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# On 2-Primal Modules

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**Abstract :** In this paper, the concept of 2-primal modules is introduced. We show that the implications between rings which are reduced, IFP, symmetric and 2-primal are preserved when the notions are extended to modules. Like for rings, for 2-primal modules, prime submodules coincide with completely prime submodules. We prove that if $M$ is a quasi-projective and finitely generated right $R$-module which is a self-generator, then $M$ is 2-primal if and only if $S = \mathrm{End}_R(M)$ is 2-primal. Some properties of 2-primal modules are also investigated.

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# 1 Introduction

Throughout this paper, a ring $R$ is an associative ring with identity and all modules are unitary right $R$-modules. Let $\beta(R)$, $\beta_{co}(R)$ and $\mathcal{N}(R)$ be the prime radical, generalized nil radical (also called the completely prime radical) and the set of all nilpotent elements of $R$, respectively. An ideal $I$ of $R$ is called 2-primal [1, Definition 1.1] if $\beta(R/I) = \mathcal{N}(R/I)$. Thus, a ring $R$ is 2-primal if and only if the zero ideal is 2-primal, i.e., $\beta(R) = \mathcal{N}(R)$. 2-primal rings were first studied in [1] (although not so-called at that time). Birkenmeier *et al.* [1, Proposition 2.1] showed that $R$ is 2-primal if and only if $\beta(R) = \beta_{co}(R)$. In [2], Marks gave a detailed study of 2-primal rings. The definition of a prime submodule has been defined by Dauns in [3] and the notion of a completely prime submodule has been defined in [4]. Using these definitions, 2-primal modules are given by Groenewald and Ssevviiri in [5]. In modifying the structure of prime ideals and prime rings, many authors transferred the notion of prime ideals to modules. There are many ways to generalize these notions. For examples, Andrunakievich (1962) [6], Beachy (1975) [7], Dauns (1978) [3], Bican *et al.* (1980) [8], C. P. Lu (1995) [9], Behboodi and Koohy (2004) [10] gave some definitions of prime submodules. In 2008, N. V. Sanh *et al.* [11] proposed a new definition of prime submodules. By using this definition, they found many beautiful properties of prime submodules that are similar to prime ideals. They could construct some new notions such as nilpotent submodules, nil submodules, a prime radical, a nil radical and a Levitzki radical of a right module $M$ over $R$ and described all properties of them as generalizations of nilpotent ideals, nil ideals, a prime radical, a nil radical and a Levitzki radical of rings.

The notion of 2-primal modules exists in literature for other algebraic structures such as near-rings (for example, see [12]). In this paper, by using the definition of prime, completely prime (strongly prime) submodules introduced by Sanh *et al.* [11,13–15] and Bac *et al.* [16], we extend it to modules by defining 2-primal (sub)modules.

The rest of this paper is organized as follows. Preliminaries concept and some properties of prime submodules are shown in Section 2. We give some results about the prime, semiprime and strongly prime submodules. We provide some properties of symmetric modules and submodules in Section 3. In Section 4, 2-primal submodules are investigated.

# 2 Preliminaries

Denote $S = \mathrm{End}_R(M)$, the endomorphism ring of the module $M$. A submodule $X$ of $M$ is called a *fully invariant* submodule if $f(X) \subset X$, for any $f \in S$.

Especially, a right ideal of $R$ is a fully invariant submodule of $R_R$ if it is a two-sided ideal of $R$. The class of all fully invariant submodules of $M$ is non-empty and closed under intersections and sums. A right $R$-module $M$ is *finitely generated* if there are $m_1, m_2, \ldots, m_k \in M$ such that $M = \sum_{i=1}^{k} m_i R$. This is equivalent to the condition that there is an epimorphism $R^k \longrightarrow M$, for some $k \in \mathbb{Z}^+$.

