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Numerical Solution of the Nonlinear Integro- Differential Equations of Multi-Arbitrary Order

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Abstract : Fractional calculus has been used for modelling many of physical and engineering processes, that many of them are described by nonlinear Fredholm integro- differential equations of multi-arbitrary (integer or fractional) order. Therefore, an efficient and suitable method for the solution of them is too important. In this paper, the generalized fractional order of the Chebyshev functions (GFCFs) based on the classical Chebyshev polynomials of the first kind have been expressed that can be used to obtain the solution of the nonlinear Fredholm integro- differential equations of multi-arbitrary order. Also, the operational matrices of fractional derivative, the product, and the dual for the GFCFs, that they convert the differential equations to a system of algebraic equations, have been constructed. Some examples are included to demonstrate the validity and applicability of the technique.

Keywords : fractional order of the Chebyshev functions; operational matrix; Fredholm integro- differential equations; Galerkin method; nonlinear IDE.

2010 Mathematics Subject Classification : 45B05; 45J05; 45Gxx; 65M70

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1 Introduction

Fractional derivatives have a long mathematical history (Since 1695 by Hopital [1]), but for numerous reasons were not used in sciences for many years, for example, the various definitions of the fractional derivative [2] and have no exact geometrical interpretation [3]. However, in recent decades, many physicists and mathematicians have been many works on this subject and have found many uses for them. For example, the nonlinear oscillation of earthquake [4], the fluid-dynamic models with fractional derivatives [5–7] can eliminate the deficiency arising from the assumption of continuum traffic flow, and differential equations with fractional order have recently proved to be valuable tools for the modeling of many physical phenomena [8]. A review of some definition and applications of fractional derivatives is given in [9] and [10]. The analytical results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors [2, 11]. During the last decades, several methods have been used to solve fractional ordinary/partial/integro-differential differential equations, such as Adomian's decomposition method [12,13], fractional-order Legendre functions [14], fractional-order Chebyshev functions of the second kind [15], Homotopy analysis method [16, 17], Bessel functions and Spectral methods [18], Legendre and Bernstein polynomials [19], and other methods [20–23].

The aim of the paper is to present a numerical method (GFCF Galerkin (Tau) method) for approximating the solution of a nonlinear integro- differential equation of the multi-arbitrary order and the second kind:

$$\sum_{j=1}^{N_1} \mu_j D^{\alpha_j} y(x) + \sum_{l=1}^{N_2} \lambda_l \int_0^{\eta} k_l(x,t)[y(t)]^{q_l} dt = g(x), \qquad 0 \leqslant x < \eta, \tag{1.1}$$

with these supplementary conditions:

$$y^{(i)}(x_0) = y_i, \quad i = 0, 1, ..., r-1, \quad s.t. \;\; r = \lceil max\{\alpha_j\} \rceil, \tag{1.2}$$

where $\eta,\ \alpha_j$ are positive real numbers; $g \in L^2([0,\eta)),\ k_l \in L^2([0,\eta)^2)$ are known functions; $y(x)$ is the unknown function; D^{α_j} are the Caputo fractional differentiation operators; $\mu_j,\ \lambda_l$ are real numbers; and $q,\ N_1,\ N_2$ are positive integers.

The rest of the paper is structured as follows: in section 2, some basic definitions and theorems of fractional calculus is expressed. In section 3, the GFCFs and their properties is expressed. Section 4 is devoted to applying the GFCFs operational matrices to obtain the solution of differential equations. In Section 5, the work method is explained. Applications of the proposed method are shown in section 6. Finally, a conclusion is provided.

2 Basical Definitions

In this section, we expressed some basic definitions and properties of fractional calculus which are further used in the paper [24].

Definition 2.1. For any real function $f(t)$, $t>0$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, is said to be in space C_μ, $\mu\in\mathbb{R}$, and it is in the space C_μ^n if and only if $f^{(n)}\in C_\mu,\ n\in N$.

Definition 2.2. The fractional derivative of $f(t)$ in the Caputo sense by the Riemann-Liouville fractional integral operator of order $\alpha>0$ is defined as

$$D^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}D^m f(s)ds, \qquad \alpha>0,$$

for $m-1<\alpha\leqslant m,\ m\in N,\ t>0$ and $f\in C_{-1}^m$.

Some properties of the operator D^α are as follows:

$$(i)\ D^\alpha C=0,$$

$$(ii)\ D^\alpha t^\gamma=\begin{cases} 0 & \gamma\in N_0 \ and\ \gamma<\alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}, & Otherwise. \end{cases} \tag{2.1}$$

$$(iii)\ D^\alpha(\sum_{i=1}^n c_i f_i(t))=\sum_{i=1}^n c_i D^\alpha f_i(t), \qquad where\ c_i\in\mathbb{R}. \tag{2.2}$$

where $f\in C_\mu,\ \mu\geq-1,\ \alpha,\beta\geq 0,\ \gamma\geq -1,\ N_0=\{0,1,2,...\}$, and c_i and C are constant.

