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A Group Action on Pandiagonal Lanna Magic Squares

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Abstract : In this paper we use a concept of group action on subgroup of S_{16} to find the number of all pandiagonal Lanna magic squares generated from a set of Myanmar numbers found in a 4x4 non-normal Lanna magic square, called Buddha Khunnung 56 Yantra. Those numbers are 1-15 with repeated 8. This magic square is a talisman from Lanna Kingdom, an ancient kingdom of Thailand. The study found that there were 384 pandiagonal Lanna magic squares.

Keywords : magic square; Lanna magic squares; pandiagonal Lanna magic squares; group action.

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1 Introduction

1.1 Magic Squares

A magic square of order n is a square with n rows and n columns filled with positive integers such that the sum of these integers in each row, in each column and in each of the two principal main diagonals is the same.

If the integers forming a $n \times n$ magic square are consecutive positive numbers from 1 to n^2, the square is said to be **a normal magic square of the nth order**. Otherwise it is **a non-normal magic square** which integers are not restricted in 1 to n^2. However, magic squares are used as a general term to cover both the normal and non-normal ones [1]. For example, the normal magic square of the 3th order is shown below.

8	1	6
3	5	7
4	9	2

The sum of numbers on each row, each column and the each principle diagonal is called the **magic constant** or **magic sum** of the magic square. For a normal magic square of order n, the magic sum is $\frac{1}{2}n(n^2+1)$. For example, the magic constants of normal magic squares of orders n = 3, 4, 5, 6, 7 and 8 are 15, 34, 65, 111, 175 and 260, respectively [1].

A normal magic square of order 3 has exactly one but it can be rotated and reflected to produce 8 trivially distinct squares. In 1675, Bernard Frenicle de Bessey was the first who found that there were exactly 880 normal magic squares of order 4 and it could be generated to $7,040$ different magic squares.

In 1973, Richard Schroeppel was the first to compute the number of magic squares of order 5. He found that there were exactly $68,826,306$ magic squares which could be $275,305,224$ of 5×5 magic squares. However, for the 66 case, the exact number of all magic squares is unknown but it was estimated to be approximately $1.7745 \pm 0.0016 \times 10^{19}$ magic squares [2].

In 2015, Jos M. Pacheco and Isabel Fernndez had a trip to Barcelona. They observed a non-normal magic square of order 4 in the Gauds Sagrada Famlia Temple in Barcelona. It is a large magic square artwork over 1m1m, located in the wall. Its magic constant is 33 and it features on rows, columnns, diagonals and 2/2 broken diagonals (not all broken diagonals), and on any 22 subsquares [3]. Their study emphasised the role of more or less hidden symmetries in preserving the magic constant.

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

A magic square is said to be **pandiagonal** (sometimes diabolic or Nasik) if the magic square has additional properties that its all broken diagonals also add up to the magic constant. For example, the magic square of order 4 below was found in Khajurado, India in 1904 [1].

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

It is a pandiagonal magic square because not only the numbers in rows, columns and pricipal diagonals add to a magic constant 34, but also the numbers in all broken diagonals add to 34, such as 7 6 10 11, 2 12 15 5, 16 13 1 4, 14 2 3 15, 1 11 16 6 and 12 8 5 9.

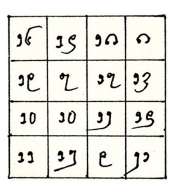
In 1937, Barkley Rosser and R. J. Walker studied on normal 4×4 magic squares. They found there were 384 pandiagonal magic squares by using abstract algebra [4].

1.2 Lanna Yantra

Lanna was a kingdom located in present-day Northern Thailand from the 13th to 18th centuries. Its center was in Chiang Mai. Lanna had their own culture, language and letters [5].

Lanna Yantra is talisman of Lanna people. It was recorded by using Lanna letters or Myanmar numbers in fabric or thin silver or copper plates. There are a lot of Lanna Yantra with different supernatural. Lanna people kept Yantra at home or brought it with themselves [6].

In 2011, Jeeraporn Kongjai [7] mentioned in her work that Atichart Kettapun and his research team interested in one of Lanna Yantra in a form of magic square containing many interesting mathematics patterns. Lanna People believed that this Yantra could help them have safe journeys [6]. The numbers in that Yantra are Myanmar numbers used widely in Lanna Kingdom. They translated these numbers into Arabic numbers as shown below.



