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Common Fixed Point Results for Three Maps One of which is Multivalued in G-Metric Spaces

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Abstract : In this work, we prove the existence of common fixed points for two single-valued maps and one multi-valued map satisfying certain contractive conditions in G-metric spaces.

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1 Introduction

Mustafa and Sims introduced the G-metric spaces as a generalization of the notion of metric spaces.

Definition 1.1. [1] Let X be a non-empty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

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(G5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, the function G is called a *generalized metric*, or more specially, a *G-metric on X*, and the pair (X,G) is called a *G-metric space*.

Example 1.2. Let (X,d) be a metric space. The function $G : X \times X \times X \to [0,+\infty)$, defined by $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, for all $x,y,z \in X$, is a G-metric on X.

Later, many results appeared in G-metric spaces (see [2–5]). Abbas and Rhoades [6] initiated the study of common fixed point in G-metric spaces. Since then the common fixed point theorem for mappings satisfying certain contractive conditions has been continually studied for decade (see [7–9]). Recently, Abbas, Nazir and Vetro [10] proved some common fixed point results for three single-valued maps in G-metric spaces. The aim of this paper is to prove the existence of the common fixed points for two single-valued and one multi-valued maps in G-metric spaces. Our results improve Theorem 2.1, 2.4, 2.8 and 2.11 of Abbas et al [10].

2 Preliminaries

We now recall some of the basic concepts and results in G-metric spaces that were introduced in [1].

Definition 2.1. [1] Let (X,G) be a G-metric space, and $\{x_n\}$ a sequence of points of X. We say that $\{x_n\}$ is *G-convergent* to $x \in X$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.2. [1] *Let (X,G) be a G-metric space, the following are equivalent:*

(1) $\{x_n\}$ *is G-convergent to x,*

(2) $G(x_n,x_n,x) \to 0$ *as* $n \to +\infty$,

(3) $G(x_n,x,x) \to 0$ *as* $n \to +\infty$,

(4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to +\infty$.

Definition 2.3. [1] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a *G-Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$. That is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 2.4. [1] *Let (X,G) be a G-metric space, the following are equivalent*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq N$.*

Definition 2.5. [1] A G-metric space (X, G) is called *G-complete* if every G-Cauchy sequence is G-convergent in (X, G).

Every G-metric on X defines a metric d_G on X given by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Recently, Kaewcharoen and Kaewkhao [11] introduced the following concepts. Let X be a G-metric space. We shall denote $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(\cdot, \cdot, \cdot)$ be the Hausdorff G-distance on $CB(X)$, i.e.,

$$H_G(A, B, C) = \max\{\sup_{x \in A} G(x, B, C), \sup_{x \in B} G(x, C, A), \sup_{x \in C} G(x, A, B)\},$$

where

$$\begin{aligned} G(x, B, C) &= d_G(x, B) + d_G(B, C) + d_G(x, C), \\ d_G(x, B) &= \inf\{d_G(x, y), y \in B\}, \\ d_G(A, B) &= \inf\{d_G(a, b), a \in A, y \in B\}. \end{aligned}$$

Recall that $G(x, y, C) = \inf\{G(x, y, z), z \in C\}$. A mapping $T : X \to 2^X$ is called a *multi-valued mapping.* A point $x \in X$ is called a *fixed point* of T if $x \in Tx$.

Proposition 2.6. *Let X be a G-metric space and $A, B, C \subset X$. For $x, y \in X$ and $a \in A$, we have*

(i) $G(x, y, y) \leq 2G(y, x, x)$,

(ii) $G(x, x, y) \leq G(x, y, A)$ *if* $y \notin A$,

(iii) $G(x, x, A) \leq G(x, y, A)$ *if* $y \notin A$,

(iv) $G(x, x, A) + G(x, x, B) \leq G(x, A, B)$,

(v) $G(x, A, A) \leq 6G(x, a, a)$,

(vi) $G(x, A, A) \leq 4G(x, y, A) \leq 4G(x, y, a)$ *if* $y \notin A$ *and* $x \neq y$,

(vii) $G(a, B, C) \leq H_G(A, B, C)$,

(viii) $G(x, y, A) \leq G(x, y, a) \leq H_G(x, y, A)$.