Definition 2.3. Suppose that $f(t)\in C(0,\eta]$ and $w(t)$ is a weight function, then

$$\parallel f(t)\parallel_w^2 \ = \ \int_0^\eta f^2(t)w(t)dt.$$

3 Generalized Fractional Order of the Chebyshev Functions (GFCF)

The Chebyshev polynomials have many properties, for example recursively, orthogonality, simple real zeros, complete for the space of polynomials. For these reasons, many authors have used these functions in their works [25–28].

Some researchers by using some transformations extended Chebyshev polynomials to infinite or semi-infinite domains. For example, by $x=\frac{t-L}{t+L}, L>0$ the rational Chebyshev functions on semi-infinite domain [29–31], by $x=\frac{t}{\sqrt{t^2+L}}, L>0$ the rational Chebyshev functions on infinite domain [32], and by $x=1-2(\frac{t}{\eta})^\alpha$, $\alpha,\eta>0$ the generalized fractional order of the Chebyshev functions (GFCF) [33] are introduced.

Darani and Nasiri in [15] have introduced the fractional-order Chebyshev functions of the second kind, then just constructed the derivative operational matrix for them and used it for solving linear fractional differential equations.

In the present work, we use the transformation $x = 1-2(\frac{t}{\eta})^\alpha$, $\alpha, \eta > 0$ on the Chebyshev polynomials of the first kind, that was introduced in [33], and can use them to solve the nonlinear integro- differential equations of multi-arbitrary order.

The GFCFs are defined on interval $[0, \eta]$, denoted by ${}_\eta FT_n^\alpha(t) = T_n(1-2(\frac{t}{\eta})^\alpha)$, and have the analytical form as follows [33]:

$$ {}_\eta FT_n^\alpha(t) = \sum_{k=0}^{n} \beta_{n,k,\eta,\alpha}.t^{\alpha k}, \qquad t \in [0, \eta], \tag{3.1}$$

where

$$\beta_{n,k,\eta,\alpha} = (-1)^k \frac{n 2^{2k}(n+k-1)!}{(n-k)!(2k)!\eta^{\alpha k}} \qquad \textit{and} \qquad \beta_{0,k,\eta,\alpha} = 1.$$

The GFCFs with $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ are orthogonal on the interval $(0, \eta)$:

$$\int_0^\eta {}_\eta FT_n^\alpha(t)\ {}_\eta FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}, \tag{3.2}$$

where δ_{mn} is the Kronecker delta function, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

Any function $y(t) \in C[0, \eta]$ can be expanded as [34]:

$$y(t) = \sum_{n=0}^{\infty} a_n\ {}_\eta FT_n^\alpha(t),$$

and using the property of orthogonality in the GFCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t) y(t) w(t) dt, \qquad n = 0, 1, 2, \cdots,$$

but in the numerical methods, we have to use first m-terms of the GFCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m-1} a_n\ {}_\eta FT_n^\alpha(t) = A^T \Phi(t), \tag{3.3}$$

where

$$A = [a_0, a_1, ..., a_{m-1}]^T, \tag{3.4}$$

$$\Phi(t) = [\ {}_\eta FT_0^\alpha(t),\ {}_\eta FT_1^\alpha(t), ...,\ {}_\eta FT_{m-1}^\alpha(t)]^T. \tag{3.5}$$

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 3.1. *Suppose that $D^{k\alpha} f(t) \in C[0, \eta]$ for $k = 0, 1, ..., m$, and ${}_\eta F_m^\alpha$ is the generated subspace by $\{{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \cdots, {}_\eta FT_{m-1}^\alpha(t)\}$. If $f_m(t) = A^T\Phi(t)$ (Eq. (3.3)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is:*

$$\| f(t) - f_m(t) \|_w \leq \frac{\eta^{m\alpha} M_\alpha}{2^m \Gamma(m\alpha + 1)} \sqrt{\frac{\pi}{\alpha\ m!}},$$

where $M_\alpha \geq |D^{m\alpha} f(t)|$, $t \in [0, \eta]$.

Proof. See Ref. [33]. □

4 Operational Matrices of the GFCFs

In this section, operational matrices the fractional derivative and the product for the GFCFs are constructed, these matrices can be used to solve the linear and nonlinear differential equations of arbitrary order.

4.1 The Fractional Derivative Operational Matrix of the GFCFs

The α-order Caputo fractional derivative operator of the vector $\Phi(t)$ in Eq. (3.5) can be expressed by

$$D^{\alpha}\Phi(t) = \mathbf{D}^{(\alpha)}\Phi(t). \tag{4.1}$$

In the following theorem, the operational matrix of fractional derivatives of the GFCFs is generalized.