16	14	18	8
19	7	17	13
10	10	12	14
11	15	9	21

However, they agreed that the number in the 3rd row the 2nd column should be 20 because of following reasons:

1. Since it was copied from generations to generations by hand writing, a mistake could occur.

2. The Myanmar numbers 1 and 2 are very similar. The main different is the length of bottom tails. Therefore, error in reading and copying numbers could happen. The table below shows the Arabic numbers and Myanmar numbers.

Arabic numbers	Myanmar Numbers	Arabic numbers	Myanmar Numbers
1	၁	6	၆
2	၂	7	၇
3	၃	8	၈
4	၄	9	၉
5	၅	0	၀

3. It is clear that if we change the number 10 in the 3rd row and the 2nd column to be 20, we will get a magic square of order 4 with the magic constant 56. Moreover, this magic square is pandiagonal. Since we have found many magic squares in Lanna Yantra, it is reasonable to change 10 to be 20.

Thus, they changed the number 10 in that position to 20 and got the following square.

16	14	18	8
19	7	17	13
10	20	12	14
11	15	9	21

The square is a magic square of order 4 with the magic constant 56. His team called it **Buddha Khunnung 56 Yantra**, as used by many people nowadays, which means the number of syllables in one of Buddhism prayer [6].

In this paper, we focus on finding the number of all pandiagonal magic squares of order 4 generated from numbers found in the Buddha Khunnung 56 Yantra by using a concept group action in Abstract Algebra.

However, we reduce the numbers 7-12 with double 14 in the Buddha Khunnung 56 Yantra to the number 1-15 with double 8 by substracting 6 from each original number. Therefore, the magic constant is changed from 56 to 32. We reduce the number for making it similar to a normal magic square and easier to study.

After reduced, the new square is the following.

10	8	12	2
13	1	11	7
4	14	6	8
5	9	2	15

We call this new square the specific **Lanna magic square**. Clearly, this square is a pandiagonal magic square of order 4 with the magic constant 32.

Moreover, we call pandiagonal magic squares of order 4 created by numbers $1-15$ with repeated number 8 that **pandiagonal Lanna magic squares**. That means the sum of each row, each column, each of 2 main diagonals and each of 6 broken diagonals is 32. An example is shown below.

2	7	13	10
12	11	1	8
3	6	14	9
15	8	4	5

2 Preliminaries

We show some important base concepts of Abstract Algebra for our study here. For more details, see Algebra by Thomas W. Hungerford [8].

Let A be the finite set {1, 2, 3, , n }. The group of all permutations of A is a symmetric group on n letters, and is denoted by S_n. Note that S_n has $n!$ elements, where $n! = n(n-1)(n-2)(3)(2)(1)$.

Let G be a group and let $a_i \in G$ for $i \in I$. The smallest subgroup of G containing $\{a_i|i \in I\}$ is the **subgroup generated by** $\{a_i|i \in I\}$ and denoted by $< a_i >$. If this subgroup is G, then $< a_i >$ **generates** G and the a_i are **generators of** G.

If there is a finite set $\{a_i|i \in I\}$ that generates G, then G is finitely generated. If $a \in G$, the subgroup $< a >$ is called the cyclic subgroup generated by a.

Theorem 2.1. *If G is a group and X is a nonempty subset of G, then the subgroup $< X >$ generated by X consists of all finite products $a_1^{n_1} a_2^{n_2 \cdots} a_m^{n_m}$ $(a_i \in X; n_i \in \mathbb{Z})$. In particular for every $a \in G, < a >= \{a_n \mid n \in \mathbb{Z}\}$.*

The **order** of an element $a \in G$ is the least positive integer n such that $a^n = 1$. If no such integer exists, the order of a is infinite. We denote it by $|a|$.

An action of a group G on a set S is a function $G \times S \to S$ (usually denoted by $(g, x) \mapsto gx$) such that for all $x \in S$ and $g_1, g_2 \in G$:

$$ex = x \text{ and } (g_1 g_2)x = g_1(g_2 x).$$

When such an action is given, we say that G **acts on the set** S.

Theorem 2.2. *Let G be a group that acts on a set S.*
1. The relation on S defined by $x \sim x' \leftrightarrow gx = x'$ for some $g \in G$ is an equivalence relation.
2. For each $x \in S, G_x = \{g \in G \mid gx = x\}$ is a subgroup of G.

The equivalence classes of the equivalence relation of Theorem 2.2(1) are called **orbits** of G on S; the orbit of $x \in S$ is denoted $\bar{x}$. The subgroup G_x is called the stabilizer of x.