Proof. It is easy to check that (i)-(iii) and (vii)-(viii) hold, so we will show that (iv), (v) and (vi) hold. Let $x, y \in X$ and $A, B \subset X$,

(iv) By (G2) and (G4), we get

$$\begin{aligned} G(x, x, A) + G(x, x, B) &= \inf\{G(x, x, a), a \in A\} + \inf\{G(x, x, b), b \in B\} \\ &\leq \inf\{G(x, a, a) + G(a, x, x), a \in A\} \\ &\quad + \inf\{G(x, b, b) + G(b, x, x), b \in B\} \\ &= \inf\{d_G(x, a), a \in A\} + \inf\{d_G(x, b), b \in B\} \\ &= d_G(x, A) + d_G(x, B) \\ &= G(x, A, B). \end{aligned}$$

(v) By (i), we obtain

$$\begin{aligned} G(x,A,A) &= 2\inf\{G(x,a,a), a\in A\} \\ &\leq 2\inf\{G(x,a,a)+G(a,x,x), a\in A\} \\ &\leq 2[G(x,a,a)+G(a,x,x)], \text{ for all } a\in A \\ &\leq 2G(x,a,a)+4G(x,a,a), \text{ for all } a\in A \\ &= 6G(x,a,a), \text{ for all } a\in A. \end{aligned}$$

(vi) Let $x\neq y$ and $y\notin A$. By (v) and (G3), we have

$$\begin{aligned} G(x,A,A) &\leq 2[G(x,a,a)+G(a,x,x)], \text{ for all } a\in A \\ &\leq 2G(x,a,y)+2G(a,x,y), \text{ for all } a\in A \\ &= 4G(x,y,a), \text{ for all } a\in A. \end{aligned}$$

Therefore, $G(x,A,A)\leq 4\inf\{G(x,y,a), a\in A\} = 4G(x,y,A)$. □

3 Main Results

Theorem 3.1. *Let (X,G) be a G-metric space. Assume that $f,g: X\to X$ and $T: X\to CB(X)$ satisfy the following condition*

$$\begin{aligned} H_G(fx,gy,Tz) \leq\ & \alpha G(x,y,z)+\beta[G(fx,x,x)+G(y,gy,y)+G(z,z,Tz)] \\ & +\gamma[G(fx,y,z)+G(x,gy,z)+G(x,y,Tz)] \end{aligned} \tag{3.1}$$

for all $x,y,z\in X$, where $\alpha,\beta,\gamma>0$ and $\alpha+4\beta+4\gamma<1$. Then f,g and T have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and T and conversely.

Proof. First, we will prove that any fixed point of f is a fixed point of g and T. Assume that $p\in X$ is such that $fp=p$. Now, we prove that $p=gp=Tp$. If it is not the case, then for $p\neq gp$ and $p\notin Tp$,

Case 1: If $gp\notin Tp$, we have

$$\begin{aligned} G(p,gp,Tp) &\leq H_G(fp,gp,Tp) \\ &\leq \alpha G(p,p,p)+\beta[G(fp,p,p)+G(p,gp,p)+G(p,p,Tp)] \\ &\quad +\gamma[G(fp,p,p)+G(p,gp,p)+G(p,p,Tp)] \\ &= (\beta+\gamma)[G(p,p,gp)+G(p,p,Tp)] \\ &\leq (\beta+\gamma)G(p,gp,Tp), \end{aligned}$$

a contradiction.

Case 2: If $gp \in Tp$, we have

$$\begin{aligned} G(p, gp, gp) &\leq H_G(fp, gp, Tp) \\ &\leq \alpha G(p, p, p) + \beta[G(fp, p, p) + G(p, gp, p) + G(p, p, Tp)] \\ &\quad +\gamma[G(fp, p, p) + G(p, gp, p) + G(p, p, Tp)] \\ &= (\beta + \gamma)[G(p, p, gp) + G(p, p, Tp)] \\ &\leq 2(\beta + \gamma)G(p, p, gp) \\ &\leq 2(\beta + \gamma)G(p, gp, gp), \end{aligned}$$

a contradiction. Therefore, $p = gp = Tp$. Analogously, following the similar arguments to those given above, we obtain a contradiction for $p \neq gp$ and $p \in Tp$ or for $p = gp$ and $p \notin Tp$. Hence in all the cases, we conclude that $p = gp \in Tp$. The same conclusion holds if $p = gp$ or $p \in Tp$.