Theorem 4.1. *Let $\Phi(t)$ be the GFCFs vector in Eq. (3.5) and $\boldsymbol{D}^{(\alpha)}$ be the $m \times m$ fractional derivative operational matrix of the α-order Caputo fractional derivatives as follows:*

$$D^{\alpha}\Phi(t) = \boldsymbol{D}^{(\alpha)}\Phi(t). \tag{4.2}$$

where

$$\boldsymbol{D}^{(\alpha)}_{i,j} = \begin{cases} \frac{2}{\sqrt{\pi}c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k,\eta,\alpha}\beta_{j,s,\eta,\alpha}\frac{\Gamma(\alpha k+1)\Gamma(s+k-\frac{1}{2})\eta^{\alpha(k+s-1)}}{\Gamma(\alpha k-\alpha+1)\Gamma(s+k)}, & i > j \\ 0 & \textit{otherwise} \end{cases} \tag{4.3}$$

for $i, j = 0, 1, ..., m-1$.

Proof. See Ref. [33]. □

4.2 The Product Operational Matrix of the GFCFs

The following property of the product of two GFCFs vectors will also be applied:

$$\Phi(t)\Phi(t)^{T}A \approx \widehat{\mathbf{A}}\Phi(t), \tag{4.4}$$

where $\widehat{\mathbf{A}}$ is the $m \times m$ product operational matrix for the vector $A = \{a_i\}_{i=0}^{m-1}$.

Theorem 4.2. *Let $\Phi(t)$ be the GFCFs vector in Eq. (3.5) and A is a vector, then the elements of $\widehat{\boldsymbol{A}}$, that is an $m \times m$ product operational matrix for the vector $A = \{a_i\}_{i=0}^{m-1}$, are obtained as*

$$\Phi(t)\Phi(t)^{T}A \approx \widehat{\boldsymbol{A}}\Phi(t), \tag{4.5}$$

where

$$\widehat{\boldsymbol{A}}_{ij} = \sum_{k=0}^{m-1} a_k \widehat{g_{ijk}}, \tag{4.6}$$

and

$$\widehat{g}_{ijk} = \begin{cases} \frac{c_k}{2c_j} & i \neq 0 \text{ and } j \neq 0 \text{ and } (k = i + j \text{ or } k = |i - j|) \\ \frac{c_k}{c_j} & (j = 0 \text{ and } k = i) \text{ or } (i = 0 \text{ and } k = j) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See Ref. [33]. □

4.3 The Dual Operational Matrix of the GFCFs

The dual operational matrix of $\Phi(t)$ is defined as follows:

$$\mathbf{B} = \int_0^\eta \Phi(t)\Phi^T(t)dt. \tag{4.7}$$

Now, the dual operational matrix of the GFCFs is generalized.

Theorem 4.3. *Let $\Phi(t)$ be the GFCFs vector in Eq.* (3.5) *and $\boldsymbol{B}$ be the $m \times m$ dual operational matrix, then for $i, j = 0, 1, ..., m - 1$:*

$$\boldsymbol{B}_{i,j} = \sum_{k=0}^{i} \sum_{s=0}^{j} \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \frac{\eta^{\alpha(k+s)+1}}{\alpha(k + s) + 1}. \tag{4.8}$$

Proof. Using Eq. (3.1), we can write

$$\begin{aligned} \mathbf{B}_{i,j} &= \int_0^\eta \phi_i(t)\phi_j(t)dt \\ &= \int_0^\eta \sum_{k=0}^{i} \beta_{i,k,\eta,\alpha} t^{\alpha k} \sum_{s=0}^{j} \beta_{j,s,\eta,\alpha} t^{\alpha s} dt \\ &= \sum_{k=0}^{i} \sum_{s=0}^{j} \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \int_0^\eta t^{\alpha(k+s)} dt, \end{aligned}$$

by integration of above equation, the theorem can be proved. □

Remark: Notice that the fractional derivative operational matrix of the GFCFs is a lower-triangular matrix that at least $50(1 + \frac{1}{m})\%$ of their elements are zero, the elements of the product operational matrix of the GFCFs are independent of values of α and η, and the dual operational matrix is symmetric.

5 Application of the Method

Consider Eq. (1.1), by previous section the two variable functions $k_l(x,t) \in L^2([0,1))^2$ can be approximated as:

$$k_l(x,t) \approx \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} k_{l,i,j}\phi(x)\phi(t),$$

or in the matrix form:

$$k_l(x,t) \approx \Phi^T(x)\mathbf{K}_l\Phi(t), \tag{5.1}$$

where $\mathbf{K}_l = [k_{l,i,j}]$ and also can be written:

$$y(x) \approx \sum_{n=0}^{m-1} a_n FT_n^{\alpha}(x) = A^T\Phi(x), \tag{5.2}$$

$$D^{\alpha}y(x) \approx \sum_{n=0}^{m-1} a_n D^{\alpha} FT_n^{\alpha}(x) = A^T\mathbf{D}^{(\alpha)}\Phi(x), \tag{5.3}$$