Theorem 2.3. *If a group G acts on a set S, then the cardinal number of the orbit of $x \in S$ is the index $[G : G_x]$.*

Definition 2.4. Let G be a group acting on a set S. G is transitive if for each $x, y \in S$, there exists $g \in G$ such that $gx = y$.

Theorem 2.5. *Let G be a transitive group acting on a set S. For $x \in S$, the orbit $\overline{x}$ of x is S (or there is only one orbit). (Theorem* 2.5 *is the exercise* 6(a) *in page* 93 *of* [8]*).*

3 Pandiagonal Lanna Magic Squares

Let L be a pandiagonal Lanna magic square written as $L =$

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

and let T_1, T_2, T_3, T_4 and T_5 be transformations of the pandiagonal Lanna magic square where
T_1 is the reflection about the a, f, k, p diagonal,
T_2 is the rotation through 90 ° counter-clockwise,
T_3 is the move of the first column to the last column,
T_4 is the move of the first row to the last row,
T_5 is the transformation of the pandiagonal Lanna magic square L into

a	d	h	e
b	c	g	f
n	o	k	j
m	p	l	i

.

If we consider the transformations T_1, T_2, T_3, T_4 and T_5 of pandiagonal Lanna magic squares L in permutation forms of subgroups of S_{16} where the numbers 1-16 represent the positions in the square

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

.

We get
T_1: (1)(6)(11)(16)(2 5)(3 9)(4 13)(7 10)(8 14)(12 15),
T_2: (1 13 16 4)(2 9 15 8)(3 5 14 12)(6 10 11 7),
T_3: (1 4 3 2)(5 8 7 6)(9 12 11 10)(13 16 15 14),
T_4: (1 13 9 5)(2 14 10 6)(3 15 11 7)(4 16 12 8) and
T_5: (1)(7)(11)(13)(2 5 4)(3 6 8)(9 16 14)(10 12 15).

When $(i_1\, i_2 \cdots i_r)$ is a cycle of length r or a r-cycle. The notation $(i_1\, i_2 \cdots i_r)$ means that i_1 is replaced by i_2, i_2 by i_3, $\cdots$, i_{r-1} by i_r and i_r by i_1. Clearly that the transformations $T_1, T_2, T_3, T_4, T_5 \in S_{16}$.

Now we will prove that a pandiagonal Lanna magic square preserves to be a pandiagonal Lanna magic square after applying the transformations T_1, T_2, T_3, T_4 and T_5.

Since L is a pandiagonal Lanna magic square, for all 4 rows, 4 columns, 2 main diagonals (i.e. a, f, k, p and d, g, j, m) and 6 broken diagonals (i.e. a, n, k, h, e, b, o, l and i, f, c, p), the sum of each one is 32.

T_1 : By applying the transformation T_1 to L, we have

$T_1L =$

a	e	i	m
b	f	j	n
c	g	k	o
d	h	l	p

.

The transformation T_1 carries rows into columns and columns into rows. Therefore, the sum of each row, column, main diagonal and broken diagonal is 32. Thus, a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square after applying T_1.

T_2 : By applying transformation T_2 to L, we have

$T_2L =$

d	h	l	p
c	g	k	o
b	f	j	n
a	e	i	m

.

Similar to the transformation T_1, the transformation T_2 carries rows into columns and columns into rows. Therefore, the sum of each row, column, main diagonal and broken diagonal is 32. Thus, a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square after applying T_2.

T_3 : By applying transformation T_3 to L, we have

$T_3L =$

b	c	d	a
f	g	h	e
j	k	l	i
n	o	p	m

.

T_3 carries rows into rows and columns into columns. Therefore, the sum of each row, column, main diagonal and broken diagonal is 32. Thus, a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square after applying T_3.

T_4 : By applying transformation T_4 to L, we have

$T_4L =$

e	f	g	h
i	j	k	l
m	n	o	p
a	b	c	d

.

Similar to the transformation T_3, the transformation T_4 carries rows into rows and columns into columns. Therefore, the sum of each row, column, main diagonal and broken diagonal is 32. Thus, a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square after applying T_4.

For T_5, we need the following lemmas.

Lemma 3.1. *The four elements of any* 2×2 *of a pandiagonal Lanna magic square add up to* 32.

