



Sensitivity Analysis for the Quasi Variational Inequality Problems on Uniformly Prox Regular Sets

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Abstract : In this paper, we present a parametric Wiener-Hopf equation which is equivalent to the parametric of quasi variational inequality problems over a class of nonconvex sets, as uniformly prox regular sets. By using a such Wiener-Hopf equation and some sufficient conditions, the sensitivity analysis of the quasi variational inequality problems on nonconvex sets is proved.

Keywords : sensitivity analysis; the quasi variational inequality problem; uniformly prox-regular set; locally Lipschitz continuous mapping; locally strongly monotone mapping.

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1 Introduction

Variational inequality theory is the well known theory, which was introduced by Stampacchia [1], in 1964. The interesting and fascinating of this theory are to provide the most natural, direct, simple, unified and efficient framework for a general treatment of vary of different fields in linear and nonlinear problem. Then,

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the variational inequality theory is simple for application in a wide class of problems, such as, industry, finance, economics, social and pure and applied sciences, see [2–6]. Consequently, the variational inequality theory is a power tool and useful for creating many researches in both theory and applications. By this reasons, this theory has been attracted the attention from many authors by using this idea and technique to develop many researches. A generalization of variational inequality is the quasi variational inequality, which was studied, in 1978, by A. Bensoussan and J. Lion [7] that depended on some problems of random impulse control. The quasi variational inequality problem has been applied in various aspects, for instance, mathematical programming, optimization and equilibrium problems of economics and transportation, social and game theory, etc. (see [8–12]). Get inspired by the variety of the previous applications, many researchers attend to focus on the quasi variational inequality problem.

On the other hand, many problems which was studied in the past was analysed in the concept of convex sets but it is not enough for applications. Then, many researchers have been needed to develop the various problems on nonconvex sets (see [13–17]). Starting from in 1995, F. H. Clarke et. al. [18] introduced and presented the nonconvex sets as the proximally smooth set. This set attracted many researchers for solving problems on nonconvex set, especially the variational inequality problems. So, this implies to obtain many qualitative paper for application. In 2003, M. Bounkhel et. al. [19] introduced a formulation of the set-valued variational inequality problems on nonconvex set as uniformly prox-regular set and showed the convergence of such solution. Subsequently, in 2013, J. Suwannawit and N. Petrot [20] considered and analyzed the quasi-variational inequality problem on uniformly prox-regular set and showed the existence and convergence of such problem and, moreover, some conditions which was considered in M. Bounkhel et. al. [19] was omitted.

Furthermore, the sensitivity analysis is still the motivation to develop the variational inequality problems. The sensitivity analysis of variational inequality, which was introduced by S. Dafermos [21] in 1988, was considered to generalize the variational inequality problem by using some conditions for the local uniqueness, continuity and differentiability of a solution of parametric variational inequalities. Then, the sensitivity analysis of variational inequality is useful for many applications, such as, for analysis and calculation the transportation equilibrium, for planning the governing system equilibrium and for designing the other equilibrium. By this results, in mathematical and engineering, the sensitivity analysis of variational inequality has still many useful because it can be obtained some new ideas for creating many researches. The following papers are the example of the sensitivity analysis in variational inequality. In 1992, R. N. Mukherjee and H. L. Verma [22] studied the sensitivity analysis of solutions of generalized variational inequality which extended S. Dafermos [23]. In 1997, M. A. Noor [24] used the Wiener-Hopf equations technique without assuming the differentiability to study a sensitivity analysis for quasi-variational inequalities on nonconvex sets. Later, in 1999, A. Moudafi and M. A. Noor [25] studied and analyzed the sensitivity analysis of variational inclusion relying on Wiener-Hopf equation techniques. After that,

by interesting concept of sensitivity analysis and nonconvex sets, this implies that, in 2013, M. A. Noor and K. I. Noor [26] studied and considered the sensitivity analysis of the general nonconvex variational inequalities by using the projection method and they considered the equivalence of the general nonconvex variational inequality with Wiener-Hopf equation. In recent times, J. K. Kim [27] showed the sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities and presented the parametric general Wiener-Hopf equation of a such problem.

By the above motivation, we are interested to consider the sensitivity analysis for the quasi variational inequality on uniformly prox-regular sets. The equivalent relation of the parametric quasi variational inequality problem and a parametric Wiener-Hopf equation is showed. We desire that our results extend and improve the literature results for the variational inequalities.

2 Preliminaries

In this section, we will recall some basic concepts and useful results which will be used in this paper.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed subset of H . For each $K \subseteq H$, we denote by $d(\cdot, K)$ for the usual distance function on H with respect to K , that is, $d(u, K) = \inf_{v \in K} \|u - v\|$, for all $u \in H$. A point $v \in K$ is called the closest point or the projection of u onto K if $d(u, K) = \|u - v\|$. The set of all such closest points is denoted by $Proj_K(u)$, that is, $Proj_K(u) = \{v \in K : d(u, K) = \|u - v\|\}$. The proximal normal cone to K at u is given by

$$N_K^P(u) = \{v \in H : \exists \rho > 0 \text{ such that } u \in Proj_K(u + \rho v)\}.$$

Notice that, $Proj_K = (I + N_K^P)^{-1}$ where I is an identity operator.

In 1995, Clarke et al. [18] presented a new class of nonconvex sets, namely proximally smooth sets, and this sets are important to apply in many fields such as optimization and dynamical systems. In recent years, many researchers studied and analyzed variational inequality problem and variational inclusion problem in the sense of proximally smooth sets (see [3, 14–17, 19, 20]). The original definition of proximally smooth set was presented in the sense of the differentiability of the distance function (see [18, 28]), and we now will recall the definition of proximally smooth sets in the following characterization, which was proved in [29].

Definition 2.1. For a given $r \in (0, +\infty]$, a subset K of H is said to be *uniformly prox-regular with respect to r* , say, *uniformly r -prox-regular set*, if for all $\bar{x} \in K$ and for all $0 \neq z \in N_K^P(\bar{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2, \text{ for all } x \in K.$$

Remark 2.2. For the case of $r = \infty$, the uniform r -prox-regularity K is equivalent to the convexity of K (see [18]). Moreover, it is known that the class of uniformly prox-regular sets is sufficiently large to include the class p -convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H , the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [28, 29].