Next, we will show that f, g and T have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} \in Tx_{3n+2}$, $n = 0, 1, 2, ...$ If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and T. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} & G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ \leq\ & H_G(fx_{3n}, gx_{3n+1}, Tx_{3n+2}) \\ \leq\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(fx_{3n}, x_{3n}, x_{3n}) + G(x_{3n+1}, gx_{3n+1}, x_{3n+1}) \\ & +G(x_{3n+2}, x_{3n+2}, Tx_{3n+2})] + \gamma[G(fx_{3n}, x_{3n+1}, x_{3n+2}) \\ & +G(x_{3n}, gx_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, Tx_{3n+2})] \\ \leq\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(x_{3n+1}, x_{3n}, x_{3n}) + G(x_{3n+1}, x_{3n+2}, x_{3n+1}) \\ & +G(x_{3n+2}, x_{3n+2}, x_{3n+3})] + \gamma[G(x_{3n+1}, x_{3n+1}, x_{3n+2}) \\ & +G(x_{3n}, x_{3n+2}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+3})] \\ \leq\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ & +G(x_{3n+1}, x_{3n+2}, x_{3n+3})] + \gamma[G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ & +G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})], \end{aligned}$$

that is

$$(1 - \beta - \gamma)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + 2\beta + 3\gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Hence,

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \lambda G(x_{3n}, x_{3n+1}, x_{3n+2}),$$

where $\lambda = \frac{\alpha+2\beta+3\gamma}{1-\beta-\gamma}$. Obviously $0 < \lambda < 1$. Repeating this process, we have for each n

$$G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \lambda G(x_n, x_{n+1}, x_{n+2}) \leq \cdots \leq \lambda^{n+1} G(x_0, x_1, x_2).$$

Now, for any l, m, n with $l > m > n$,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \cdots + G(x_{l-1}, x_{l-1}, x_l) \\ &\leq G(x_n, x_{x+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) \\ &\quad + \cdots + G(x_{l-2}, x_{l-1}, x_l) \\ &\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{l-2}]G(x_0, x_1, x_2) \\ &\leq \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_2). \end{aligned}$$

The same is holds if $l = m > n$ and if $l > m = n$ we have

$$G(x_n, x_m, x_l) \leq \frac{\lambda^{n-1}}{1-\lambda} G(x_0, x_1, x_2).$$

Consequently, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. This show that the sequence $\{x_n\}$ is a G-Cauchy in the complete space X. Thus, $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned} &G(fu, x_{3n+2}, x_{3n+3}) \\ \leq\ & H_G(fu, gx_{3n+1}, Tx_{3n+2}) \\ \leq\ & \alpha G(u, x_{3n+1}, x_{3n+2}) + \beta[G(fu, u, u) + G(x_{3n+1}, gx_{3n+1}, x_{3n+1}) \\ & + G(x_{3n+2}, x_{3n+2}, Tx_{3n+2})] + \gamma[G(fu, x_{3n+1}, x_{3n+2}) \\ & + G(u, gx_{3n+1}, x_{3n+2}) + G(u, x_{3n+1}, Tx_{3n+2})] \\ \leq\ & \alpha G(u, x_{3n+1}, x_{3n+2}) + \beta[G(fu, u, u) + G(x_{3n+1}, x_{3n+2}, x_{3n+1}) \\ & + G(x_{3n+2}, x_{3n+2}, x_{3n+3})] + \gamma[G(fu, x_{3n+1}, x_{3n+2}) \\ & + G(u, x_{3n+2}, x_{3n+2}) + G(u, x_{3n+1}, x_{3n+3})]. \end{aligned}$$

Taking limit $n \to \infty$, we obtain that

$$G(fu, u, u) \leq (\beta + \gamma) G(fu, u, u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $u \in Tu$.

