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Euler-Taylor Matrix Method for Solving Linear Volterra-Fredholm Integro - Differential Equations with Variable Coefficients

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Abstract : In this paper, we present a numerical method for solving the high-order linear Volterra-Fredholm integro - differential equations with constant arguments and variable coefficients.The proposed method is based on the Euler polynomials and collocation points which transforms the integro - differential equation into a matrix equation. The matrix equation corresponds to a system of algebraic equations for which the unknown are Euler coefficients. Some examples are provided to illustrate the validity of the method.

Keywords : integro - differential equations; Euler polynomials.
2010 Mathematics Subject Classification : 34K28; 45A99.

1 Introduction

The integro-differential equations and their solutions play a major role in science, economics and engineering [1–5]. The integro - differential equations with constant arguments and variable coefficients are usually difficult to solve analytically, so a numerical method is needed. There are various techniques for solving Fredholm integro - differential equations, e.g. Laguerre collocation method (LCM) [1], Fibonacci collocation method (FCM) [3], Taylor matrix method [6], Chebyshev collocation method [7], Legendre spectral collocation method [8] and hybrid Euler-Taylor matrix method [9]. In addition, Adomain decomposition method [10], Taylor-series expansion method [11], Romberg extrapolation on quadrature method [12] and the Legendre spectral collocation method [13] were applied to solving Volterra integral equations. Our goal is to develop a hybrid Euler-Taylor matrix method [9] for solving the generalized high-order linear Volterra-Fredholm integro - differential equations with constant arguments and variable coefficients in the form

$$\begin{aligned}\sum_{k=0}^{m_1}\sum_{j=0}^{n_1} p_{kj}(x)y^{(k)}(x+\tau_{kj}) &= f(x)+\sum_{r_1=0}^{m_2}\sum_{s_1=0}^{n_2}\int_a^b K_{r_1s_1}(x,t)y^{(r_1)}(t+\lambda_{r_1s_1})dt\\ &+ \sum_{r_2=0}^{m_3}\sum_{s_2=0}^{n_3}\int_a^x \hat{K}_{r_2s_2}(x,t)y^{(r_2)}(t+\gamma_{r_2s_2})dt, \qquad (1.1)\end{aligned}$$

with mixed conditions

$$\sum_{i=0}^{m_1-1}\left[\alpha_{il}y^{(i)}(a)+\beta_{il}y^{(i)}(b)+\gamma_{il}y^{(i)}(c)\right]=\mu_i, \qquad (1.2)$$

$$l=0,1,\dots,m_1-1, \quad m_1\geq m_2,m_3,$$

where $y(x)$ is an unknown function to be determined. Also, $p_{kj}(x), K_{r_1s_1}(x,t)$, $K_{r_2s_2}(x,t)$ and $f(x)$ are continuous functions defined on the interval $0\leq x,t\leq b<\infty$; α_{il},β_{il} and γ_{il} are appropriated constants. Moreover, the functions $K_{r_1s_1}(x,t)$ and $\hat{K}_{r_2s_2}(x,t)$ can be represented by Maclaurin series.

This paper is organized as follows: In Section 2, we present the basic concepts of Euler polynomials and their properties. The method of constructing the Euler matrix for solving the linear Volterra-Fredholm integro - differential equations with constant arguments and variable coefficients is described in Section 3. The numerical illustrations are provided in Section 4.

2 Preliminaries

Euler polynomials are very useful. They are classical tools for numerical methods and have a lot of applications in many fields in mathematics. Their basic properties can be summarized as follows [14–19].

Euler polynomials $E_n(x)$ can be defined in terms of exponential generating function as follows

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{n}E_n(x)\frac{t^n}{n!}, \quad |t|<\pi.$$

The recursive formula of Euler polynomials are constructed by using the following relation

$$E_n(x)+\sum_{k=0}^{n}\binom{n}{k}E_k(x)=2x^n, \quad n=0,1,2,\dots \qquad (2.1)$$

Also, Euler polynomials $E_n(x)$ can be defined as polynomials of degree $n \geq 0$ satisfying the following properties

$$\begin{aligned} E_n(x+1) + E_n(x) &= 2x^n \quad n = 1, 2, \ldots, && (2.2) \\ E_n'(x) &= nE_{n-1}(x), \quad n = 1, 2, \ldots. && (2.3) \end{aligned}$$

If we take $n = 0$, the first Euler polynomial is $E_0(x) = 1$. By using (2.1) or (2.2) the next four Euler polynomials are given below,

$$E_1(x) = 1, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \quad E_4(x) = x^4 - 2x^3 + x.$$