Following Sanh et al. [11], a fully invariant proper submodule $X$ of $M$ is called a *prime submodule* of $M$ if for any ideal $I$ of $S = \mathrm{End}_R(M)$, and any fully invariant submodule $U$ of $M$, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A fully invariant submodule $X$ of $M$ is called a *completely prime submodule* (called a *strongly prime submodule* in [11]) of $M$ if for any $\varphi \in S = \mathrm{End}_R(M)$ and any $m \in M$, if $\varphi(m) \in X$, then either $\varphi(M) \subset X$ or $m \in X$.

**Definition 2.1.** [13, Definition 2.1]. A submodule $X$ of a right $R$-module $M$ is said to have the "insertion factor property" (briefly, an IFP-submodule) if for any endomorphism $\phi$ of $M$and any element $m \in M$, if $\phi(m) \in X$, then $\phi Sm \in X$. A right ideal $I$ is an IFP-right ideal if it is an IFP-submodule of $R_R$, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subseteq I$. A right $R$-module $M$ is called an IFP-module if 0 is an IFP-submodule of $M$. A ring $R$ is IFP if 0 is an IFP-ideal.

**Definition 2.2.** [14, Definition 2.1] A fully invariant submodule $X$ of a right $R$-module $M$ is called a semiprime submodule if it is an intersection of prime submodules of $M$. A right $R$-module $M$ is called a semiprime module if 0 is a semiprime submodule of $M$. Consequently, $R$ is a semiprime ring if $R_R$ is semiprime. By symmetry, $R$ is a semiprime ring if $R_R$ is a semiprime left $R$-module.

**Definition 2.3.** [17] A fully invariant proper submodule $X$ of $M$ is called completely semiprime if for any $\psi \in S$ and $m \in M$, $\psi^2(m) \in X$ implies $\psi Sm \subseteq X$.

# 3 Symmetric Modules and Submodules

**Definition 3.1.** [18, Definition 2.2] A submodule of a right $R$-module $M$ is a symmetric submodule if for endomorphisms $\phi$ and $\alpha$ of $S$ and any element $m \in M$, if $\phi\alpha m \in X$, then $\alpha\phi m \in X$. The right $R$-module $M$ is a symmetric module if for endomorphisms $\phi$ and $\alpha$ of $S$ and any element $m \in M$, if $\phi\alpha m = 0$, then $\alpha\phi m = 0$.

We have some properties of symmetric submodules as follows.

**Proposition 3.2.** *Let $M$ be a right $R$-module. If $X$ is a fully invariant submodule which is symmetric, then $X$ has IFP.*

*Proof.* Let $\alpha \in S = \mathrm{End}_R(M)$ and $m \in M$ such that $\alpha(m) \in X$. Since $X$ is a fully invariant submodule, we have for every $\beta \in S$ that $\beta\alpha(m) \in X$. Since $X$ is symmetric, we have $\alpha\beta(m) \in X$ and thus $\alpha Sm \subseteq X$, i.e., $X$ has IFP. □

**Proposition 3.3.** *Let $M$ be a right $R$-module. The submodule $X$ is completely prime if and if it is prime and symmetric.*

*Proof.* Suppose that $X$ is a prime and symmetric submodule and $\alpha(m) \in X$. Since $X$ is a fully invariant submodule, we can see that $\beta\alpha(m) \in X$ for all $\beta \in S$. This follows that $\alpha\beta(m) \in X$ for all $\beta \in S$, by the symmetric property of $X$. Hence, $\alpha S(m) \in X$, proving that $X$ has IFP. Applying [16, Theorem 2.11], we conclude that $X$ is a completely prime submodule.

For the converse, suppose that $X$ is completely prime and $\beta\alpha(m) \in X$. It is well-known in [16, Proposition 2.3] that if $X$ is a completely prime submodule, then $X$ is also a prime submodule. Since $\beta\alpha(m) \in X$, by the definition of completely prime submodules, we have either $\alpha(m) \in X$ or $\beta(M) \subseteq X$. If $\alpha(m) \in X$, then $\alpha S(m) \subseteq X$, showing that $\alpha\beta(m) \in X$. If $\beta(M) \subseteq X$, then $\alpha\beta(m) \in X$, implying that $X$ is symmetric. □

It is routine to prove the following result.