$$g(x) \approx \sum_{n=0}^{m-1} g_n FT_n^{\alpha}(x) = G^T\Phi(x). \tag{5.4}$$

and also we have

$$[y(x)]^2 \approx A^T\widehat{\mathbf{A}}\Phi(x),$$

that it is easy to show by induction:

$$[y(x)]^{q_l} \approx A^T(\widehat{\mathbf{A}})^{q_l-1}\Phi(x) = \Phi^T(x)((\widehat{\mathbf{A}})^{q_l-1})^T A, \qquad q_l = 1,2,3,\cdots. \tag{5.5}$$

where $\widehat{\mathbf{A}}$ is the product operational matrix in Eq. (4.4).

Using the above equations, we have:

$$\begin{aligned}
\int_0^1 k_l(x,t)[y(t)]^{q_l}dt &= \int_0^1 \Phi^T(x)\mathbf{K}_l\Phi(t)\Phi^T(t)(\widehat{\mathbf{A}}^{q_l-1})^T A dt \\
&= \Phi^T(x)\mathbf{K}_l\left(\int_0^1 \Phi(t)\Phi^T(t)dt\right)(\widehat{\mathbf{A}}^{q_l-1})^T A \\
&= \Phi^T(x)\mathbf{K}_l\mathbf{B}(\widehat{\mathbf{A}}^{q_l-1})^T A \\
&= A^T(\widehat{\mathbf{A}})^{q_l-1}\mathbf{B}^T\mathbf{K}_l^T\Phi(x),
\end{aligned} \tag{5.6}$$

where $\mathbf{B}$ is the dual operational matrix in Eq. (4.7).

By substituting the above approximations into Eq. (1.1), we obtain:

$$\sum_{j=1}^{N_1}\mu_j A^T\mathbf{D}^{(\alpha_j)}\Phi(x) + \sum_{l=1}^{N_2}\lambda_l A^T(\widehat{\mathbf{A}})^{q_l-1}\mathbf{B}^T\mathbf{K}_l^T\Phi(x) = G^T\Phi(x), \tag{5.7}$$

now, by multiplying two sides of Eq. (5.7) in $\Phi^T(x)$ then integration in the interval $[0, \eta]$, according to orthogonal of the GFCFs we get (Galerkin method):

$$A^T \left(\sum_{j=1}^{N_1} \mu_j \mathbf{D}^{(\alpha_j)} + \sum_{l=1}^{N_2} \lambda_l (\widehat{\mathbf{A}})^{q_l - 1} \mathbf{B}^T \mathbf{K}_l^T \right) = G^T, \tag{5.8}$$

which is a linear or nonlinear system of algebraic equations. By solving this system, we can obtain the approximate solution of Eq. (1.1) according to Eq. (5.2).

We define the residual function as follows:

$$Res(x) = A^T \left(\sum_{j=1}^{N_1} \mu_j \mathbf{D}^{(\alpha_j)} + \sum_{l=1}^{N_2} \lambda_l (\widehat{\mathbf{A}})^{q_l - 1} \mathbf{B}^T \mathbf{K}_l^T \right) \Phi(x) - G^T \Phi(x).$$

In the Galerkin (Tau) method that is derived from the weighted residual method, the test functions are chosen the same trail functions. Then, the inner product of the residual function and the test functions is set equal to zero, i.e. $\langle Res(x), \phi_n(x) \rangle = 0$, $n = 0, \cdots, m-1$, and algebraic equations obtain for the calculation of the coefficient of a_n.

Furthermore, the computations in this paper have been done by Maple 18.

6 Illustrative Examples

In this section, by using the present method we solve some well-known examples of multi-arbitrary order to show efficiently and applicability the GFCFs Galerkin method based on the Spectral method. Their outputs are compared with the corresponding analytical solution.

Example 6.1. Consider the following fractional nonlinear integro- differential equation [35, 36]:

$$\begin{aligned} D^\alpha y(x) - \int_0^1 xty^2(t)dt &= 1 - \frac{x}{4}, \qquad 0 \leqslant x < 1, \quad 0 < \alpha \leqslant 1 \\ y(0) &= 0. \end{aligned}$$

The exact solution of this problem for $\alpha = 1$ is $y(x) = x$. By applying the technique described in last section, the equation is obtained as

$$A^T[\mathbf{D}^{(\alpha)} - \widehat{\mathbf{A}}\mathbf{B}^T\mathbf{K}^T]\Phi(x) = G^T\Phi(x),$$

where $\mathbf{D}^{(\alpha)}$, $\widehat{\mathbf{A}}$, $\mathbf{B}^T$, $\mathbf{K}^T$ and G^T are obtained from Eqs. (4.2), (4.4), (4.7), (5.1) and (5.4), respectively. Using the above equations and Galerkin method, $m-1$ algebraic equations can be generated as

$$A^T[\mathbf{D}^{(\alpha)} - \widehat{\mathbf{A}}\mathbf{B}^T\mathbf{K}^T] = G^T. \tag{6.1}$$