For the sake of simplicity, from now on, we will use the following notation: for each $r \in (0, +\infty]$, we write

$$K_r := \{x \in H : d(x, K) < r\}.$$

Lemma 2.3. [29] Let $r \in (0, +\infty]$ and K be a nonempty closed subset of H . If K is a uniformly r -prox-regular set, then the following holds

- (i) For all $x \in K_r$, $Proj_K(x) \neq \emptyset$;
- (ii) For all $s \in (0, r)$, $Proj_K$ is a $\frac{r}{r-s}$ -Lipschitz on K_s ;
- (iii) The proximal normal cone is closed as a set-valued mapping.

Definition 2.4. A set-valued mapping $C : H \rightarrow 2^H$ is said to be a κ -Lipschitz continuous if there exists a real number $\kappa > 0$ such that

$$|d(y, C(x)) - d(y', C(x'))| \leq \|y - y'\| + \kappa\|x - x'\|$$

for all $x, x', y, y' \in H$.

Lemma 2.5. [19] Let $r \in (0, +\infty]$ and let $C : H \rightarrow 2^H$ be a κ -Lipschitz set valued mapping with uniformly r -prox regular values then the following closedness property holds: "For any $x_n \rightarrow x^*, y_n \rightarrow y^*$ and $u_n \rightarrow u^*$ with $y_n \in C(x_n)$ and $u_n \in N_{C(x_n)}(y_n)$, one has $u^* \in N_{C(x^*)}(y^*)$."

Let $T : H \rightarrow H$ be a mapping and $C : H \rightarrow \mathcal{C}(H)$ be a set-valued mapping, where $\mathcal{C}(H)$ is a family of all nonempty closed subsets of H . In [20], the authors considered the existence theorems of the following quasi variational inequality problem on uniformly prox regular sets: find $x^* \in C(x^*)$ such that

$$-Tx^* \in N_{C(x^*)}^P(x^*). \tag{NQVI}$$

Furthermore, in [19], M. Bounkhel showed that, for $r \in (0, +\infty]$, the problem (NQVI) is equivalent to the following problem: find $x^* \in C(x^*)$ such that

$$\langle Tx^*, x - x^* \rangle + \frac{\|Tx^*\|}{2r} \|x - x^*\|^2 \geq 0, \text{ for all } x \in C(x^*), \tag{NQVI'}$$

where $C : H \rightarrow [Cl(H)]_r$ is a set-valued mapping that $[Cl(H)]_r$ denotes for class of all uniformly r -prox regular subsets of H .

In this work, we are interested to study sensitivity analysis of the quasi variational inequality problem on uniformly-prox regular sets. That is, we now introduce the parametric problem of (NQVI) as follows. Let Ω be a subset of H , in which the parameter λ take values. Let $T : H \times \Omega \rightarrow H$ be a nonlinear mapping and $C : H \times \Omega \rightarrow 2^H / \{\emptyset\}$ be a set-valued mapping. For each $\lambda \in \Omega$, the parametric quasi variational inequality problem on uniformly prox-regular sets is to find $x^* \in C(x^*, \lambda) =: C_\lambda(x^*)$ such that

$$0 \in T(x^*, \lambda) + N_{C_\lambda(x^*)}^P(x^*), \quad (PNQVI_\lambda)$$

where $C(\cdot, \lambda)$ is uniformly prox-regular sets.

In this work, we will concern with the following class of mappings.

Definition 2.6. A mapping $T : H \times \Omega \rightarrow H$ is said to be a ξ_λ -locally Lipschitz continuous if for each $\lambda \in \Omega$, there exists a real number $\xi_\lambda > 0$ such that

$$\|T(x, \lambda) - T(y, \lambda)\| \leq \xi_\lambda \|x - y\|,$$

for all $x, y \in H$. For the case $\xi_\lambda \in (0, 1)$, the mapping T is said to be a ξ_λ -locally contractive mapping.

Definition 2.7. A mapping $T : H \times \Omega \rightarrow H$ is said to be a β_λ -locally strongly monotone if for each $\lambda \in \Omega$, there exists a real number $\beta_\lambda > 0$ such that

$$\langle T(x, \lambda) - T(y, \lambda), x - y \rangle \geq \beta_\lambda \|x - y\|^2,$$

for all $x, y \in H$.

3 Main Results

We will start by the following lemma, which is important tool to obtain our main results.

Lemma 3.1. Let $\lambda \in \Omega$. Let $T : H \times \Omega \rightarrow H$ and $C_\lambda : H \times \{\lambda\} \rightarrow [Cl(H)]_r$ be mappings. Then, we have the following statements:

1. if x^* is a solution of the problem $(PNQVI_\lambda)$, then for any constants $\eta > 0$ we have

$$x^* = Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)),$$

2. if there is a constant $\eta > 0$ such that

$$x^* = Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)),$$

then, x^* is a solution of the problem $(PNQVI_\lambda)$.

Proof. 1. Let x^* be a solution of the problem $(PNQVI_\lambda)$. We obtain that

$$0 \in T(x^*, \lambda) + N_{C_\lambda(x^*)}^P(x^*).$$

Let $\eta > 0$ be given. We see that

$$x^* - \eta T(x^*, \lambda) \in x^* + N_{C_\lambda(x^*)}^P(x^*),$$

this implies

$$x^* - \eta T(x^*, \lambda) \in \left(I + N_{C_\lambda(x^*)}^P \right) (x^*).$$

This means,

$$x^* = \left(I + N_{C_\lambda(x^*)}^P \right)^{-1} (x^* - \eta T(x^*, \lambda)).$$

Hence,

$$x^* = Proj_{C_\lambda(x^*)} (x^* - \eta T(x^*, \lambda)),$$

this proves (i).