**Proposition 3.4.** *Let $M$ be a right $R$-module. If $X$ is a symmetric submodule, then $I_X$ is a symmetric ideal of $S$.*

We give an example of symmetric modules.

**Example 3.5.** Let p be any prime integer and $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$ a $\mathbb{Z}$-module. Then the endomorphism ring $S$ of the module $M$ is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$. It is evident that $M$ is a symmetric module.

The following result gives a relationship between a symmetric module and its endomorphism ring.

**Theorem 3.6.** *If $M$ is a symmetric module, then $S$ is a symmetric ring. The converse is true if for all $m \in M$, there exists $g \in S$ such that $mR = g(M)$ or $M$ is a free $R$-module.*

*Proof.* Suppose that $f, g, h \in S$ such that $fgh = 0$. This implies that $fgh(m) = 0$ for all $m \in M$. Since $M$ is a symmetric module, we have $fgh(m) = gfh(m) = 0$. Hence, $gfh = 0$, proving that $S$ is a symmetric ring. For the converse, suppose that $fh(m) = 0$, then $fh(mR) = 0$. Applying the hypothesis, we have $fhg(M) = 0$ for some $g \in S$. This implies that $fhg = 0$. Since $S$ is a symmetric ring, we have $hfg = 0$. Hence, $hfg(m) = 0$ for all $m \in M$. This follows that $hf(mR) = 0$, showing that $hf(m) = 0$. From this, we can conclude that $M$ is a symmetric module.

Let $F = M$ be a free $R$-module. Clearly, for every $m \in F$ there exists $f \in S = \text{End}_R(M)$ such that $fF = Rm$. This implies that $F = M$ is a symmetric module. □

We provide a couple of lemmas as follows.

**Lemma 3.7.** *A completely semiprime submodule is symmetric.*

*Proof.* Assume that $fg(m) \in X$. Since $X$ is a fully invariant submodule, we have $(gfg)^2(m) = gfggfg(m) \in X$. This implies that $gfgS(m) \subseteq X$ because $X$ is a completely semiprime submodule. From $gfgS(m) \subseteq X$, we have $gfgf(m) \in X$. Again, since $X$ is a completely semiprime submodule, we have $gfS(m) \subseteq X$. It follows that $gf(m) \in X$, concluding that $X$ is a symmetric submodule. □

**Lemma 3.8.** *Let $X$ be a submodule of a right $R$-module $M$. If $X$ is a symmetric submodule and $M$ is quasi-projective, then $M/X$ is a symmetric submodule. Conversely, if $M/X$ is symmetric and $X$ is fully invariant, then $X$ is a symmetric submodule of $M$.*