Also, for $y(0) = 0$, one has

$$A^T \Phi(0) = 0. \tag{6.2}$$

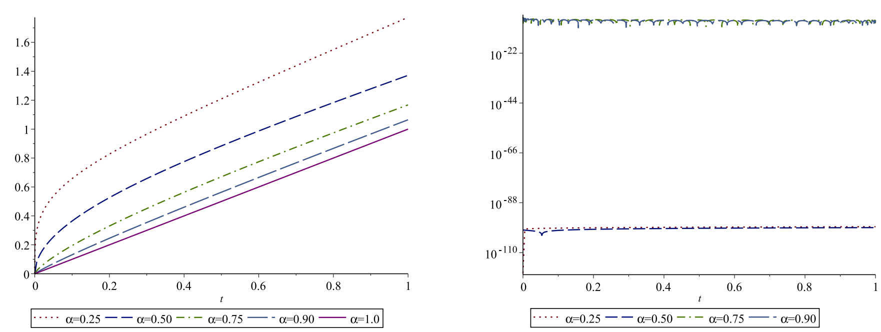
The Eqs. (6.1) and (6.2) generate a set of nonlinear algebraic equations.

Fig. 1(a) shows the numerical results for the values various of $0 < \alpha \leqslant 1$. The comparisons show that as $\alpha \to 1$, the approximate solutions tend to $y(x) = x$, which is the exact solution of the equation in the case of $\alpha = 1$. The residual errors in this cases are shown in Fig. 1(b). Table 1 shows the obtained values of $y(x)$ by the present method with various values α. We define the residual error as follows

$$Res(x) = A^T \mathbf{D}^{(\alpha)} \Phi(x) - \int_0^1 x t y_m^2(t) dt - (1 - \frac{x}{4})$$

Table 1: Obtained values of $y(x)$ by the present method with various values α for Example 6.1

x	$\alpha = 0.25$, $m = 13$		$\alpha = 0.50$, $m = 7$		$\alpha = 1.00$, $m = 4$	
	$y(x)$	$Res(x)$	$y(x)$	$Res(x)$	$y(x)$	Abs. Err.
0.1	0.658062583965183	5.4e-100	0.364522952461285	6.3e-101	0.10000	9.37e-104
0.2	0.827349212023551	7.9e-100	0.526400101930032	1.9e-100	0.20000	3.75e-103
0.3	0.965165762796598	1.02e-99	0.658039376089674	3.0e-100	0.30000	8.43e-103
0.4	1.090386646323213	1.25e-99	0.775234680306955	4.2e-100	0.40000	1.50e-102
0.5	1.209246980182119	1.46e-99	0.883952262100326	5.4e-100	0.50000	2.34e-102
0.6	1.324569576356220	1.68e-99	0.987177676009430	6.5e-100	0.60000	3.37e-102
0.7	1.437856268459510	1.89e-99	1.086641092417285	7.6e-100	0.70000	4.59e-102
0.8	1.549989720809763	2.09e-99	1.183441789016069	8.7e-100	0.80000	6.00e-102
0.9	1.661524824469015	2.29e-99	1.278323958921395	9.8e-100	0.90000	7.59e-102
1.0	1.772827573935424	2.50e-99	1.371815388009805	1.10e-99	1.00000	9.37e-102



(a) The approximate solutions (b) The residual error functions

Figure 1: The approximate solutions and the residual error functions for Example 6.1 with various values α

Example 6.2. Next, we consider the following fractional nonlinear integro-differential equation [35]:

$$
\begin{aligned}
D^{\frac{1}{2}}y(x) - \int_0^1 xty^4(t)dt &= g(x), \qquad 0 \leqslant x < 1, \\
y(0) &= 0.
\end{aligned}
$$

where $g(x) = \frac{1}{\Gamma(1/2)}(\frac{8}{3}\sqrt{x^3} - 2\sqrt{x}) - \frac{x}{1260}$. The exact solution of this problem is $y(x) = x^2 - x$.