Finally, suppose that v is another common fixed point of f, g and T, then

$$\begin{aligned} G(u, v, v) &\leq H_G(fu, gv, Tv) \\ &\leq \alpha G(u, v, v) + \beta[G(fu, u, u) + G(v, gv, v) + G(v, v, Tv)] \\ &\quad + \gamma[G(fu, v, v) + G(u, gv, v) + G(u, v, Tv)] \\ &\leq \alpha G(u, v, v) + \beta[G(u, u, u) + G(v, v, v) + G(v, v, v)] \\ &\quad + \gamma[G(u, v, v) + G(u, v, v) + G(u, v, v)] \\ &= (\alpha + 3\gamma) G(u, v, v), \end{aligned}$$

which gives that $G(u, v, v) = 0$, and $u = v$. We can conclude that u is a unique common fixed point of f, g and T. □

Theorem 3.2. *Let (X, G) be a G-metric space. Assume that $f, g : X \to X$ and $T : X \to CB(X)$ satisfy the following condition*

$$G(fx, gy, Tz) \leq aG(x, y, z) + bG(x, fx, fx) + cG(y, gy, gy) + dG(z, Tz, Tz) \quad (3.2)$$

for all $x, y, z \in X$, where $0 < a + 2b + 2c + 6d < 1$. Then f, g and T have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and T and conversely.

Proof. First, we will show that any fixed point of f is a fixed point of g and T. Assume that $p \in X$ is such that $fp = p$. Now, we prove that $p = gp = Tp$. If it is not the case, then for $p \neq gp$ and $p \notin Tp$,

Case 1: If $gp \notin Tp$, we have

$$\begin{aligned} G(p, gp, Tp) &\leq H_G(fp, gp, Tp) \\ &\leq aG(p, p, p) + bG(p, fp, fp) + cG(p, gp, gp) + dG(p, Tp, Tp) \\ &\leq aG(p, p, p) + bG(p, p, p) + cG(p, gp, Tp) + 4dG(p, gp, Tp) \\ &\leq (c + 4d)G(p, gp, Tp), \end{aligned}$$

which is a contradiction.

Case 2: If $gp \in Tp$, we have

$$\begin{aligned} G(p, gp, gp) &\leq H_G(fp, gp, Tp) \\ &\leq aG(p, p, p) + bG(p, fp, fp) + cG(p, gp, gp) + dG(p, Tp, Tp) \\ &\leq aG(p, p, p) + bG(p, p, p) + cG(p, gp, gp) + 6dG(p, gp, gp) \\ &\leq (c + 6d)G(p, gp, gp), \end{aligned}$$

which is a contradiction. Similarly to the previous case, we obtain a contradiction for $p \neq gp$ and $p \in Tp$ or for $p = gp$ and $p \notin Tp$. Hence in all the cases, we conclude that $p = gp \in Tp$. The same conclusion holds if $p = gp$ or $p \in Tp$.

Let now show that f, g and T have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} \in Tx_{3n+2}$, $n = 0, 1, 2, ...$ If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and T. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} & G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ \leq\ & H_G(fx_{3n}, gx_{3n+1}, Tx_{3n+2}) \\ \leq\ & aG(x_{3n}, x_{3n+1}, x_{3n+2}) + bG(x_{3n}, fx_{3n}, fx_{3n}) + cG(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) \\ & +dG(x_{3n+2}, Tx_{3n+2}, Tx_{3n+2}) \\ \leq\ & aG(x_{3n}, x_{3n+1}, x_{3n+2}) + bG(x_{3n}, x_{3n+1}, x_{3n+1}) + cG(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ & +4dG(x_{3n+1}, x_{3n+2}, x_{3n+3}), \end{aligned}$$

that is

$$(1-c-4d)G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq (a+b)G(x_{3n},x_{3n+1},x_{3n+2}).$$

Hence,

$$G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq \lambda G(x_{3n},x_{3n+1},x_{3n+2}),$$

where $\lambda = \frac{a+b}{1-c-4d}$. Obviously $0 < \lambda < 1$. Continue this process, we obtain for each n,

$$G(x_{n+1},x_{n+2},x_{n+3}) \leq \lambda G(x_n,x_{n+1},x_{n+2}) \leq \cdots \leq \lambda^{n+1} G(x_0,x_1,x_2).$$