*Proof.* Suppose that $X$ is a symmetric submodule of $M$ and $\bar{\phi}\bar{\varphi}(\bar{m}) = \bar{0}$, where $\bar{\phi}, \bar{\varphi} \in \bar{S}$ =$\text{End}_R(M/X)$ and $\bar{m} \in M/X$. By the quasi-projectivity of $M$, there are $\phi, \varphi \in S$ such that $\nu\phi = \bar{\phi}\nu$, and $\nu\varphi = \bar{\varphi}\nu$, where $\nu : M \to M/X$ is the natural epimorphism. It follows that $\phi\varphi(m) \in X$. Since $X$ is symmetric, we have $\varphi\phi(m) \in X$. Thus, $\bar{\varphi}\bar{\phi}(\bar{m}) = \bar{\varphi}\bar{\phi}\nu(m) = \bar{\varphi}\nu\phi(m) = \nu\varphi\phi(m) = \bar{0}$. Hence, $\bar{\varphi}\bar{\phi}(\bar{m}) = \bar{0}$ and consequently $M/X$ is a symmetric module. For the converse, suppose that $X$ is a fully invariant submodule of $M$ and $M/X$ is symmetric. Let $\phi\varphi(m) \in X$ with $\phi, \varphi \in S$ and $m \in M$. Since $M$ is quasi-projective, there are $\bar{\phi}, \bar{\varphi} \in \bar{S}$ such that $\nu\phi = \bar{\phi}\nu$ and $\nu\varphi = \bar{\varphi}\nu$. From $\phi\varphi(m) \in X$, we have $\nu\phi\varphi(m) = \bar{0}$. Therefore, $\bar{\phi}\bar{\varphi}(\bar{m}) = \bar{\phi}\bar{\varphi}\nu(m) = \nu\phi\varphi(m) = \bar{0}$. By using the fact that $M/X$ is symmetric, we have $\bar{\varphi}\bar{\phi}(\bar{m}) = \bar{0}$ . This implies that $\bar{\varphi}\bar{\phi}(\bar{m}) = \bar{\varphi}\bar{\phi}\nu(m) = \bar{\varphi}\nu\phi(m) = \nu\varphi\phi(m) = \bar{0}$. Thus, $\varphi\phi(m) \in X$, proving that $X$ is a symmetric submodule of $M$. □

The following result is given in [16].

**Lemma 3.9.** [16, Lemma 2.8] *Let $M, N$ be right $R$-modules and $f : M \longrightarrow N$ be an epimorphism. Suppose that $Kerf$ is a fully invariant submodule of $M$. Then,*

(1) *For any $\varphi \in S$, there exists $\phi \in \bar{S}$ =End$(N)$ such that $\phi f = f\varphi$.*

(2) *If $V$ is a fully invariant submodule of $N$, then $U = f^{-1}(V)$ is a fully invariant submodule of $M$.*

We provide some properties of symmetric submodules.

**Lemma 3.10.** *Let $M$ be a quasi-projective module, and $P$ a symmetric submodule of $M$. If $A \subset P$ is a fully invariant submodule of $M$, then $P/A$ is a symmetric submodule of $M/A$.*

*Proof.* Let $\varphi, \phi \in \bar{S} = End_R(M/A)$, and $m + A \in M/A$ such that $\varphi\phi(m + A) \in P/A$. By the quasi-projectivity of $M$, we can find endomorphisms $f, g \in S$ such that $\varphi\nu = \nu f$ and $\phi\nu = \nu g$ where $\nu : M \longrightarrow M/A$ is the natural epimorphism. From $fg(m) + A = \nu fg(m) = \varphi\phi\nu(m) = \varphi\phi(m + A) \in P/A$, we see that $fg(m) \in P$. By the symmetric property of $P$, we have $gf(m) \in P$. This implies that $\varphi\phi(m+A) = fg(m) + A \in P/A$, showing that $P/A$ is symmetric. □

**Proposition 3.11.** *Let $M$ be a quasi-projective module, and $f : M \longrightarrow N$ be an epimorphism such that $Kerf$ is a fully invariant submodule of $M$. Then,*
*(1) If $Y$ is a symmetric submodule of $N$, then $X = f^{-1}(Y)$ is a symmetric submodule of $M$.*
*(2) If $X$ is a symmetric submodule of $M$, then $f(X)$ is a symmetric submodule of $N$.*

*Proof.* (1) By Lemma 3.9, $X = f^{-1}(Y)$ is a fully invariant submodule of $M$. It is easy to see that $X$ is different from $M$. Suppose that $\varphi, \phi \in S$ and $m \in M$ such that $\varphi\phi(m) \in X$. We will show that $\phi\varphi(m) \in X$. From Lemma 3.9, there exists $\gamma, \beta \in S' =$End$(N)$ such that $\gamma f = f\varphi$ and $\beta f = f\phi$. From $\varphi\phi(m) \in X$, we see that $f\varphi\phi(m) \in f(X) = Y$. Since $\gamma f = f\varphi$, and $\beta f = f\phi$, we have $\gamma\beta f(m) \in Y$. By symmetric property of $Y$, we can see that $\beta\gamma f(m) \in Y$. From $\gamma f = f\varphi$ and $\beta f = f\phi$, we have $f\phi\varphi(m) \in Y$. This implies that $\phi\varphi(m) \in X$, proving that $X$ is a symmetric submodule.