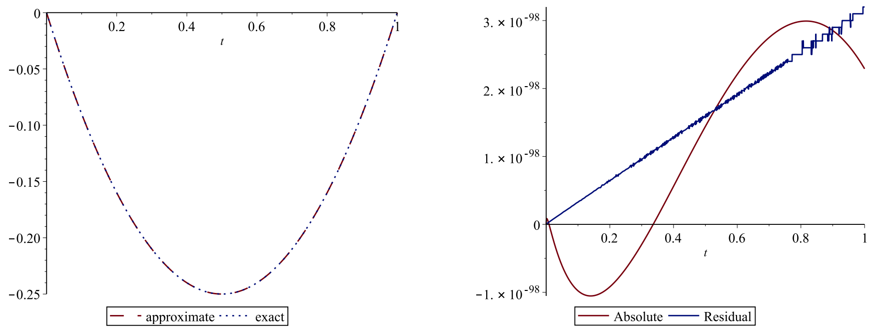
By applying the technique described in the last section, the equation is obtained as

$$
\begin{aligned}
A^T[\mathbf{D}^{(\frac{1}{2})} - \widehat{\mathbf{A}}^3\mathbf{B}^T\mathbf{K}^T] &= G^T, \\
A^T\Phi(0) &= 0,
\end{aligned}
\tag{6.3}
$$

The absolute and residual errors and the approximate solution with $m = 20$ are displayed in Fig. 2, we can see that the approximate solution is in a good agreement with the exact solution. Table 2 shows the absolute and residual errors with $m = 20$.

Table 2: The absolute and residual errors with $m = 20$, $\alpha = 0.50$ for Example 6.2

x	absolute error	residual error
0.1	9.779e-99	3.26e-99
0.2	9.192e-99	6.50e-99
0.3	2.948e-99	9.70e-99
0.4	5.603e-99	1.32e-98
0.5	1.441e-98	1.60e-98
0.6	2.206e-98	1.95e-98
0.7	2.749e-98	2.21e-98
0.8	2.987e-98	2.60e-98
0.9	2.852e-98	2.80e-98
1.0	2.290e-98	2.80e-98



(a) Comparison of the exact solution and the approximate solution

(b) The absolute and residual errors

Figure 2: (a) Comparison of the exact solution and the approximate solution, (b) The absolute and residual errors, for Example 6.2 with $m = 20$ and $\alpha = 0.50$

Example 6.3. Consider the following nonlinear Fredholm integro- differential equation of order $\alpha = \frac{1}{3}$ [35]:

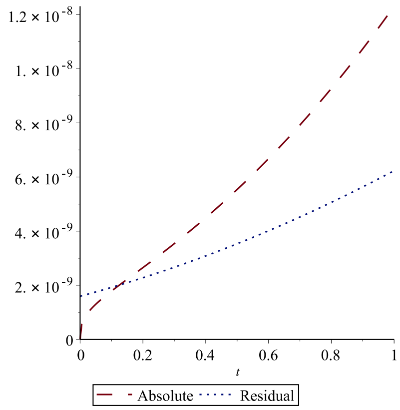
$$\begin{aligned} D^{\frac{1}{3}}y(x) - \int_0^1 (x+t)^2 y^3(t)dt &= g(x), \qquad 0 \leqslant x < 1, \\ y(0) &= 0. \end{aligned}$$

where $g(x) = \frac{2}{\Gamma(8/3)}x^{\frac{5}{3}} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}$. The exact solution of this problem is $y(x) = x^2$.

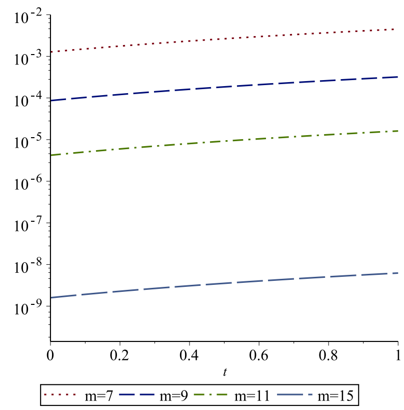
By applying the technique described in the last section, the equation is obtained as

$$\begin{aligned} A^T[\mathbf{D}^{(\frac{1}{3})} - \widehat{\mathbf{A}}^2 \mathbf{B}^T \mathbf{K}^T] &= G^T. \qquad (6.4) \\ A^T \Phi(0) &= 0, \end{aligned}$$

The absolute and residual errors with $m = 15$ are displayed in Fig. 3(a). To show the convergence of the present method to solve this example in the Fig. 3(b), we showed that by increasing the m the residual function decreases.



(a) The absolute and residual errors



(b) Log residual functions for m =7, 9, 11, 15, to show the convergence rate

Figure 3: (a) The absolute and residual errors, (b) Log residual functions to show the convergence rate of GFCF method

Example 6.4. Consider the following nonlinear Fredholm integro- differential equation of order $\alpha = \frac{3}{2}$:

$$\begin{aligned} D^{\frac{3}{2}}y(x) - \int_0^1 e^{x+t}y^2(t)dt &= g(x), \qquad 0 \leqslant x < 1, \\ y(0) &= -2, \\ y'(0) &= 0, \end{aligned}$$

where $g(x) = \frac{8}{\sqrt{\pi}}x^{\frac{3}{2}} + (748 - 277e)e^x$. The exact solution of this problem is $y(x) = x^3 - 2$.