2. Assume that there exists a constant $\eta > 0$ such that $x^* = Proj_{C_\lambda(x^*)} (x^* - \eta T(x^*, \lambda))$. That is

$$x^* = \left(I + N_{C_\lambda(x^*)}^P \right)^{-1} (x^* - \eta T(x^*, \lambda)).$$

It follows that

$$x^* - \eta T(x^*, \lambda) \in x^* + N_{C_\lambda(x^*)}^P(x^*).$$

Therefore,

$$0 \in T(x^*, \lambda) + N_{C_\lambda(x^*)}^P(x^*),$$

this completes the proof. □

Next, we will present the parametric Wiener-Hopf equation of the quasi variational inequality problem on uniformly prox-regular sets. Let $(\lambda, x^*, \eta) \in \Omega \times H \times (0, \infty)$ be fixed. We are interesting to find $z^* := z^*(\lambda, x^*, \eta) \in H$ such that

$$0 = Q_{C_\lambda(x^*)}(z^*) + \eta T(Proj_{C_\lambda(x^*)}(z^*), \lambda), \tag{WH}(\lambda, x^*, \eta)$$

where $Q_{C_\lambda(x)} = I + Proj_{C_\lambda(x)}$, for all $x \in H$.

In [20], under some suitable control conditions, the authors showed that the problem (NQVI) has a unique solution. By employing such obtained result, for a fixed $\lambda \in \Omega$, we let x^* be a unique solution of the problem $(PNQVI_\lambda)$. Next, we will show that, for a given $(\lambda, x^*, \eta) \in \Omega \times H \times (0, \infty)$, the problem $(WH)(\lambda, x^*, \eta)$ has a unique solution. To do this, the following lemma is important.

Lemma 3.2. *Let $T : H \times \Omega \rightarrow H$ be a mapping and λ be fixed in Ω . Assume that the mapping $Proj_{C_\lambda}$ satisfies (3.3). Then, the parametric quasi variational inequality problem on uniformly prox regular sets $(PNQVI_\lambda)$ has a solution x^* if and only if the parametric Wiener-Hopf equation $(WH(\lambda, x^*, \eta))$ has a solution, $z^* \in H$ with*

$$z^* = x^* - \eta T(x^*, \lambda) \quad (3.1)$$

and

$$z^* = Proj_{C_\lambda(x^*)}(z^*) - \eta T(x^*, \lambda), \quad (3.2)$$

where η is a positive constant.

Proof. (\Rightarrow) Suppose that x^* is a solution of $(PNQVI_\lambda)$. Let η be a fixed positive real number. We now show that $z^* \in H$, which satisfies (3.1) and (3.2), is a solution of the problem $(WH(\lambda, x^*, \eta))$. By Lemma 3.1, we know that

$$x^* = Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)).$$

Subsequently, it follows, from (3.1) and (3.2), that

$$\begin{aligned} z^* &= Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)) - \eta T(Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)), \lambda), \\ &= Proj_{C_\lambda(x^*)}(z^*) - \eta T(Proj_{C_\lambda(x^*)}(z^*), \lambda). \end{aligned}$$

Using this one, since $Q_{C_\lambda(x^*)} = I - Proj_{C_\lambda(x^*)}$, we obtain that

$$Q_{C_\lambda(x^*)}(z^*) = -\eta T(Proj_{C_\lambda(x^*)}(z^*), \lambda).$$

This yields the required result.

(\Leftarrow) Let $x^* \in H$ be such that z^* , which are satisfied by (3.1) and (3.2), is a solution of the problem $(WH(\lambda, x^*, \eta))$ for some $\eta > 0$. Firstly, we note that (3.1) and (3.2) implies $x^* \in C_\lambda(x^*)$. Next, we will show that such x^* is a solution of the problem $(PNQVI_\lambda)$. Since z^* is a solution of the problem $(WH(\lambda, x^*, \eta))$, we know that

$$0 = (I - Proj_{C_\lambda(x^*)})(z^*) + \eta T(Proj_{C_\lambda(x^*)}(z^*), \lambda).$$

Thus, by using (3.1) and (3.2), we obtain that

$$\begin{aligned} 0 &= z^* - Proj_{C_\lambda(x^*)}(z^*) + \eta T(Proj_{C_\lambda(x^*)}(z^*), \lambda) \\ &= x^* - \eta T(x^*, \lambda) - Proj_{C_\lambda(x^*)}(z^*) + \eta T(z^* + \eta T(x^*, \lambda), \lambda) \\ &= x^* - \eta T(x^*, \lambda) - Proj_{C_\lambda(x^*)}(z^*) + \eta T(x^*, \lambda). \end{aligned}$$

This implies that

$$x^* = Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)).$$

By Lemma 3.1 (ii), we conclude that x^* is a solution of the problem $(PNQVI_\lambda)$. This completes the proof. \square

Remark 3.3. Let $\lambda \in \Omega$ be fixed and $\eta \in \left(0, \frac{1}{\xi_\lambda(\psi_\lambda + 1)}\right)$. We define the mapping $g_{\eta,\lambda} : H \rightarrow H$ by

$$g_{\eta,\lambda}(x) = \eta T(x, \lambda), \quad \text{for all } x \in H,$$

where $T : H \times \Omega \rightarrow H$ is a ξ_λ -locally Lipschitz continuous mapping. Also, for a fixed $z \in H$, we define a mapping $h_{z,\lambda} : H \rightarrow H$ by

$$h_{z,\lambda}(x) = Proj_{C_\lambda(z+x)}(z) - x,$$

for all $x \in H$. If there is $\psi_\lambda \in [0, 1)$ such that

$$\|Proj_{C_\lambda(x)}(z) - Proj_{C_\lambda(y)}(z)\| \leq \psi_\lambda \|x - y\| \tag{3.3}$$

for all $x, y, z \in H$, then we can check that the mapping $h_{z,\lambda} \circ g_{\eta,\lambda}$ is a contractive mapping. Subsequently, $h_{z,\lambda} \circ g_{\eta,\lambda}$ has a unique fixed point, that is,