(2) Note that $f(X)$ is a fully invariant submodule of $N$. Suppose that $f(X) = N = f(M)$. Then we have $M \subset X + Kerf = X$, a contradiction. This implies that $f(X)$ is different from $N$. Let $\gamma\alpha(n) \in f(X)$, where $\gamma, \alpha \in S' =$End$(N)$. Since $M$ is a quasi-projective module, there are $\varphi, \phi \in S$ such that $\gamma f = f\varphi$ and $\alpha f = f\phi$. From this, we see that $\gamma\alpha(n) = \gamma\alpha(f(f^{-1})(n)) = \gamma f\phi(f^{-1}(n)) = f\varphi\phi(f^{-1}(n)) \subset f(X)$. It follows that $\varphi\phi(f^{-1}(n)) \subset X + Kerf = X$. If $X$ is a symmetric submodule, then we have $\phi\varphi(f^{-1}(n)) \in X$. Thus $\alpha\gamma(n) \in f(X)$. This shows that $f(X)$ is a symmetric submodule. □

# 4 On 2-Primal Submodules

For a right $R$-module $M$, let $P(M)$ be the intersection of all prime submodules of $M$ and $C(M)$, the intersection of all completely prime submodules of $M$. $P(M)$ is the prime radical of $M$ and $C(M)$ is the completely prime radical of $M$. By applying the fact that a ring is 2-primal if and only if $\beta(R) = \beta_{co}(R)$, we now give the definition of a 2-primal module as follows:

**Definition 4.1.** A submodule $X$ of an $R$-module $M$ is 2-primal if $P(M/X) = C(M/X)$. An $R$-module $M$ is 2-primal if $P(M) = C(M)$.

Recall from [16] that in a duo module, every prime submodule is completely prime submodule and a submodule $X$ is a completely prime submodule if and only if it is prime and IFP. These results lead the following lemma.

**Lemma 4.2.**

(1) *Duo modules are* 2*-primal.*

(2) *IFP modules are* 2*-primal.*

(3) *Symmetric modules are* 2*-primal.*

The following result gives a relationship between a 2-primal module and its endomorphisms ring.

**Theorem 4.3.** *Let $M$ be a quasi-projective and finitely generated right $R$-module which is a self-generator. Then $M$ is 2-primal if and only if $S = End_R(M)$ is 2-primal.*

*Proof.* Suppose that $M$ is 2-primal, i.e., $P(M) = C(M)$, where $P(M)$ is the prime radical of $M$ and $\mathcal{C}(M)$ is the completely prime radical of $M$. By applying [19, 18.4], we have $\beta(S) = I_{P(M)}$ and $\mathcal{N}(S) = I_{C(M)}$, where $\beta(S)$ is the prime radical of $S$ and $\mathcal{N}(S)$ is the generalized nil radical (completely prime radical) of $S$. Since $P(M) = C(M)$, we have $\beta(S) = I_{P(M)} = I_{C(M)} = \mathcal{N}(S)$ and consequently $\beta(S) = \mathcal{N}(S)$. Hence, $S$ is 2-primal. For the converse, assume that $S$ is 2-primal, i.e., $\beta(S) = \mathcal{N}(S)$. Applying [19, 18.4] again, we have $I_{P(M)} = \beta(S) = \mathcal{N}(S) = I_{C(M)}$ and consequently $P(M) = C(M)$. Hence, $M$ is 2-primal. □

We recall two propositions appeared in [11].