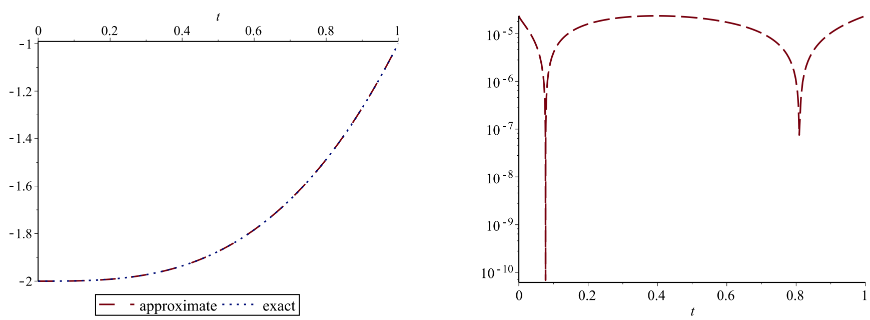
By applying the technique described in the last section, the equation is obtained as

$$\begin{aligned} A^T[\mathbf{D}^{(\frac{3}{2})} - \widehat{\mathbf{A}}\mathbf{B}^T\mathbf{K}^T] &= G^T, \\ A^T\Phi(0) &= -2, \\ A^T\mathbf{D}^{(1)}\Phi(0) &= 0, \end{aligned} \tag{6.5}$$

The residual error and the approximate solution with $m = 7$ are displayed in Fig. 4, we can see the approximate solution is in a good agreement with the exact solution. Table 3 shows the absolute and residual errors with $m = 7$.

Table 3: The absolute and residual errors with $m = 7$ and $\alpha = \frac{3}{4}$ for Example 6.4

t	absolute error	residual error
0.1	2.18955e-7	3.97608e-6
0.2	1.01085e-7	1.58680e-5
0.3	4.83845e-7	2.17587e-5
0.4	1.48242e-6	2.34365e-5
0.5	2.78456e-6	2.17600e-5
0.6	4.25067e-6	1.72455e-5
0.7	5.72279e-6	1.02422e-5
0.8	7.03034e-6	1.00414e-6
0.9	7.99348e-6	1.02741e-5
1.0	8.42526e-6	2.34381e-5



(a) Comparison of the solutions (b) The residual error function

Figure 4: (a) Comparison of the exact solution and approximate solution, (b) The residual error function, for Example 6.4 with $m = 7$ and $\alpha = \frac{3}{4}$

Example 6.5. Consider the following nonlinear Fredholm integro- differential equation with multi order:

$$\begin{aligned} 2D^{\frac{4}{3}}y(x) - D^{\frac{1}{6}}y(x) - 56\int_0^1 (x+t)y^3(t)dt &= g(x), \qquad 0 \leqslant x < 1, \\ y(0) &= 1, \\ y'(0) &= -3, \end{aligned}$$

where $g(x) = \frac{6}{\Gamma(\frac{2}{3})}x^{\frac{2}{3}} - \frac{18}{55\Gamma(\frac{5}{6})}x^{\frac{5}{6}}(4x-11) + 6x + 9$. The exact solution of this problem is $y(x) = x^2 - 3x + 1$.

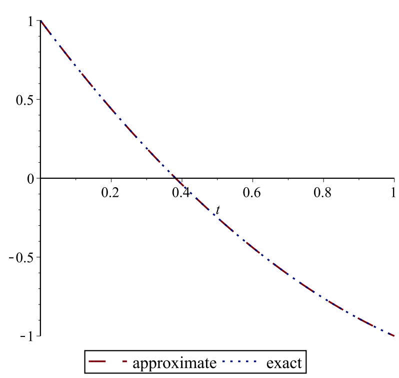
By applying the technique described in the last section, the equation is obtained as

$$\begin{aligned} A^T[2\mathbf{D}^{(\frac{4}{3})} - \mathbf{D}^{(\frac{1}{6})} - 56\widehat{\mathbf{A}}^2\mathbf{B}^T\mathbf{K}^T] &= G^T, \\ A^T\Phi(0) &= 1, \\ A^T\mathbf{D}^{(1)}\Phi(0) &= -3, \end{aligned} \tag{6.6}$$

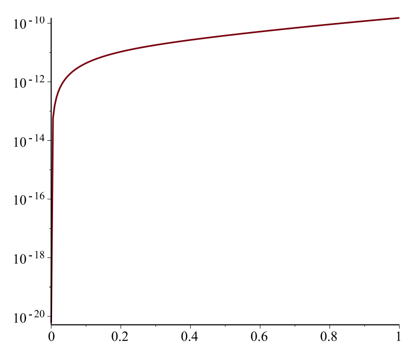
The residual error and the approximate solution with $m = 20$ are displayed in Fig. 5. Table 4 shows the absolute and residual errors with $m = 20$.