3 Main Results

In this section, we consider the equation (1.1) and find the corresponding matrix forms of each term in the corresponding equation. Let $y_N(x)$ be an approximate solution of (1.1) expressed in the truncated Euler series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n E_n(x), \quad 0 \leq a \leq x \leq b,$$

where a_n; $n = 0, 1, \ldots N$ are the Euler coefficient unknowns, and $E_n(x)$; $n = 0, 1, \ldots N$ are Euler polynomials. We can rewrite the approximate solution $y_N(x)$ to the matrix equation

$$y_N(x) = \mathbf{E}(x)\mathbf{A}, \tag{3.1}$$

where

$$\mathbf{E}(x) = [E_0(x) \ E_1(x) \ \ldots \ E_N(x)],$$

and

$$\mathbf{A} = [a_0 \ a_1 \ \ldots \ a_N]^T.$$

From the second property (2.3) and equation (3.1), the relation between the matrix $\mathbf{E}(x)$ and its derivative $\mathbf{E}'(x)$ is

$$\mathbf{E}'(x) = \mathbf{E}(x)\mathbf{M},$$

and then, it follows that

$$\begin{aligned} \mathbf{E}''(x) &= \mathbf{E}'(x)\mathbf{M} = \mathbf{E}(x)\mathbf{M}^2 \\ \mathbf{E}'''(x) &= \mathbf{E}''(x)\mathbf{M} = \mathbf{E}(x)\mathbf{M}^3 \\ &\vdots \\ \mathbf{E}^{(k)}(x) &= \mathbf{E}^{(k-1)}(x)\mathbf{M} = \mathbf{E}(x)\mathbf{M}^k, \quad k = 0, 1, 2, \ldots, && (3.2) \end{aligned}$$

where

$$\mathbf{M}^0 \quad \text{is identity matrix and} \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Furthermore, if in equation (2.1) n varies from 0 to N, we obtain the linear matrix equation as follows

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \dots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \dots & 0 \\ & & \vdots & & \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \dots & 1 \end{bmatrix} \begin{bmatrix} E_0(x) \\ E_1(x) \\ E_2(x) \\ \vdots \\ E_N(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{bmatrix}$$

or briefly

$$\mathbf{T}\mathbf{E}^T(x) = \mathbf{X}^T \Rightarrow \mathbf{E}(x)\mathbf{T}^T = \mathbf{X}(x),$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \dots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \dots & 0 \\ & & \vdots & & \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \dots & 1 \end{bmatrix}$$

and the standard basis matrix

$$\mathbf{X}(x) = [1 \ x \ x^2 \ \dots \ x^N].$$

Since $\mathbf{T}^T$ is a upper triangular matrix with nonzero diagonal elements, it is nonsingular and hence $(\mathbf{T}^T)^{-1}$ exists. Thus, the Euler matrix can be given directly from

$$\mathbf{E}(x) = \mathbf{X}(x)(\mathbf{T}^T)^{-1}. \tag{3.3}$$

By putting $x \to x + \tau_{kj}$ in the equation (3.3), we have

$$\mathbf{E}(x+\tau_{kj}) = \mathbf{X}(x)\mathbf{D}(\tau_{kj})(\mathbf{T}^T)^{-1}, \tag{3.4}$$

where

$\mathbf{D}(0)$ is identity matrix and

$$\mathbf{D}(\tau_{kj}) = \begin{bmatrix} \binom{0}{0}(\tau_{kj})^0 & \binom{1}{0}(\tau_{kj})^1 & \dots & \binom{N}{0}(\tau_{kj})^N \\ 0 & \binom{1}{1}(\tau_{kj})^0 & \dots & \binom{N}{1}(\tau_{kj})^{N-1} \\ & & \vdots & \\ 0 & 0 & \dots & \binom{N}{N}(\tau_{kj})^0 \end{bmatrix}.$$

From the matrix relations (3.1), (3.2), (3.3) and (3.4), we obtain

$$y_N^{(k)}(x) \cong \mathbf{X}(x)(\mathbf{T}^T)^{-1}\mathbf{M}^k\mathbf{A}, \quad k = 0, 1, 2, \dots \tag{3.5}$$

and

$$y^{(k)}(x + \tau_{kj}) \cong \mathbf{X}(x)\mathbf{D}(\tau_{kj})(\mathbf{T}^T)^{-1}\mathbf{M}^k\mathbf{A}, \quad k = 0, 1, 2, \dots \tag{3.6}$$