**Proposition 4.4.** [11, Proposition 2.1] *Let $M$ be a right $R$-module which is a self-generator. Then we have the following:*

(1) *If $X$ is a minimal prime submodule of $M$, then $I_X$ is a minimal prime ideal of $S$.*

(2) *If $P$ is a minimal prime ideal of $S$, then $X := P(M)$ is a minimal prime submodule of $M$ and $I_X = P$.*

**Proposition 4.5.** [11, Proposition 1.8] *If $P$ is a prime submodule of a right $R$-module $M$, then $P$ contains a minimal prime submodule of $M$.*

Using Proposition 4.5, we have the following result.

**Proposition 4.6.** *Let $M$ be a right $R$-module. Then*

$$P(M) = \bigcap\{P : P \text{ is a minimal prime submodule of } M\}.$$

*Proof.* Let $P_m(M) = \bigcap\{P : P \text{ is a minimal prime submodule of } M\}$. Applying Proposition 4.5, it follows that $P_m(M) \subseteq P(M)$. It is easy to see that $P(M) \subseteq P_m(M)$, showing our proof. □

Clearly $P(M) \subseteq C(M)$. Since $P(M) = \bigcap\{P : P$ is a minimal prime submodule of $M\}$ and every minimal prime submodule of $M$ is completely prime, we have $C(M) \subseteq P(M)$. Hence, $M$ is 2-primal. We summarize our discussion in the following proposition.

**Proposition 4.7.** *Let $M$ be a right $R$-module. If every minimal prime submodule of $M$ is completely prime, then $M$ is 2-primal.*

We give a characterization of 2-primal modules as follows.

**Theorem 4.8.** *Let $M$ be a quasi-projective and finitely generated right $R$-module which is a self-generator. Then $M$ is 2-primal if and only if every minimal prime submodule of $M$ is completely prime.*

*Proof.* We only have to show that if $M$ is 2-primal, then every minimal prime submodule of $M$ is completely prime. Let $X$ be a minimal prime submodule of $M$. From Proposition 4.4, $I_X = P$ is a minimal prime ideal of $S$. From Theorem 4.3 and from [20, Proposition 1.11], we conclude that $P$ is completely prime. By applying [16, Proposition 2.13], $X$ is completely prime. This completes the proof. □

**Proposition 4.9.** *Let $M$ be a quasi-projective right $R$-module. The submodule $X$ is completely prime if and only if $X$ is prime and 2-primal.*

*Proof.* Suppose that $X$ is completely prime. Then $X$ is prime and has IFP. Therefore, $X$ is 2-primal. Now assume that $X$ is prime and 2-primal. This implies that $P(M/X) = \mathcal{C}(M/X)$. Since $X$ is prime, $M/X$ is a prime module and $0 = P(M/X) = \mathcal{C}(M/X)$. This implies that $\mathcal{C}(M/X)$ is completely semiprime. Hence, $M/X$ is a completely semiprime module, i.e., $X$ is a completely semiprime submodule of $M$., as required. □

Let $M$ be a right $R$-module and $X$, a submodule of $M$. Denote $\mathfrak{P}(X) = \bigcap\{P$ is a prime submodule of $M$ such that $X \subseteq P\}$. Let $v$: $M \to M/X$. If $x \in \mathfrak{P}(X)$, then $\bar{x} = v(x) = (x + X) \in P(M/X)$.

**Proposition 4.10.** *Let $M$ be a quasi-projective right $R$-module. The submodule $X$ is 2-primal if and only if $\mathfrak{P}(X)$ is completely semiprime.*