$$\begin{aligned} & \| (h_{z,\lambda} \circ g_{\eta,\lambda})(x) - (h_{z,\lambda} \circ g_{\eta,\lambda})(y) \| \\ &= \| h_{z,\lambda}(g_{\eta,\lambda}(x)) - h_{z,\lambda}(g_{\eta,\lambda}(y)) \| \\ &= \| h_{z,\lambda}(\eta T(x, \lambda)) - h_{z,\lambda}(\eta T(y, \lambda)) \| \\ &= \| Proj_{C_\lambda(z+\eta T(x,\lambda))}(z) - \eta T(x, \lambda) - Proj_{C_\lambda(z+\eta T(y,\lambda))}(z) + \eta T(y, \lambda) \| \\ &\leq \| Proj_{C_\lambda(z+\eta T(x,\lambda))}(z) - Proj_{C_\lambda(z+\eta T(y,\lambda))}(z) \| + \eta \| T(x, \lambda) - T(y, \lambda) \| \\ &\leq \psi_\lambda \| z + \eta T(x, \lambda) - z - \eta T(y, \lambda) \| + \eta \| T(x, \lambda) - T(y, \lambda) \| \\ &= \psi_\lambda \| \eta T(x, \lambda) - \eta T(y, \lambda) \| + \eta \| T(x, \lambda) - T(y, \lambda) \| \\ &= \eta \psi_\lambda \| T(x, \lambda) - T(y, \lambda) \| + \eta \| T(x, \lambda) - T(y, \lambda) \| \\ &\leq \eta \xi_\lambda (\psi_\lambda + 1) \| x - y \|. \end{aligned}$$

Hence $\eta \xi_\lambda (\psi_\lambda + 1) < 1$, and so $h_{z,\lambda} \circ g_{\eta,\lambda}$ is a contractive mapping.

Now, we will consider the sensitivity analysis of the quasi variational inequality problem $(PNQVI_\lambda)$ on uniformly prox-regular sets. More precisely, we assume that for some $\bar{\lambda} \in \Omega$, the problem $(WH(\bar{\lambda}, x^*, \eta))$ has a solution \bar{z} , and X is a closure of a ball in H centered at \bar{z} . We want to investigate the conditions under which, for each λ in a neighborhood of $\bar{\lambda}$, the associated problem $(WH(\lambda, x^*, \eta))$ has a unique solution z near \bar{z} and the function $z := z(\lambda)$ is a continuous or Lipschitz continuous.

The following assumption will be proposed as the sufficient conditions.

Assumption A Let $\lambda \in \Omega$ and $r_\lambda \in (0, +\infty]$. Let $T : X \times \Omega \rightarrow X$ and $C : H \times \Omega \rightarrow 2^H / \{\emptyset\}$ be nonlinear mappings which satisfy:

- (i) $T(\cdot, \lambda)$ is a β_λ -locally strongly monotone and ξ_λ -locally Lipschitz continuous mapping;
- (ii) $C(\cdot, \lambda)$ is uniformly r_λ -prox regular sets and a κ_λ -locally Lipschitz continuous mapping;

(iii) there is $\psi_\lambda \in [0, 1)$ such that

$$\|Proj_{C_\lambda(x)}(z) - Proj_{C_\lambda(y)}(z)\| \leq \psi_\lambda \|x - y\|$$

for all $x, y, z \in H$;

(iv) there is $r_\lambda^* \in \left(0, \frac{r_\lambda}{1-\kappa_\lambda}\right)$ such that for each $z \in X$,

$$M_{z,\lambda}^{r_\lambda^*} = \{x \in (PNQVI_\lambda) | d(z, C_\lambda(x)) \leq r_\lambda^*\}$$

is a nonempty set.

Remark 3.4. By Assumption \mathcal{A} , we can check that $M_{z,\lambda}$ is a closed set. Indeed, if $x_n \in M_{z,\lambda}^{r_\lambda^*}$ and $x_n \rightarrow x \in H$. Then, by using the condition (ii), we see that

$$\begin{aligned} d(x, C_\lambda(x)) &= |d(x, C_\lambda(x)) - d(x_n, C_\lambda(x_n))| \\ &\leq \|x - x_n\| + \kappa_\lambda \|x - x_n\|, \end{aligned}$$

by taking $n \rightarrow \infty$, we have $d(x, C_\lambda(x)) = 0$. This implies that $x \in C_\lambda(x)$. Further, since $-T(x_n, \lambda) \in N_{C_\lambda(x_n)}^P(x_n)$ and T is continuous mapping, by using Lemma 2.5, we get $-T(x, \lambda) \in N_{C_\lambda(x)}^P(x)$. Thus $x \in (PNQVI_\lambda)$. Next, by using Assumption \mathcal{A} (ii), we obtain that

$$\begin{aligned} d(z, C_\lambda(x)) &\leq \kappa_\lambda \|x - x_n\| + d(z, C_\lambda(x_n)) \\ &\leq \kappa_\lambda \|x - x_n\| + r_\lambda^*. \end{aligned}$$

Taking $n \rightarrow \infty$, we have $d(z, C_\lambda(x)) \leq r_\lambda^*$. Hence, $x \in M_{z,\lambda}$.

Now, we define a mapping $f_{z,\lambda} : M_{z,\lambda}^{r_\lambda^*} \rightarrow M_{z,\lambda}^{r_\lambda^*}$ by

$$f_{z,\lambda}(x) = Proj_{C_\lambda(x)}(z), \quad (3.4)$$

for all $x \in M_{z,\lambda}^{r_\lambda^*}$. If Assumption \mathcal{A} (iii) holds, then it is easy to check that $f_{z,\lambda}$ has a unique fixed point. Indeed,

$$\begin{aligned} \|f_{z,\lambda}(x) - f_{z,\lambda}(y)\| &= \|Proj_{C_\lambda(x)}(z) - Proj_{C_\lambda(y)}(z)\| \\ &\leq \psi_\lambda \|x - y\|. \end{aligned}$$