Now we transform the kernel function $K_{r_1 s_1}(x,t)$ and $\hat{K}_{r_2 s_2}(x,t)$ to the matrix form by using Maclaurin's expansion as

$$K_{r_1 s_1}(x,t) = \mathbf{X}(x)\mathbf{K}_{r_1 s_1}\mathbf{X}^T(t) \quad \text{and} \quad \hat{K}_{r_2 s_2}(x,t) = \mathbf{X}(x)\hat{\mathbf{K}}_{r_2 s_2}\mathbf{X}^T(t), \tag{3.7}$$

where

$$\mathbf{K}_{r_1 s_1} = [k_{r_1 s_1}^{mn}]; \quad k_{r_1 s_1}^{mn} = \frac{1}{m!n!}\frac{\partial^{m+n}}{\partial x^m \partial t^n} K_{r_1 s_1}(0,0) \quad m, n = 0, 1, 2, \dots, N$$

and

$$\hat{\mathbf{K}}_{r_2 s_2} = [k_{r_2 s_2}^{mn}]; \quad k_{r_2 s_2}^{mn} = \frac{1}{m!n!}\frac{\partial^{m+n}}{\partial x^m \partial t^n} \hat{K}_{r_2 s_2}(0,0) \quad m, n = 0, 1, 2, \dots, N.$$

Thus, the matrix representation for the integral parts are obtained by

$$\int_a^b K_{r_1 s_1}(x,t) y^{(r_1)}(t + \lambda_{r_1 s_1}) dt \simeq \mathbf{X}(x)\mathbf{K}_{r_1 s_1}\mathbf{G}_{r_1 s_1}\mathbf{D}(\lambda_{r_1 s_1})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_1}\mathbf{A}, \tag{3.8}$$

where

$$\begin{aligned} \mathbf{G}_{r_1 s_1} &= [g_{r_1 s_1}^{mn}] = \int_a^b \mathbf{X}^T(t)\mathbf{X}(t) dt; \\ g_{r_1 s_1}^{mn} &= \frac{b^{m+n+1} - a^{m+n+1}}{m+n+1}, \quad m, n = 0, 1, \dots, N. \end{aligned}$$

Similarly, we get the matrix form for $\int_a^x \hat{K}_{r_2 s_2}(x,t) y^{(r_1)}(t + \gamma_{r_2 s_2}) dt$ as follows

$$\int_a^x \hat{K}_{r_2 s_2}(x,t) y^{(r_1)}(t + \gamma_{r_2 s_2}) dt \simeq \mathbf{W}_{r_2 s_2}(x)\mathbf{D}(\gamma_{r_2 s_2})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_2}\mathbf{A}, \tag{3.9}$$

where

$$\mathbf{W}_{r_2 s_2}(x) = \mathbf{X}(x)\hat{\mathbf{K}}_{r_2 s_2}\mathbf{H}_{r_2 s_2}(x),$$

and

$$\begin{aligned}\mathbf{H}_{r_2s_2}(x) &= [h^{mn}_{r_2s_2}(x)] = \int_a^x \mathbf{X}^T(t)\mathbf{X}(t)dt;\\ h^{mn}_{r_2s_2}(x) &= \frac{x^{m+n+1}-a^{m+n+1}}{m+n+1}, \quad m,n=0,1,\ldots,N.\end{aligned}$$

Substituting the matrix forms (3.5) - (3.9) into equation (1.1), we obtain

$$\begin{pmatrix} \sum_{k=0}^{m_1}\sum_{j=0}^{n_1} p_{kj}(x)\mathbf{X}(x)\mathbf{D}(\tau_{kj})(\mathbf{T}^T)^{-1}\mathbf{M}^k \\ -\sum_{r_1=0}^{m_2}\sum_{s_1=0}^{n_2}\mathbf{X}(x)\mathbf{K}_{r_1s_1}\mathbf{G}_{r_1s_1}\mathbf{D}(\lambda_{r_1s_1})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_1} \\ -\sum_{r_2=0}^{m_3}\sum_{s_2=0}^{n_3}\mathbf{W}_{r_2s_2}(x)\mathbf{D}(\gamma_{r_2s_2})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_2}\end{pmatrix}\mathbf{A} = f(x). \quad (3.10)$$