Table 4: The absolute and residual errors with $m = 20$ and $\alpha = \frac{1}{6}$ for Example 6.5

t	absolute error	residual error
0.1	7.19310e-9	-4.33445e-12
0.2	5.67658e-9	-1.06538e-11
0.3	3.50843e-9	-1.80383e-11
0.4	9.8974e-10	-2.68854e-11
0.5	1.76571e-9	-3.78585e-11
0.6	4.69932e-9	-5.17159e-11
0.7	7.77568e-9	-6.92530e-11
0.8	1.09712e-8	-9.12790e-11
0.9	1.42691e-8	-1.18605e-10
1.0	1.76569e-8	-1.52041e-10



(a) Comparison of the solutions



(b) The residual error function

Figure 5: (a) Comparison of the exact solution and approximate solution, (b) The residual error function, for Example 6.5 with $m = 20$ and $\alpha = \frac{1}{6}$

Example 6.6. Consider the following nonlinear Fredholm integro- differential equation with multi order:

$$
\begin{aligned}
D^3y(x)+2D^2y(x)-D^{\frac{1}{2}}y(x)+\int_0^1 t^2y(t)dt-\frac{360}{113}\int_0^1 xt^3y^2(t)dt &= g(x),\\
y(0) &= 0,\\
y'(0) &= 4,\\
y''(0) &= 0,
\end{aligned}
$$

where $g(x)=\frac{43}{6}-\frac{8}{5\Gamma(\frac{1}{2})}x^{\frac{1}{2}}(2x^2+5)$. The exact solution of this problem is $y(x)=x^3+4x$.

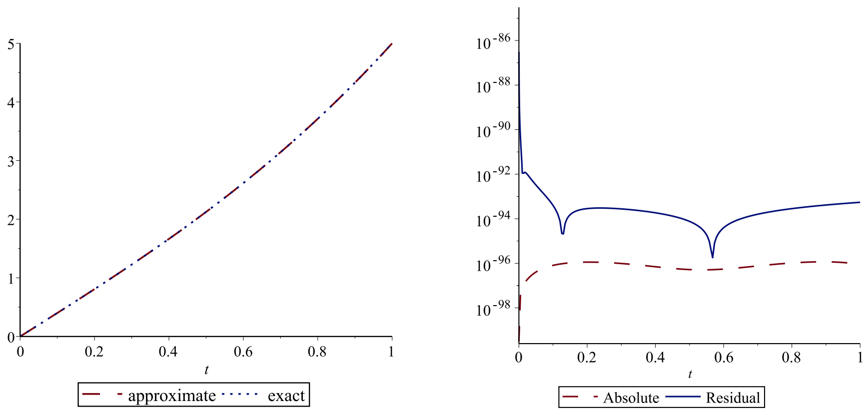
By applying the technique described in the last section, the equation is obtained as

$$
\begin{aligned}
A^T[\mathbf{D}^{(3)}+2\mathbf{D}^{(2)}-\mathbf{D}^{(\frac{1}{6})}+\mathbf{B}^T\mathbf{K}_1^T-\frac{360}{113}\widehat{\mathbf{A}}\mathbf{B}^T\mathbf{K}_2^T] &= G^T, \qquad (6.7)\\
A^T\Phi(0) &= 0,\\
A^T\mathbf{D}^{(1)}\Phi(0) &= 4,\\
A^T\mathbf{D}^{(2)}\Phi(0) &= 0,
\end{aligned}
$$

The absolute and residual errors and the approximate solution with $m=15$ are displayed in Fig. 6. Table 5 shows the absolute and residual errors with $m=15$. We can see that the approximate solution is in a good agreement with the exact solution.

Table 5: The absolute and residual errors with $m=15$ and $\alpha=0.50$ for Example 6.6

x	Absolute error	Residual error
0.1	8.100e-97	3.980e-94
0.2	1.132e-96	2.856e-94
0.3	9.837e-97	2.771e-94
0.4	7.031e-97	1.855e-94
0.5	5.252e-97	7.627e-94
0.6	5.514e-97	3.995e-94
0.7	7.632e-97	1.599e-94
0.8	1.038e-96	2.848e-94
0.9	1.166e-96	4.142e-94
1.0	8.602e-97	5.482e-94



(a) Comparison of the solutions (b) The absolute and residual errors

Figure 6: (a) Comparison of the exact solution and approximate solution, (b) The absolute and residual errors, for Example 6.6 with $m = 15$ and $\alpha = 0.50$

7 Conclusion

In this paper, the generalized fractional order of the Chebyshev functions (GFCF) of the first kind is expressed, next, the operational matrices of fractional derivative, the product, and the dual for these orthogonal functions are obtained. These matrices can be used to solve the linear and nonlinear Fredholm integro- differential equations of the multi-arbitrary order. As shown, the method converges and has an appropriate accuracy and stability, that the sufficient accuracy is due choosing the basic of fractional. Illustrative examples show that this method has good results.

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