Assume that $d(z, C_\lambda(z)) < (1 - \kappa_\lambda)r_\lambda^*$, we have $Proj_{C_\lambda(x)}(z) \in M_{z,\lambda}^{r_\lambda^*}$. Indeed,

$$\begin{aligned} d(z, C_\lambda(Proj_{C_\lambda(x)}(z))) &\leq d(z, C_\lambda(z)) + \kappa_\lambda \|Proj_{C_\lambda(x)}(z) - z\| \\ &< (1 - \kappa_\lambda)r_\lambda^* + \kappa_\lambda r_\lambda^* \\ &= r_\lambda^*. \end{aligned}$$

Remark 3.5. If $x \in M_{z,\lambda}^{r_\lambda^*}$, for each $\tilde{z} \in B(z, \epsilon)$, then

$$d(\tilde{z}, C_\lambda(\tilde{z})) < (1 - \kappa_\lambda)r_\lambda^*,$$

where $\epsilon = (0, \Gamma)$ with $\Gamma = \frac{(1-\kappa_\lambda)r_\lambda^* - (1+\kappa_\lambda)\|z-x\|}{1+\kappa_\lambda}$, and x is a solution of (PNQVI $_\lambda$) such that

$$\|z - x\| < \frac{(1 - \kappa_\lambda)r_\lambda^*}{1 + \kappa_\lambda}.$$

Moreover, If $X = B(z, \epsilon)$, then $f_{z,\lambda}(M_{z,\lambda}) \subseteq M_{z,\lambda}$,

Proof.

$$\begin{aligned} d(\tilde{z}, C_\lambda(\tilde{z})) &\leq (1 + \kappa_\lambda)\|\tilde{z} - z\| + (1 + \kappa_\lambda)\|z - x\| + d(x, C_\lambda(x)) \\ &= (1 + \kappa_\lambda)\|\tilde{z} - z\| + (1 + \kappa_\lambda)\|z - x\| \\ &< (1 + \kappa_\lambda) \left(\frac{(1 - \kappa_\lambda)r_\lambda^* - (1 + \kappa_\lambda)\|z - x\|}{1 + \kappa_\lambda} \right) + (1 + \kappa_\lambda)\|z - x\| \\ &= (1 - \kappa_\lambda)r_\lambda^* - (1 + \kappa_\lambda)\|z - x\| + (1 + \kappa_\lambda)\|z - x\| \\ &= (1 - \kappa_\lambda)r_\lambda^*. \end{aligned} \quad \square$$

Remark 3.6. If we define $G_{z,\eta}^\lambda : H \rightarrow H$ by

$$G_{z,\eta}^\lambda(x) = z + \eta T(x, \lambda), \text{ for all } x \in H.$$

If T is a ξ_λ -locally Lipschitz continuous mapping, then

$$\begin{aligned} \|G_{z,\eta}^\lambda(x_1) - G_{z,\eta}^\lambda(x_2)\| &= \|z + \eta T(x_1, \lambda) - z - \eta T(x_2, \lambda)\| \\ &= \eta \|T(x_1, \lambda) - T(x_2, \lambda)\| \\ &\leq \eta \xi_\lambda \|x_1 - x_2\|. \end{aligned}$$

Thus, if and $\eta \in (0, \frac{1}{\xi_\lambda})$, we see that $G_{z,\eta}^\lambda$ is a contractive mapping. Hence, if we define $\Delta_{z,\lambda} = \{x|x - z = \eta T(x, \lambda), \text{ for some } \eta > 0\}$, then $\Delta_{z,\lambda}$ is nonempty set. Observing, if we choose $0 < \eta < \frac{r}{\delta_\lambda}$, where $\delta_\lambda = \sup\{\|T(x, \lambda)\| : x \in H\}$, then $\|x - z\| < r$. Therefore, the best choice of η for using in our results is $\eta \in (0, \min\{\frac{1}{\xi_\lambda(\psi_\lambda+1)}, \frac{r}{\delta_\lambda}\})$.

Now, we consider the case when the solutions of parametric Wiener-Hopf equations (WH(λ, x^*, η)) lie in the interior of X . For each $\lambda \in \Omega$ and a positive constant η , we define a mapping $F_\lambda^\eta : X \rightarrow X$ by

$$F_\lambda^\eta(z) = x - \eta T(x, \lambda), \text{ for all } z \in X, \tag{3.5}$$

where x is a unique fixed point of $f_{z,\lambda}$ which was defined in (3.4).

Theorem 3.7. Let $\lambda \in \Omega$ and $r_\lambda \in (0, +\infty]$. Let $T : X \times \Omega \rightarrow X$ and $C : H \times \Omega \rightarrow 2^H / \{\emptyset\}$ be nonlinear mappings. Assume that Assumption \mathcal{A} holds and

$$\left| \eta - \frac{\beta_\lambda}{\xi_\lambda^2} \right| < \frac{\sqrt{\beta_\lambda^2 t_{r_\lambda}^2 - \xi_\lambda^2 (t_{r_\lambda}^2 - (1 - \psi_\lambda)^2)}}{\xi_\lambda^2 t_{r_\lambda}^*} \quad (3.6)$$

where $t_{r_\lambda}^* = \frac{r_\lambda}{r_\lambda - r_\lambda^*}$ and $r_\lambda^* \in \left(0, r_\lambda \left(1 - \frac{\sqrt{\xi_\lambda^2 - \beta_\lambda^2}}{\xi_\lambda (1 - \psi_\lambda)} \right) \right]$. Then F_λ^η , which is defined in (3.5), has a unique fixed point.