Following similar arguments to those given in Theorem 3.1, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$. Hence, $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned}
& G(fu,x_{3n+2},x_{3n+3}) \\
\leq\ & H_G(fu,gx_{3n+1},Tx_{3n+2}) \\
\leq\ & aG(u,x_{3n+1},x_{3n+2}) + bG(u,fu,fu) + cG(x_{3n+1},gx_{3n+1},gx_{3n+1}) \\
& +dG(x_{3n+2},Tx_{3n+2},Tx_{3n+2}) \\
\leq\ & aG(u,x_{3n+1},x_{3n+2}) + bG(u,fu,x_{3n+1}) + cG(x_{3n+1},x_{3n+2},x_{3n+2}) \\
& +4dG(x_{3n+1},x_{3n+2},x_{3n+3}).
\end{aligned}$$

Taking limit $n \to \infty$, we obtain that

$$G(fu,u,u) \leq bG(fu,u,u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $u \in Tu$.

We next prove the uniqueness, suppose that v is another common fixed point of f,g and T, then

$$\begin{aligned}
G(u,v,v) &\leq H_G(fu,gv,Tv) \\
&\leq aG(u,v,v) + bG(u,fu,fu) + cG(v,gv,gv) + dG(v,Tv,Tv) \\
&\leq aG(u,v,v) + bG(u,u,u) + cG(v,v,v) + 6dG(v,v,v) \\
&\leq aG(u,v,v),
\end{aligned}$$

which gives that $G(u,v,v) = 0$, and $u = v$. Hence u is a unique common fixed point of f,g and T. □

Definition 3.3. Let $f : X \to X$ be a single-valued mapping, $T : X \to CB(X)$ a multi-valued mapping on G-metric space X. Then, f and T are said to be *commuting mappings* if $fTx \subset Tfx$ for all $x \in X$.

Example 3.4. Let $X = [0,1]$ and $G(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$ be a G-metric on X. Define $f,g : X \to X$ and $T : X \to CB(X)$ as

$$f(x) = \begin{cases} \frac{x}{24} & \text{if } x \in [0,\frac{1}{2}) \\ \frac{x}{20} & \text{if } x \in [\frac{1}{2},1], \end{cases} \quad g(x) = \begin{cases} \frac{x}{16} & \text{if } x \in [0,\frac{1}{2}) \\ \frac{x}{12} & \text{if } x \in [\frac{1}{2},1], \end{cases}$$

and

$$T(x) = \begin{cases} [0, \frac{x}{10}] & \text{if } x \in [0, \frac{1}{2}) \\ [0, \frac{x}{6}] & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Note that f, g and T are discontinuous maps. Also

$$\begin{aligned} fg(\frac{1}{2}) &= f(\frac{1}{24}) = \frac{1}{576}, & gf(\frac{1}{2}) &= f(\frac{1}{40}) = \frac{1}{640}, \\ gT(\frac{1}{2}) &= g([0, \frac{1}{12}]) = [0, \frac{1}{192}], & Tg(\frac{1}{2}) &= T(\frac{1}{24}) = [0, \frac{1}{240}], \\ fT(\frac{1}{2}) &= g([0, \frac{1}{12}]) = [0, \frac{1}{288}], & Tf(\frac{1}{2}) &= T(\frac{1}{40}) = [0, \frac{1}{400}], \end{aligned}$$

which shows that f, g and T does not commute to each other. For $x, y, z \in [0, \frac{1}{2})$,

$$\begin{aligned} H_G(fx, gy, Tz) &= \max\{|\frac{x}{24} - \frac{z}{10}|, \frac{x}{24}, |\frac{x}{24} - \frac{y}{16}|, |\frac{y}{16} - \frac{z}{10}|, \frac{y}{16}\} \\ &= \frac{1}{16}\max\{|\frac{2x}{3} - \frac{8z}{5}|, \frac{2x}{3}, |\frac{2x}{3} - y|, |y - \frac{8z}{5}|, y\} \\ &\leq \frac{1}{16}[\max\{|x - z|, |y - z|, |z - x|\} + x + y + z] \\ &= \frac{1}{16}\max\{|x - z|, |y - z|, |z - x|\} + \frac{x}{16} + \frac{y}{16} + \frac{z}{16} \\ &= \frac{1}{16}\max\{|x - z|, |y - z|, |z - x|\} + \frac{3}{46}(\frac{23x}{24}) + \frac{1}{15}(\frac{15y}{16}) \\ &\quad + \frac{5}{288}(\frac{18z}{5}) \\ &= aG(x, y, z) + bG(x, fx, fx) + cG(y, gy, gy) + dG(z, Tz, Tz). \end{aligned}$$

Thus (3.2) is satisfied for $0 < a + 2b + 2c + 6d = 0.43 < 1$. The same conclusion holds in all cases. 0 is the unique common fixed point of f, g and T. Also any fixed point of f is a fixed point of g and T and conversely.