In order to use equation (3.10) the collocation points are defined by

$$x_i = a + \frac{b-a}{N}i, \quad i=0,1,2,\ldots,N.$$

We have the system of the matrix equations

$$\begin{pmatrix} \sum_{k=0}^{m_1}\sum_{j=0}^{n_1} p_{kj}(x_i)\mathbf{X}(x_i)\mathbf{D}(\tau_{kj})(\mathbf{T}^T)^{-1}\mathbf{M}^k \\ -\sum_{r_1=0}^{m_2}\sum_{s_1=0}^{n_2}\mathbf{X}(x_i)\mathbf{K}_{r_1s_1}\mathbf{G}_{r_1s_1}\mathbf{D}(\lambda_{r_1s_1})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_1} \\ -\sum_{r_2=0}^{m_3}\sum_{s_2=0}^{n_3}\mathbf{W}_{r_2s_2}(x_i)\mathbf{D}(\gamma_{r_2s_2})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_2}\end{pmatrix}\mathbf{A} = f(x_i).$$

Therefore, the fundamental matrix equation are obtained as

$$\begin{pmatrix} \sum_{k=0}^{m_1}\sum_{j=0}^{n_1} \mathbf{P}_{kj}\mathbf{X}\mathbf{D}(\tau_{kj})(\mathbf{T}^T)^{-1}\mathbf{M}^k \\ -\sum_{r_1=0}^{m_2}\sum_{s_1=0}^{n_2}\mathbf{X}\mathbf{K}_{r_1s_1}\mathbf{G}_{r_1s_1}\mathbf{D}(\lambda_{r_1s_1})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_1} \\ -\sum_{r_2=0}^{m_3}\sum_{s_2=0}^{n_3}\mathbf{W}_{r_2s_2}\mathbf{D}(\gamma_{r_2s_2})(\mathbf{T}^T)^{-1}\mathbf{M}^{r_2}\end{pmatrix}\mathbf{A} = \mathbf{F}, \quad (3.11)$$

where

$$\mathbf{P}_{kj} = diag[p_{kj}(x_0)\ p_{kj}(x_1)\ \ldots\ p_{kj}(x_N)],$$

$$\mathbf{X} = \begin{bmatrix}\mathbf{X}(x_0)\\ \mathbf{X}(x_1)\\ \mathbf{X}(x_2)\\ \vdots \\ \mathbf{X}(x_N)\end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^N \\ 1 & x_1 & x_1^2 & \ldots & x_1^N \\ & & \vdots & & \\ 1 & x_N & x_N^2 & \ldots & x_N^N\end{bmatrix},$$

$$\mathbf{W}_{r_2s_2} = \begin{bmatrix}\mathbf{W}_{r_2s_2}(x_0)\\ \mathbf{W}_{r_2s_2}(x_1)\\ \mathbf{W}_{r_2s_2}(x_2)\\ \vdots \\ \mathbf{W}_{r_2s_2}(x_N)\end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} f(x_0)\\ f(x_1)\\ f(x_2)\\ \vdots \\ f(x_N)\end{bmatrix}.$$

The matrix equation (3.11) corresponds to a system of $(N+1)$ algebraic equations for (N+1) unknown Euler coefficients $a_0, a_1, \ldots, a_N$. Briefly, one can write (3.11) in form

$$\mathbf{U}\mathbf{A} = \mathbf{F} \quad \text{or} \quad [\mathbf{U};\mathbf{F}], \quad (3.12)$$

where

$$\mathbf{U} = [u_{mn}], \qquad m, n = 0, 1, \ldots, N.$$

On the other hand, we can obtain the matrix forms for the mixed condition (1.2) as

$$\sum_{i=0}^{m_1-1} [\alpha_{il}\mathbf{X}(a) + \beta_{il}\mathbf{X}(b) + \gamma_{il}\mathbf{X}(c)](\mathbf{T}^T)^{-1}\mathbf{M}^i\mathbf{A} = [\mu_i]; \quad i = 0, 1, \ldots, m_1 - 1,$$

or briefly

$$\mathbf{V}_i\mathbf{A} = [\mu_i] \quad \text{or} \quad [\mathbf{V}_i; \mu_i], \tag{3.13}$$

where

$$\mathbf{V}_i = [v_{i0}\ v_{i1}\ \ldots\ v_{iN}], \quad i = 0, 1, \ldots, m_1 - 1.$$

Consequently, to obtain the solution of equation (1.1) with conditions (1.2), by replacing the m rows of matrix (3.12) by the last row matrices (3.13), we have

$$\tilde{\mathbf{U}}\mathbf{A} = \tilde{\mathbf{F}} \quad \text{or} \quad [\tilde{\mathbf{U}}; \tilde{\mathbf{F}}].$$