Proof. Given $\tilde{z}, \hat{z} \in X, \lambda \in \Omega$ and for some constants η . By using (3.5), we have

$$\begin{aligned} & \|F_\lambda^\eta(\tilde{z}) - F_\lambda^\eta(\hat{z})\|^2 \\ &= \|(\tilde{x} - \eta T(\tilde{x}, \lambda)) - (\hat{x} - \eta T(\hat{x}, \lambda))\|^2 \\ &= \|(\tilde{x} - \hat{x}) - \eta(T(\tilde{x}, \lambda) - T(\hat{x}, \lambda))\|^2 \\ &\leq \|\tilde{x} - \hat{x}\|^2 - 2\eta \langle T(\tilde{x}, \lambda) - T(\hat{x}, \lambda), \tilde{x} - \hat{x} \rangle + \eta^2 \|T(\tilde{x}, \lambda) - T(\hat{x}, \lambda)\|^2 \\ &\leq \|\tilde{x} - \hat{x}\|^2 - 2\eta\beta_\lambda \|\tilde{x} - \hat{x}\|^2 + \eta^2 \xi_\lambda^2 \|\tilde{x} - \hat{x}\|^2 \\ &= (1 - 2\eta\beta_\lambda + \eta^2 \xi_\lambda^2) \|\tilde{x} - \hat{x}\|^2. \end{aligned}$$

This implies that $\|F_\lambda^\eta(\tilde{z}) - F_\lambda^\eta(\hat{z})\| \leq \sqrt{1 - 2\eta\beta_\lambda + \eta^2 \xi_\lambda^2} \|\tilde{x} - \hat{x}\|$. By using (3.5) and Assumption \mathcal{A} (iii), we see that

$$\begin{aligned} & \|\tilde{x} - \hat{x}\| \\ &= \|\text{Proj}_{C_\lambda(\tilde{x})}(\tilde{z}) - \text{Proj}_{C_\lambda(\hat{x})}(\hat{z})\| \\ &\leq \|\text{Proj}_{C_\lambda(\tilde{x})}(\tilde{z}) - \text{Proj}_{C_\lambda(\tilde{x})}(\hat{z})\| + \|\text{Proj}_{C_\lambda(\tilde{x})}(\hat{z}) - \text{Proj}_{C_\lambda(\hat{x})}(\hat{z})\| \\ &\leq t_{r_\lambda}^* \|\tilde{z} - \hat{z}\| + \psi_\lambda \|\tilde{x} - \hat{x}\|. \end{aligned}$$

Thus,

$$(1 - \psi_\lambda) \|\tilde{x} - \hat{x}\| \leq t_{r_\lambda}^* \|\tilde{z} - \hat{z}\|.$$

This implies that

$$\|\tilde{x} - \hat{x}\| \leq \frac{t_{r_\lambda}^*}{1 - \psi_\lambda} \|\tilde{z} - \hat{z}\|.$$

Hence,

$$\begin{aligned} \|F_\lambda^\eta(\tilde{z}) - F_\lambda^\eta(\hat{z})\| &\leq \frac{t_{r_\lambda}^* \sqrt{1 - 2\eta\beta_\lambda + \eta^2 \xi_\lambda^2}}{1 - \psi_\lambda} \|\tilde{z} - \hat{z}\| \\ &= \theta \|\tilde{z} - \hat{z}\| \end{aligned}$$

where $\theta = \frac{t_{r_\lambda}^* \sqrt{1 - 2\eta\beta_\lambda + \eta^2 \xi_\lambda^2}}{1 - \psi_\lambda}$. By the assumption of η and r_λ^* , we can check that $\theta < 1$. Then, we conclude that F_λ^η is a contractive mapping. Therefore, F_λ^η has a unique fixed point. \square

Theorem 3.8. *Let $\lambda \in \Omega$ and η be a positive constant. Assume that all of assumptions of Theorem 3.7 hold and if x^* is a solution of the problem $(PNQVI_\lambda)$. Then, z^* , which is a solution of the problem $(WH(\lambda, x^*, \eta))$, is a fixed point of F_λ^η .*

Proof. By Theorem 3.7, we see that F_λ^η has a fixed point. Assume that x^* is a solution of the problem $(PNQVI_\lambda)$. By Lemma 3.1 (i) and Lemma 3.2, we obtain that z^* is the solution of the problem $(WH(\lambda, x^*, \eta))$, where

$$z^* = x^* - \eta T(x^*, \lambda),$$

and

$$x^* = Proj_{C_\lambda(x^*)}(x^* - \eta T(x^*, \lambda)).$$

This implies that

$$x^* = Proj_{C_\lambda(x^*)}(z^*).$$

From Definition of F_λ^η in (3.5), we have

$$F_\lambda^\eta(z^*) = z^*$$

Hence, z^* is a fixed point of F_λ^η . This completes the proof. □

Remark 3.9. *Let $\lambda \in \Omega$, we see that the mapping F_λ^η has a unique fixed point $z := z(\lambda)$, that is,*

$$z(\lambda) = F_\lambda^\eta(z).$$

By assumption, for $\lambda = \bar{\lambda}$, the function \bar{z} is a solution of parametric Wiener-Hopf equation $(WH(\bar{\lambda}, x^, \eta))$. Using Theorem 3.8, we see that \bar{z} , for $\lambda = \bar{\lambda}$, is a fixed point of $F_\lambda^\eta(z)$ and also it is a fixed point of $F_\lambda^\eta(z)$. Subsequently, we conclude that*

$$\bar{z} = z(\bar{\lambda}) = F_\lambda^\eta(z(\bar{\lambda})).$$

Theorem 3.10. *Let $\lambda \in \Omega$ and $r_\lambda \in (0, +\infty]$. Let $T : X \times \Omega \rightarrow X$ and $C : H \times \Omega \rightarrow 2^H/\{\emptyset\}$ be nonlinear mappings. Assume that all of assumptions of Theorem 3.8 hold and for fixed $x, z \in X$ the operator $T(x, \cdot)$ is a locally Lipschitz continuous with a constant δ and the map $\lambda \rightarrow Proj_{C_\lambda(x)}(z)$ is continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$ with a constant τ . Then, the function $z(\lambda)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$.*

Proof. Let $\lambda, \bar{\lambda} \in \Omega$ and η is a positive constant. By using Theorem 3.7 and Theorem 3.8, we have

$$\begin{aligned} \|z(\lambda) - z(\bar{\lambda})\| &= \|F_\lambda^\eta(z(\lambda)) - F_\lambda^\eta(z(\bar{\lambda}))\| \\ &\leq \|F_\lambda^\eta(z(\lambda)) - F_\lambda^\eta(z(\bar{\lambda}))\| + \|F_\lambda^\eta(z(\bar{\lambda})) - F_\lambda^\eta(z(\bar{\lambda}))\| \\ &\leq \theta \|z(\lambda) - z(\bar{\lambda})\| + \|F_\lambda^\eta(z(\bar{\lambda})) - F_\lambda^\eta(z(\bar{\lambda}))\|. \end{aligned}$$