Example 3.5. Let $X = [0, 1]$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ be a G-metric on X. Define $f, g : X \to X$, $T : X \to CB(X)$ by $f(x) = \frac{x}{4}$, $g(x) = \frac{x}{8}$ and $T(x) = [0, \frac{x}{16}]$. without loss of generality, we assume that $z \leq y \leq x$. Consider

$$\begin{aligned} G(x, y, z) &= \max\{|x - y|, |y - z|, |z - x|\} = x - z, \\ G(x, fx, fx) &= \max\{|x - \frac{x}{4}|, |\frac{x}{4} - \frac{x}{4}|, |\frac{x}{4} - x|\} = \frac{3x}{4}, \\ G(y, gy, gy) &= \max\{|y - \frac{y}{8}|, |\frac{y}{8} - \frac{y}{8}|, |\frac{y}{8} - y|\} = \frac{7y}{8}, \\ G(z, Tz, Tz) &= 2d_G(z, Tz) = 2\inf_{t \in Tz}\{G(z, t, t) + G(t, z, z)\} \\ &= 4\inf_{t \in Tz}|z - t| = 4(z - \frac{z}{10}) = \frac{15z}{4}. \end{aligned}$$

Now,

$$\begin{aligned}
H_G(fx, gy, Tz) &= \max\{|\frac{x}{4} - \frac{y}{8}|, \frac{x}{4}, \frac{y}{8}\} \\
&= \frac{x}{4} \\
&\leq \frac{1}{4}(x-z) + \frac{z}{4} \\
&\leq \frac{1}{4}(x-z) + \frac{x}{12} + \frac{y}{12} + \frac{z}{12} \\
&\leq \frac{1}{4}(x-z) + \frac{1}{9}(\frac{3x}{4}) + \frac{2}{21}(\frac{7y}{8}) + \frac{1}{45}(\frac{15z}{4}) \\
&= aG(x,y,z) + bG(x,fx,fx) + cG(y,gy,gy) + dG(z,Tz,Tz).
\end{aligned}$$

Thus (3.2) is satisfied for $0 < a + 2b + 2c + 6d = 0.79 < 1$. 0 is the unique common fixed point of f, g and T. Also any fixed point of f is a fixed point of g and T and conversely.

Theorem 3.6. *Let (X, G) be a G-metric space. Assume that $f, g : X \to X$ and $T : X \to CB(X)$ satisfy the following condition*

$$G(fx, gy, Tz) \leq a[G(y, fx, fx) + G(z, gy, gy) + G(x, Tz, Tz)] \tag{3.3}$$

for all $x, y, z \in X$, where $0 < a < \frac{1}{12}$. Then f, g and T have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and T and conversely.

Proof. First, we will prove that any fixed point of f is a fixed point of g and T. Assume that $p \in X$ is such that $fp = p$. Now, we show that $p = gp = Tp$. If it is not the case, then for $p \neq gp$ and $p \notin Tp$,

Case 1: If $gp \notin Tp$, we have

$$\begin{aligned}
G(p, gp, Tp) &\leq H_G(fp, gp, Tp) \\
&\leq a[G(p, fp, fp) + G(p, gp, gp) + G(p, Tp, Tp)] \\
&\leq a[G(p, p, p) + G(p, gp, Tp) + 4G(p, gp, Tp)] \\
&\leq 5aG(p, gp, Tp),
\end{aligned}$$

which is a contradiction.