*Proof.* Let $X$ be 2-primal, i.e., $P(M/X) = \mathcal{C}(M/X)$. We show $\mathfrak{P}(X)$ is completely semiprime. Let $f \in S =\text{Hom}_R(M)$ and $m \in M$ such that $f^2(m) \in \mathfrak{P}(X)$. Hence, $v(f^2(m)) = f^2(m) + X \in P(M/X)$. Since $M$ is quasi-projective, there exists $\bar{f} \in \bar{S} =\text{Hom}_R(M/X)$ such that $vf = \bar{x}v$. From this, we can see that $\bar{f}^2(\bar{m}) = \bar{f}^2 v(m) = vf^2(m) \in P(M/X)$. By quasi-projectivity, if $\gamma \in S$, then there exists $\bar{\gamma} \in S$ such that $v\gamma = \bar{\gamma}v$. Using the fact that $P(M/X)$ is completely semiprime, we must have $\bar{f}\ \bar{\gamma}\ \bar{m} \in P(M/X)$. Hence, $vf\gamma m = \bar{f}v\gamma m = \bar{f}\ \bar{\gamma}\ vm = \bar{f}\ \bar{\gamma}\ \bar{m} \in P(M/X)$. It follows that $f\gamma m \in \mathfrak{P}(X)$. Since $\gamma \in S$ was arbitrary, we have $fSm \subseteq \mathfrak{P}(X)$. Therefore, $\mathfrak{P}(X)$ is completely semiprime.

For the converse, we will show that if $\mathfrak{P}(X)$ is completely semiprime, then $P(M/X)$ is completely semiprime which will give $\mathcal{C}(M/X) \subseteq P(M/X) \subseteq \mathcal{C}(M/X)$. Let $\bar{f} \in \bar{S}$ and $\bar{m} \in M/X$ such that $\bar{f}^2\bar{m} \in P(M/X)$. As discussion above, since $M$ is quasi-projective, there exists $f \in S$ such that $vf = \bar{f}v$. This shows that $vf^2m = \bar{f}^2vm = \bar{f}^2\bar{m} \in P(M/X)$ and consequently $f^2m \in \mathfrak{P}(X)$. Because $\mathfrak{P}(X)$ is completely semiprime, we have $fSm \subseteq \mathfrak{P}(X)$ . Using quasi-projectivity again, if $\bar{\gamma} \in S$, then there is $\gamma \in S$ such that $v\gamma = \bar{\gamma}v$. From this equation, we can see that $\bar{f}\ \bar{\gamma}\ \bar{m} = \bar{f}\ \bar{\gamma}vm = \bar{f}v\gamma m = vf\gamma m \in P(M/X)$. Hence,
$\bar{f}\ \bar{S}\ \bar{m} \subseteq P(M/X)$ and $P(M/X)$ is completely semiprime. □

**Proposition 4.11.** *Let $M$ be a quasi-projective right $R$-module. Then $M$ is 2-primal if and only if $M/P(M)$ is a completely semiprime module.*

*Proof.* Let $M$ be a 2-primal module, i.e., $P(M) = \mathcal{C}(M)$. Hence, $P(M)$ is a completely semiprime submodule of $M$. Using [17, Lemma 2.2], $M/P(M)$ is a completely semiprime module. For the converse, suppose that $M/P(M)$ is a completely semiprime module. Again, from [17, Lemma 2.2], it follows that $P(M)$ is a completely semiprime submodule of $M$. Therefore, $\mathcal{C}(M) \subseteq P(M) \subseteq \mathcal{C}(M)$ and we have $P(M) = \mathcal{C}(M)$. Thus, $M$ is a 2-primal module. □

**Lemma 4.12.** *Let $M$ be a right $R$-module, and $X$ a prime submodule. If $A$ and $B$ are left ideals of $S$ and $m \in M$ such that $ABm \subseteq P$, then $Am \subseteq P$ or $Bm \subseteq P$.*

*Proof.* Let $A$ and $B$ be left ideals of $S$ and $m \in M$ such that $ABm \subseteq P$. From [11, Theorem 1.2 (4)] and $ASBm \subseteq ABm \subseteq P$, we have either $AM \subseteq P$ or $Bm \subseteq P$. Hence, either $Am \subseteq P$ or $Bm \subseteq P$, completing our proof. □

**Lemma 4.13.** *Let $M$ be a right $R$-module, and $X$ a submodule with IFP. If $f, g \in S$ and $m \in M$ such that $fg(m) \in X$, then $\langle f\rangle\langle g\rangle(m) \subseteq X$.*