Then, by using the definition of F_λ^η , we have

$$\begin{aligned} & \|F_\lambda^\eta(z(\bar{\lambda})) - F_{\bar{\lambda}}^\eta(z(\bar{\lambda}))\| \\ &= \|\text{Proj}_{C_\lambda(x(\bar{\lambda}))}(z(\bar{\lambda})) - \eta T(x(\bar{\lambda}), \lambda) - \text{Proj}_{C_{\bar{\lambda}}(x(\bar{\lambda}))}(z(\bar{\lambda})) + \eta T(x(\bar{\lambda}), \bar{\lambda})\| \\ &\leq \|\text{Proj}_{C_\lambda(x(\bar{\lambda}))}(z(\bar{\lambda})) - \text{Proj}_{C_{\bar{\lambda}}(x(\bar{\lambda}))}(z(\bar{\lambda}))\| + \eta \|T(x(\bar{\lambda}), \lambda) - T(x(\bar{\lambda}), \bar{\lambda})\| \\ &\leq \tau \|\lambda - \bar{\lambda}\| + \eta \delta \|\lambda - \bar{\lambda}\| \\ &= (\tau + \eta \delta) \|\lambda - \bar{\lambda}\|. \end{aligned}$$

Thus,

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \theta \|z(\lambda) - z(\bar{\lambda})\| + (\tau + \eta \delta) \|\lambda - \bar{\lambda}\|.$$

We see that

$$(1 - \theta) \|z(\lambda) - z(\bar{\lambda})\| \leq (\tau + \eta \delta) \|\lambda - \bar{\lambda}\|.$$

Hence,

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{(\tau + \eta \delta)}{(1 - \theta)} \|\lambda - \bar{\lambda}\|.$$

We conclude that $z(\lambda)$ is a Lipschitz continuous at $\lambda = \bar{\lambda}$. \square

Next, we will present the main result of this paper.

Theorem 3.11. *Let \bar{x} be a solution of parametric quasi variational inequality problem in uniformly prox-regular sets $(PNQVI_{\bar{x}})$ and let \bar{z} be a solution of parametric Wiener-Hopf equation $(WH(\bar{\lambda}, x^*, \eta))$, for $\lambda = \bar{\lambda}$. Assume that Assumption A holds. If for fixed $x, z \in X$, the operator $T(x, \cdot)$ is a locally Lipschitz continuous and $\lambda \rightarrow \text{Proj}_{C_\lambda(x)}(z)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$. Then, there exists a neighborhood $N \subseteq \Omega$ of $\bar{\lambda}$ such that for $\lambda \in N$ the parametric Wiener-Hopf equation $(WH(\lambda, x^*, \eta))$ has a solution, $z(\lambda)$, in the interior of X , $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$.*

Proof. Follow from Theorem 3.7, Theorem 3.8 and Theorem 3.10. \square

Theorem 3.12. *Let $\lambda \in \Omega$ and $r_\lambda \in (0, +\infty]$. Let $T : X \times \Omega \rightarrow X$ and $C : H \times \Omega \rightarrow 2^H / \{\emptyset\}$ be nonlinear mappings. Assume that all of assumptions of Theorem 3.10 hold and $\eta \xi_\lambda < 1$. Then, the function $x(\lambda)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$.*

Proof. Let $\lambda, \bar{\lambda} \in \Omega$ and η is a positive constant. By using Theorem 3.10, we have

$$\begin{aligned} \|x(\lambda) - x(\bar{\lambda})\| &= \|z(\lambda) + \eta T(x(\lambda), \lambda) - z(\bar{\lambda}) - \eta T(x(\bar{\lambda}), \bar{\lambda})\| \\ &\leq \|z(\lambda) - z(\bar{\lambda})\| + \eta \|T(x(\lambda), \lambda) - T(x(\bar{\lambda}), \bar{\lambda})\| \\ &\leq \|z(\lambda) - z(\bar{\lambda})\| + \eta (\|T(x(\lambda), \lambda) - T(x(\lambda), \bar{\lambda})\| \\ &\quad + \|T(x(\lambda), \bar{\lambda}) - T(x(\bar{\lambda}), \bar{\lambda})\|) \\ &\leq \left(\frac{\tau + \eta \delta}{1 - \theta} \right) \|\lambda - \bar{\lambda}\| + \eta (\delta \|\lambda - \bar{\lambda}\| + \xi_{\bar{\lambda}} \|x(\lambda) - x(\bar{\lambda})\|). \end{aligned}$$

This implies that

$$\text{Then, } (1 - \eta\xi_\lambda)\|x(\lambda) - x(\bar{\lambda})\| \leq \left(\frac{\tau + \eta\delta}{1 - \theta} + \eta\delta \right) \|\lambda - \bar{\lambda}\|.$$

$$\|x(\lambda) - x(\bar{\lambda})\| \leq \left(\frac{\tau + \eta\delta(2 - \theta)}{(1 - \theta)(1 - \eta\xi_\lambda)} \right) \|\lambda - \bar{\lambda}\|.$$

We conclude that $x(\lambda)$ is a Lipschitz continuous at $\lambda = \bar{\lambda}$. \square

Theorem 3.13. *Let \bar{x} be a solution of parametric quasi variational inequality problem in uniformly prox-regular sets $(PNQVI_{\bar{\lambda}})$ and let \bar{z} be a solution of parametric Wiener-Hopf equation $(WH(\bar{\lambda}, x^*, \eta))$, for $\lambda = \bar{\lambda}$. Assume that Assumption **A** holds and $\eta\xi_\lambda < 1$. If for fixed $x, z \in X$ the operator $T(x, \cdot)$ is a locally Lipschitz continuous and $\lambda \rightarrow \text{Proj}_{C_\lambda(x)}(z)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$. Then, there exists a neighborhood $M \subseteq \Omega$ of $\bar{\lambda}$ such that for $\lambda \in M$, the parametric quasi variational inequality problem in uniformly prox-regular sets $(PNQVI_\lambda)$ has a solution, $x(\lambda)$, in the interior of X , $x(\bar{\lambda}) = \bar{x}$ and $x(\lambda)$ is a continuous or Lipschitz continuous at $\lambda = \bar{\lambda}$.*

Proof. Follow from Theorem 3.7, Theorem 3.8 and Theorem 3.12. \square

4 Conclusion

In this work, we study and consider the parametric quasi variational inequality problem on uniformly prox-regular sets and a parametric Wiener-Hopf equation. The equivalent relation of the both problem is studied. Furthermore, we use this equivalence to consider the sensitivity analysis of the quasi variational inequality problem on nonconvex sets. We desire that the results which presented here will be useful and valuable for researchers who study the branch of variational inequality and related applications.