Case 2: If $gp \in Tp$, we have

$$\begin{aligned}
G(p, gp, gp) &\leq H_G(fp, gp, Tp) \\
&\leq a[G(p, fp, fp) + G(p, gp, gp) + G(p, Tp, Tp)] \\
&\leq a[G(p, p, p) + G(p, gp, gp) + 6G(p, gp, gp)] \\
&\leq 7aG(p, gp, gp),
\end{aligned}$$

which is a contradiction. Analogously, following the similar arguments to those given above, we obtain a contradiction for $p \neq gp$ and $p \notin Tp$ and $gp \in Tp$

or for $p \neq gp$ and $p \in Tp$ or for $p = gp$ and $p \notin Tp$. Hence in all the cases, we conclude that $p = gp \in Tp$. The same conclusion holds if $p = gp$ or $p \in Tp$.

Next, we will show that f, g and T have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} \in Tx_{3n+2}$, $n = 0, 1, 2, ...$ If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and T. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
& G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
\leq\ & H_G(fx_{3n}, gx_{3n+1}, Tx_{3n+2}) \\
\leq\ & a[G(x_{3n+1}, fx_{3n}, fx_{3n}) + G(x_{3n+2}, gx_{3n+1}, gx_{3n+1}) \\
& +G(x_{3n}, Tx_{3n+2}, Tx_{3n+2})] \\
\leq\ & a[G(x_{3n+1}, x_{3n+1}, x_{3n+1}) + G(x_{3n+2}, x_{3n+2}, x_{3n+2}) \\
& +6G(x_{3n}, x_{3n+3}, x_{3n+3})] \\
\leq\ & 6a[G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+3}, x_{3n+3})] \\
\leq\ & 6a[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})]
\end{aligned}$$

that is

$$(1 - 6a)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq 6aG(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Hence,

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \lambda G(x_{3n}, x_{3n+1}, x_{3n+2}),$$

where $\lambda = \frac{6a}{1-6a}$. Obviously $0 < \lambda < 1$. Continue the above process, we obtain for each n,

$$G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \lambda G(x_n, x_{n+1}, x_{n+2}) \leq \cdots \leq \lambda^{n+1} G(x_0, x_1, x_2).$$

As the proof of Theorem 3.1, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence, $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned}
& G(fu, x_{3n+2}, x_{3n+3}) \\
\leq\ & H_G(fu, gx_{3n+1}, Tx_{3n+2}) \\
\leq\ & a[G(x_{3n+1}, fu, fu) + G(x_{3n+2}, gx_{3n+1}, gx_{3n+1}) + G(u, Tx_{3n+2}, Tx_{3n+2})] \\
\leq\ & a[G(x_{3n+1}, fu, fu) + G(x_{3n+2}, x_{3n+2}, x_{3n+2}) + 6G(u, x_{3n+3}, x_{3n+3})].
\end{aligned}$$

Taking limit $n \to \infty$, we obtain that

$$G(fu, u, u) \leq aG(u, fu, fu) \leq 2aG(fu, u, u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $u \in Tu$.

Now we show that u is unique. For this, assume that there exists another point

$v \in X$ such that $v = fv = gv \in Tv$, then

$$\begin{aligned} G(u,v,v) &\leq H_G(fu,gv,Tv) \\ &\leq a[G(v,fu,fu)+G(v,gv,gv)+G(u,Tv,Tv)] \\ &\leq a[G(v,u,u)+G(v,v,v)+6G(u,v,v)] \\ &\leq a[2G(u,v,v)+6G(u,v,v)] \\ &\leq 8aG(u,v,v), \end{aligned}$$

which implies that $G(u,v,v) = 0$, and $u = v$. Hence u is a unique common fixed point of f, g and T. □

Theorem 3.7. *Let (X, G) be a G-metric space. Assume that $f, g : X \to X$ and $T : X \to CB(X)$ satisfy the following condition*

$$G(fx,gy,Tz) \leq k[G(x,fx,fx)+G(y,gy,gy)+G(z,Tz,Tz)] \tag{3.4}$$

for all $x, y, z \in X$, where $0 < k < \frac{1}{8}$. Then f, g and T have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and T and conversely.

Proof. First, we will prove that any fixed point of f is a fixed point of g and T. Assume that $p \in X$ is such that $fp = p$. Now, we prove that $p = gp = Tp$. If it is not the case, then for $p \neq gp$ and $p \notin Tp$,

Case 1: If $gp \notin Tp$, we have

$$\begin{aligned} G(p,gp,Tp) &\leq H_G(fp,gp,Tp) \\ &\leq k[G(p,fp,fp)+G(p,gp,gp)+G(p,Tp,Tp)] \\ &\leq k[G(p,p,p)+G(p,gp,Tp)+4G(p,gp,Tp)] \\ &\leq 5kG(p,gp,Tp), \end{aligned}$$

which is a contradiction.