Proof. For a pandiagonal Lanna magic square L, the sum of each row, column and diagonal is 32. Therefore,

$$\begin{aligned}(a+f+k+p)+(d+g+j+m)+(e+f+g+h)+&\\ (i+j+k+l)-(a+e+i+m)-(d+h+l+p) &= 64\\ 2(f+g+j+k) &= 64\\ f+g+j+k &= 32.\end{aligned}$$

By using of transformations T_3 and T_4, this result can be applied to any square of order two. □

Lemma 3.2. *The sum of two opposite corners of any* 3×3 *of a pandiagonal Lanna magic square is* 16.

Proof. For a pandiagonal Lanna magic square L,

$$\begin{aligned}(a+b+c+d)+(i+j+k+l)+(a+e+i+m)+(c+g+k+o)+&\\ (a+f+k+p)+(a+n+k+h)-(e+b+o+l)-(i+f+c+p)-&\\ (m+j+g+d)-(i+n+c+h) &= 64\\ a+k &= 16.\end{aligned}$$

By using of transformations T_3 and T_4, this result can be applied to any pair of two opposite corners of all 3×3 of a pandiagonal Lanna magic square. □

Now we can prove that the transformation T_5 preserves a pandiagonal Lanna magic square properties.

T_5 : By applying the transformation T_5 to L, we have

$$T_5L = \begin{array}{|c|c|c|c|}\hline a & d & h & e\\ \hline b & c & g & f\\ \hline n & o & k & j\\ \hline m & p & l & i\\ \hline\end{array}\,.$$

By Lemma 3.1 and 3.2, it is easily seen that the transformation T_5 applied to a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square.

We can now conclude that a pandiagonal Lanna magic square remains a pandiagonal Lanna magic square after applied transformations T_1, T_2, T_3, T_4 and T_5.

Let $G =< T_1, T_2, T_3, T_4, T_5 >$ be a subgroup of S_{16} generated by T_1, T_2, T_3, T_4, and T_5, e be the identity of G, and $\mathbb{L}$ be a set of all pandiagonal Lanna magic squares.

Define $F : G \times \mathbb{L} \to \mathbb{L}$ by $(g, L) \mapsto gL = L \circ g, \forall g \in G, L \in \mathbb{L}$.

Since $T_iL = L \circ T_i \in \mathbb{L}$ $\forall i \in \{1, 2, 3, 4, 5\}$ and $G =< T_1, T_2, T_3, T_4, T_5 >$, we have F is an action of the group G.

Theorem 3.3. *All pandiagonal Lanna magic squares can be derived from a single one (the Lanna magic square) by successive transformations of* $T_1, T_2, T_3, T_4,$ *and* T_5.

Proof. For any pandiagonal Lanna magic square, by using of T_3 and T_4, 1 can be brought into the second row and the second column. So, there is no loss of generality in taking $f = 1$. By Lemma 3.2, we have $p = 15$. Then g and h can be at most 13 and 14. Thus, $g + h \leq 27$ and we get $e \geq 4$. Similarly, $g \geq 4, h \geq 4, b \geq 4, j \geq 4, n \geq 4, a \geq 4, k \geq 4$ and $p \geq 4$.

Using $b + c + f + g = 32, e + f + i + j = 32$ and Lemma 3.1, we also get $c \geq 4, i \geq 4$. Thus, 2 and 3 can only occur in d, l, m and o.

If 2 occurs in o, it can be brought into l by taking T_5 twice. If it occurs in l, it can be brought into d by taking T_3 twice. If 2 occurs in m, it can be brought into d by taking T_1. Hence, we take $d = 2$ and we then get $j = 14$. If 3 occurs in l, it can be brought into o by taking T_5. If 3 occurs in m, it can be brought into o by taking T_3 twice. So, one can take $o = 3$ and then $e = 13$. By Lemma 3.1, we also get $i = 4$ and $c = 12$.

Now $m = 15 - a$, $g = a + 1$, $n = a - 1$, $b = 18 - a$. By Lemma 3.1, $k = 16 - a$, $l = a - 2$ and $h = 17 - a$. So, we have to find values of a such that $15 - a, a + 1, a - 1, 18 - a, 16 - l, a - 2$ and $17 - a$ are 5, 6, 7, 8, 9, 10 and 11 in some order. By substitution, a can be only 7 or 10.

For $a = 7$, we get

7	11	12	2
13	1	8	10
4	14	9	5
8	6	3	15

and for $a = 10$, we get exactly the same as the Lanna magic square

10	8	12	2
13	1	11	7
4	14	6	8
5	9	3	15

.