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References

- [1] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris 258 (1964) 4413-4416.
- [2] Q.H. Ansari, Y.J. Yao, A fixed point theorem and its applications to a system of variational inequalities, Bull. Aust. Math. Soc. 59 (1999) 433-442.
- [3] Y.J. Cho, J.K. Kim, R.U. Verma, A class of nonlinear variational inequalities involving partially relaxed monotone mappings and general auxiliary principle, Dyn. Syst. Appl. 11 (2002) 333-338.

- [4] M.B. Donato, M. Milasi, C. Vitanza, On the study of economic equilibrium with variational inequality arguments, *Journal of Optimization Theory and Applications* 168 (2) (2016) 646-660.
- [5] J.S. Pang, Asymmetric variational inequality problems over product sets: applications and iterative methods, *Math. Program.* 31 (1985) 206-219.
- [6] X. Zou, D. Gong, L. Wang, Z. Chen, A novel method to solve inverse variational inequality problems based on neural networks. *Neurocomputing* 173 (2016) 1163-1168.
- [7] A. Bensoussan, J.L. Lions, *Applications Des Inequations Variationnelles En Controle Stochastique*, Dunod, Paris, 1978.
- [8] A. Bensoussan, M. Goursat, J.L. Lions, Control impulsif et inequations quasi-variationnelles stationnaires, *Comptes Rendus de l'Academie des Sciences* 276 (1973) 1279-1284.
- [9] S.S. Chang, N.J. Huang, Generalized random multivalued quasi-complementarity problems, *Indaian J. Math.* 35 (1993) 305-320.
- [10] N.J. Huang, On the generalized implicit quasi-variational inequalities, *Journal of Mathematical Analysis and Applications* 216 (1997) 197-210.
- [11] I. Stupia, L. Sanguinetti, G. Bacci, L. Vandendorpe, Power control in networks with heterogeneous users: a quasi-variational inequality approach, *IEEE Transactions on signal processing processing*, 63 (21) (2015) 5691-5705.
- [12] S. Suantai, N. Petrot, Existence and stability of iterative algorithms for the system of nonlinear quasi-mixed equilibrium problems, *Applied Mathematics Letters* 24 (2011) 308-313.
- [13] F. Bernard, L. Thibault, N. Zlateva, Prox-regular sets and epigraphs in uniformly convex Banach spaces: various regularities and other properties, *Trans. Amer. Math. Soc.* 363 (2011) 2211-2247.
- [14] M.A. Noor, Iterative schemes for nonconvex variational inequalities, *Journal of Optimization Theory and Applications* 121 (2004) 385-395.
- [15] N. Petrot, Some existence theorems for nonconvex variational inequalities problems, *Abstract and Applied Analysis* 2010 (2010) doi:10.1155/2010/472760.
- [16] N. Petrot, J. Suwannawit, Existence theorems for some systems of quasi-variational inequalities problems on uniformly prox-regular sets, *Mathematical Inequalities & Applications* 16 (4) (2013) 1229-1242.
- [17] M.A. Noor, N. Petrot, J. Suwannawit, Existence theorems for multivalued variational inequality problems on uniformly prox-regular sets, *Optimization Letters* (2012) doi:10.1007/s11590-012-0545-x.
- [18] F.H. Clarke, R.J. Stern, P.R. Wolenski, Proximal smoothness and the lower C^2 property, *Journal of Convex Analysis* 2 (1/2) (1995) 117-144.

- [19] M. Bounkhel, L.T. Tadj, A. Hamdi, Iterative schemes to solve nonconvex variational problems, *Journal of Inequalities in Pure and Applied Mathematics* 4 (2003) 14 pages.
- [20] J. Suwannawit, N. Petrot, Existence theorems for quasi variational inequality problem on proximally smooth sets, *Abstract and Applied Analysis* 2013 (2013) doi:org/10.1155/2013/612819.
- [21] S. Dafermos, Sensitivity analysis in variational inequalities, *Mathematics of Operations Research* 13 (1988) 421-434.
- [22] R.N. Mukherjee, H.J. Verma, Sensitivity analysis of generalized variational inequalities, *J. Math. Anal. Appl.* 167 (1992) 299-304.
- [23] A. Moudafi, Projection methods for a system of nonconvex variational inequalities, *Nonlinear Analysis* 71 (2009) 517-520.
- [24] M.A. Noor, Sensitivity analysis for quasi variational inequalities, *J. Optim. Theory Appl.* 95 (1997) 399-407.
- [25] A. Moudafi, M.A. Noor, Sensitivity analysis for variational inclusions by Wiener-Hopf equation techniques, *J. Appl. Math. Scoth. Anal.* 12 (3) (1999) 223-232.
- [26] M.A. Noor, K.I. Noor, Sensitivity analysis of some quasi variational inclusions, *J. Adv. Math. Stud.* 6 (1) (2013) 43-52.
- [27] J.K. Kim, Sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities in Banach spaces, *J. Comp. Anal. Appl.* 22 (2) (2017) 327-335.
- [28] R.A. Poliquin, R.T. Rockafellar, L. Thibault, Local differentiability of distance functions, *Transactions of the American Mathematical Society* 352 (2000) 5231-5249.
- [29] Z.F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, 1998.

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