*Proof.* Let $f, g \in S$ and $m \in M$ such that $fg(m) \in X$. Since $X$ is a submodule with IFP, as in [16, Theorem 2.6 (2)] we can see that $(g(m) : X) = \{h \in S : hg(m) \in X\}$ is a two-sided ideal of $S$. Now, since $f \in (g(m) : X)$, we have $\langle f\rangle g(m) \subseteq X$. Using the fact that $X$ has IFP, we get $\langle f\rangle gSm \subseteq X$ and $\langle f\rangle SgSm \subseteq X$. Since $\langle f\rangle$ is an ideal of $S$, we can prove that $\langle f\rangle Sg(m) \subseteq X$. Hence, $\langle f\rangle\langle g\rangle(m) \subseteq X$. □

**Definition 4.14.** Let $M$ be a right $R$-module. A submodule $X$ of $M$ is called semi-symmetric if for $f \in S$ and $m \in M$, $f^2(m) \in X$ implies $\langle f\rangle^2(m) \subseteq X$.

As a direct consequence, we have the following corollary.

**Corollary 4.15.** *Let $M$ be a right $R$-module. If $P$ is an IFP submodule of $M$, then $P$ is a semi-symmetric submodule of $M$.*

**Theorem 4.16.** *Let $M$ be a right $R$-module. If $P$ is a semi-symmetric submodule of $M$, then $P$ is a 2-primal submodule of $P$.*

*Proof.* It is enough to show that if $P$ is a prime submodule which is also semi-symmetric, then it is a completely prime submodule of $M$. Suppose $P$ is both prime and semi-symmetric. Let $f \in S = \mathrm{Hom}_R(M)$ and $m \in M$ such that $f(m) \in P$. This implies that $f^2(m) \in P$. Since $P$ is a semi-symmetric submodule, we have $\langle f\rangle^2(m) \subseteq P$. By the primeness of $P$, we can see that $\langle f\rangle(m) \subseteq P$. If $m \in P$, then we are done. Assume that $m \notin P$. Now $\langle f\rangle Sm \subseteq \langle f\rangle(m) \subseteq P$ and from [11, Theorem 1.2] we conclude that $\langle f\rangle M \subseteq P$. Hence, $P$ is completely prime, as required. □

**Proposition 4.17.** *Let $M$ be a right $R$-module. If $X$ is a submodule of $M$ such that $X \subseteq P(M)$, then $M$ is 2-primal if and only if $X$ is 2-primal.*

*Proof.* Since $X \subseteq P(M)$, $X \subseteq P$, where $P$ is any prime submodule of $M$. Hence, $P(M) = \mathfrak{P}(X)$. Assume that $M$ is 2-primal, i.e., $\mathcal{C}(M) = P(M) = \mathfrak{P}(X)$. Thus, $\mathfrak{P}(X)$ is a completely semiprime submodule of $M$. By using Proposition 4.10, $X$ is 2-primal. For the converse, assume now $X$ is 2-primal. It follows from Proposition 4.10 that $\mathfrak{P}(X)$ is a completely semiprime submodule of $M$. Since $P(M) = \mathfrak{P}(X)$, $P(M)$ is a completely semiprime submodule of $M$. Hence, $\mathcal{C}(M) \subseteq P(M) \subseteq \mathcal{C}(M)$ and $M$ is 2-primal. □

# 5 Conclusion

In this paper, we studied some properties of symmetric submodules and modules [Propositions 3.2, 3.3, 3.11; Lemmas 3.7, 3.8, 3.10]. The relationship between a symmetric module and its endomorphism ring is provided [Theorem 3.6]. We proved that if $M$ is a quasi-projective and finitely generated right $R$-module which is a self-generator, then $M$ is 2-primal if and only if $S = End_R(M)$ is 2-primal [Theorem 4.3]. We also gave some characterizations of 2-primal modules [Theorem 4.8, Propositions 4.9, 4.10, 4.11].

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