on a desktop computer(CPU:Intel Pentium 4 3.00 GHz, Memory:1.00 GB).

Two examples of the complementarity problem are adopted in this paper:

$$x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0$$

where $F(x) = Mx + q$.

Example1

$$M = \begin{bmatrix} 1 & 2 & \cdots & \cdots & 2 \\ 0 & 1 & 2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, q = (-1,-1,\cdots,-1)^T.$$

Example2

$$M = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \ddots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4(n-1)+1 \end{bmatrix}, q = (-1,-1,\cdots,-1)^T.$$

In our test, we take $\Omega = \mathbb{R}^n_{++}, \beta = 1, m_1 = 0.9, m_2 = 0.3, \gamma = 1.8$ and the stop as soon as $\|e(x^k, 1)\| \leq 10^{-7}$. We denote compare the method generated by Algorithm 3.1 with the method proposed by He et al. [8].The test results for Example 1 are reported in Tables 1 and 2 and the test results for Example 2 are reported in Tables 3 and 4. It num is the number of iterations and CPU(second) is computation time.

Table1

Example 1: Numerical results for starting point $x^0 = (1, 0, 1, 0,)^T$

	method in [8]		Algorithm 3.1	
dimension N	it num	cpu	it num	cpu
100	1030	0.42	187	0.15
200	2111	0.84	266	0.32
300	3194	2.12	328	0.53
400	4277	7.84	374	1.34
500	5360	13.53	429	2.01
1000	-	-	606	7.65
1500	-	-	745	19.95

" − " represents that the CPU time is longer than 100 s.

Table2

Example 1: Numerical results for starting point $x^0 = (2, 2, 2,)^T$

	method in [8]		Algorithm 3.1	
dimension N	it num	cpu	it num	cpu
100	1516	0.48	186	0.20
200	3048	1.57	262	0.25
300	4579	3.25	319	0.39
400	6111	11.01	370	1.26
500	7643	19.57	413	1.92
1000	-	-	577	7.25
1500	-	-	710	19.45

" − " represents that the CPU time is longer than 100 s.

Table3

Example 2: Numerical results for starting point $x^0 = (1, 1, 1,)^T$

	method in [8]		Algorithm 3.1	
dimension N	it num	cpu	it num	cpu
10	2377	0.62	56	0.10
50	59835	17.31	132	0.12
100	-	-	194	0.15
200	-	-	281	0.32
500	-	-	452	1.84
1000	-	-	631	8.32

" − " represents that the CPU time is longer than 100 s.

Table4

Example 2: Numerical results for starting point $x^0 = (2, 2, 2,)^T$

	method in [8]		Algorithm 3.1	
dimension N	it num	cpu	it num	cpu
10	2435	0.64	50	0.01
50	61361	16.03	137	0.05
100	245504	80.87	197	0.09
200	-	-	284	0.18
500	-	-	461	1.90
1000	-	-	641	7.93

" $-$ " represents that the CPU time is longer than 100 s.

The numerical results show that the method generated by Algorithm 3.1 is more effective than the method presented in [8], which can achieve the solution with fewer iteration and time.

Next, the example of mapping F which is not monotone but is pseudomonotone (see [16]). Thus, our algorithm can apply to solve the problem but the algorithm in [8] can not.

Example3

$F(x) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 - 4x_2x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_3 \end{bmatrix}$ and $\Omega = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 5, i = 1, \ldots, 4\}$. The problem has only one solution $x^* = (5, 5, 5, 5)^T$.

The initial point is generated randomly. And because of the dimension of this problem is very small, the calculation is too fast, so it is not appropriate to compare performance of the methods by time. We compare the average of number of iterations for each method by calculating 500 times, then estimate the average of It num for each method. The test result for example 3 has been shown in the table below.

Table5

Example 3: Numerical results for a random start point in $[-10, 10]^4$

	method in [8]	Algorithm 3.1
Average it num	NAN	12

"NAN" represents that the It num is endless calculation.

6 Conclusions

We have proposed modified iterative algorithm for finding the solution of the variational inequality problems by using hybrid projection method which is generalization of the method in [8]. The convergence of the proposed algorithm is obtained by using weaker condition of monotonicity of the mapping and the numerical result of the hybrid iterative algorithm is also effective than the algorithm in [8].

Acknowledgement(s) : The authors would like to thanks to the Naresuan University for supporting by grant fund under Grant No. R2560C183.

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(Received 7 February 2018)
(Accepted 15 March 2018)