Case 2: If $gp \in Tp$, we have

$$\begin{aligned} G(p,gp,gp) &\leq H_G(fp,gp,Tp) \\ &\leq k[G(p,fp,fp)+G(p,gp,gp)+G(p,Tp,Tp)] \\ &\leq k[G(p,p,p)+G(p,gp,gp)+6G(p,gp,gp)] \\ &\leq 7kG(p,gp,Tp), \end{aligned}$$

which is a contradiction. Analogously, following the similar arguments to those given above, we obtain a contradiction for $p \neq gp$ and $p \notin Tp$ and $gp \in Tp$ or for $p \neq gp$ and $p \in Tp$ or for $p = gp$ and $p \notin Tp$. Hence in all the cases, we conclude that $p = gp \in Tp$. The same conclusion holds if $p = gp$ or $p \in Tp$.

Next, we will show that f, g and T have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} \in Tx_{3n+2}$, $n = 0, 1, 2, \ldots.$ If $x_n = x_{n+1}$ for some n, with $n = 3m$, then

$p = x_{3m}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g an dT. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
& G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
\leq\ & H_G(fx_{3n}, gx_{3n+1}, Tx_{3n+2}) \\
\leq\ & k[G(x_{3n}, fx_{3n}, fx_{3n}) + G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) \\
& +G(x_{3n+2}, Tx_{3n+2}, Tx_{3n+2})] \\
\leq\ & k[G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + 4G(x_{3n+1}, x_{3n+2}, x_{3n+3})] \\
\leq\ & k[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + 4G(x_{3n+1}, x_{3n+2}, x_{3n+3})]
\end{aligned}$$

that is

$$(1 - 5k)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq kG(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Hence,

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \lambda G(x_{3n}, x_{3n+1}, x_{3n+2}),$$

where $\lambda = \frac{k}{1-5k}$. Obviously $0 < \lambda < 1$. Continue the procedure to obtain for each n,

$$G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \lambda G(x_n, x_{n+1}, x_{n+2}) \leq \cdots \leq \lambda^{n+1} G(x_0, x_1, x_2).$$

Following similar arguments to those given in Theorem 3.1, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence, $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned}
& G(fu, x_{3n+2}, x_{3n+3}) \\
\leq\ & H_G(fu, gx_{3n+1}, Tx_{3n+2}) \\
\leq\ & k[G(u, fu, fu) + G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(x_{n+2}, Tx_{3n+2}, Tx_{3n+2})] \\
\leq\ & k[G(u, fu, fu) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + 6G(x_{n+2}, x_{3n+3}, x_{3n+3})].
\end{aligned}$$

Taking limit $n \to \infty$, we obtain that

$$G(fu, u, u) \leq kG(u, fu, fu) \leq 2kG(fu, u, u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $u \in Tu$.

To prove the uniqueness, suppose that v is another common fixed point of f, g and T, then

$$\begin{aligned}
G(u, v, v) &\leq H_G(fu, gv, Tv) \\
&\leq k[G(u, fu, fu) + G(v, gv, gv) + G(v, Tv, Tv)] \\
&\leq k[G(u, u, u) + G(v, v, v) + 6G(v, v, v)] \\
&\leq 0,
\end{aligned}$$

which gives that $G(u, v, v) = 0$, and $u = v$. Hence, u is a unique common fixed point of f, g and T. □

Remark 3.8.

(1) *Theorem* 3.1 *improves Theorem* 2.1 *of* [10] *in case* $\alpha + 3\beta + 4\gamma < 1$.

(2) *Theorem* 3.2 *improves Theorem* 2.4 *of* [10] *in case* $0 < a + b + c + d < 1$.

(3) *Theorem* 3.6 *improves Theorem* 2.8 *of* [10] *in case* $0 \leq a \leq \frac{1}{2}$.

(4) *Theorem* 3.7 *improves Theorem* 2.11 *of* [10] *in case* $0 < k < \frac{1}{3}$.

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