The square $a = 7$ can be the same as the square $a = 10$ by applying $T_2^2 T_1 T_2^3 T_5 T_3^2$.

Thus, all pandiagonal Lanna magic squares can be obtained from the Lanna magic square by applying application $T_1, T_2, T_3, T_4,$ and T_5 . □

From Theorem 3.3, we can say that the group $G =< T_1, T_2, T_3, T_4, T_5 >$ acts on $\mathbb{L}$ is transitive.

The proof of the next theorem is similar to Theorem 4 in [4]. However, for the sake of completeness, we describe below.

Theorem 3.4. *The order of subgroup of* S_{16} *generated by* T_1, T_2, T_3, T_4 *and* T_5 *is* 384.

Proof. If T_X and T_Y are two transformations, we denote T_XT_Y as the transformation effected by applying T_Y first and then T_X.

Consider (a) $T_2T_1 = T_1T_2^3$, (b) $T_3T_1 = T_1T_4$, (c) $T_4T_1 = T_1T_3$, (d) $T_3T_2 = T_2T_4$, (e) $T_4T_2 = T_2T_3^3$, (f) $T_4T_3 = T_3T_4$, (g) $T_5T_1 = T_1T_2^2T_3T_4T_5$, (h) $T_5^2T_1 = T_2^2T_3T_4T_5^2$, (i) $T_5T_2 = T_2^3T_3T_4T_5$, (j) $T_5^2T_2 = T_1T_2T_3T_4T_5^2$, (k) $T_5T_3 = T_3^3T_5^2$, (l) $T_5^2T_3 = T_2^2T_3T_4^2T_5$, (m) $T_5T_4 = T_1T_2^2T_4T_5$, (n) $T_5^2T_4 = T_1T_2^2T_4T_5$ and (o) $T_1^2 = T_2^4 = T_3^4 = T_4^4 = T_5^3$. They are identical transformations. By inspection from (a)-(o), for any product of T_1, T_2, T_3, T_4 and T_5, all can get T_1 to the left, then T_2 next to T_1, then T_3, T_4 and T_5 are on the right. So, any product of T_1, T_2, T_3, T_4 and T_5 is equal to the form $T_1^\alpha T_2^\beta T_3^\gamma T_4^\delta T_5^\epsilon$, [4].

It is clearly that T_2, T_3, T_4 are independent. Moreover, T_5 or T_5^2 is not the product of T_1, T_2, T_3 and T_4 since all of T_1, T_2, T_3 and T_4 carry rows into rows, columns into columns, columns into rows or rows into colums which can not yield neither T_5 nor T_5^2. Besides the transformations T_2, T_3 and T_4 preserve the orientation so, T_1 is not a product of T_2, T_3 anad T_4.

Therefore, $T_1^\alpha T_2^\beta T_3^\gamma T_4^\delta T_5^\epsilon = T_1^a T_2^b T_3^c T_4^d T_5^e$ if and only if $\alpha \equiv a$, $\beta \equiv b$, $\gamma \equiv c$, $\delta \equiv d$ and $\epsilon \equiv e$, [4].

Hence, the order of subgroups generated by $T_1, T_2, T_3, T_4,$ and T_5 is $2 \times 4 \times 4 \times 4 \times 3 = 384$. □

Theorem 3.5. *There are* 384 *pandiagonal Lanna magic squares.*

Proof. Since G acts on $\mathbb{L}$, for each $L \in \mathbb{L}$ there is only the identity e of G such that $gL = L$. So, $G_L =< e >$.

From Theorem 3.3 the group G acts on $\mathbb{L}$ is transitive. Therefore, by Theorem 2.5 for $L \in \mathbb{L}$ we have that the orbit of L is $\mathbb{L}$.

From Theorem 2.3, the cardinal number of the orbit $L \in \mathbb{L}$ is $[G : G_L]$ and from Theorem 3.4, $|G| = 384$. We have $|\mathbb{L}| = |L| = [G : G_L] = [G :< e >] = |G| = 384$.

Hence, there are 384 pandiagonal Lanna magic squares. □

4 Conclusion

In this paper, we try to figure out the number of possible pandiagonal Lanna magic squares. We start from descibing all 5 transformations and using a concept of group action in Abstract Algebra to help us generate all pandiagonal Lanna magic squares. Finally, we found that there were 384 pandiagonal Lanna magic squares.

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