If the $rank\tilde{\mathbf{U}} = rank[\tilde{\mathbf{U}}; \tilde{\mathbf{F}}] = N + 1$ then the unknown Euler coefficients matrix $\mathbf{A}$ is uniquely determined and $\mathbf{A} = \tilde{\mathbf{U}}^{-1}\tilde{\mathbf{F}}$. Therefore, the system (1.1) with conditions (1.2) has a unique solution. However, when $|\tilde{\mathbf{U}}| = 0$, if the $rank\tilde{\mathbf{U}} = rank[\tilde{\mathbf{U}} : \tilde{\mathbf{F}}]$, then we may find a particular solution. Otherwise if the $rank\tilde{\mathbf{U}} \neq rank[\tilde{\mathbf{U}} : \tilde{\mathbf{G}}] < N + 1$, then it is not a solution.

The approximate solution $y_N(x)$ are obtained by proposed method has been compared with that of obtained by other method on the basis of L_∞ error. It can be defined as

$$L_\infty = \max_{a \le x \le b} |y_N(x) - y_{\text{exact}}(x)|.$$

This comparison has been discussed in Section 4.

4 Illustrative Examples

Example 4.1.

Let us consider the integro - differential equation with variable coefficients [5]

$$y'(x) - y(x) + xy'(x-1) + y(x-1) = (x-2) + \int_{-1}^{1} (x+t)y(t-1)dt$$

with mixed condition

$$y(-1) - 2y(0) + y(1) = 0.$$

Then, for $N=2$, the collocation points are $x_0=-1,\ x_1=0,\ x_2=1$.
The fundamental matrix equation for this problem is defined by

$$\Big(\mathbf{P}_{00}\mathbf{X}(\mathbf{T}^T)^{-1}+\mathbf{X}(\mathbf{T}^T)^{-1}\mathbf{M} + \mathbf{X}\mathbf{D}(-1)(\mathbf{T}^T)^{-1} + \mathbf{P}_{11}\mathbf{X}\mathbf{D}(-1)(\mathbf{T}^T)^{-1}\mathbf{M} - \mathbf{X}\mathbf{K}_{00}\mathbf{G}_{00}\mathbf{D}(-1)(\mathbf{T}^T)^{-1}\Big)\mathbf{A}=\mathbf{F}$$

where

$$\mathbf{P}_{00}=\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix},\quad \mathbf{P}_{11}=\begin{bmatrix} -1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{X}=\begin{bmatrix} 1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\end{bmatrix},\quad \mathbf{M}=\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix},$$

$$(\mathbf{T}^T)^{-1}=\begin{bmatrix} 1 & -\frac{1}{2} & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix},\quad \mathbf{D}(-1)=\begin{bmatrix} 1 & -1 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{K}_{00}=\begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{G}_{00}=\begin{bmatrix} 2 & 0 & \frac{2}{3}\\ 0 & \frac{2}{3} & 0\\ \frac{2}{3} & 0 & \frac{2}{5}\end{bmatrix},$$

$$\text{and}\quad \mathbf{F}=\begin{bmatrix} -3\\ -2\\ -1\end{bmatrix}.$$

From (3.12), The augmented matrix is obtained as

$$[\mathbf{U};\mathbf{F}]=\begin{bmatrix} 2 & -\frac{14}{3} & \frac{38}{3} & ; & -3\\ 0 & -\frac{2}{3} & 3 & ; & -2\\ -2 & \frac{10}{3} & -\frac{8}{3} & ; & -1\end{bmatrix}.$$

Hence, the augmented matrix based on the condition
$y(-1)-2y(0)+y(1)=0$ is

$$[\tilde{\mathbf{U}}:\tilde{\mathbf{F}}]=\begin{bmatrix} 2 & -\frac{14}{3} & \frac{38}{3} & ; & -3\\ 0 & -\frac{2}{3} & 3 & ; & -2\\ 0 & 0 & 2 & ; & 0\end{bmatrix}.$$

By solving this system, the unknown Euler coefficients matrix is obtained as

$$\mathbf{A}=\begin{bmatrix} \frac{11}{2} & 3 & 0\end{bmatrix}^T.$$

Therefore, the solutions of this problem becomes

$$y_2(x)=\begin{bmatrix} 1 & x-\frac{1}{2} & x^2-x\end{bmatrix}\begin{bmatrix} \frac{11}{2} & 3 & 0\end{bmatrix}^T=3x+4,$$

which is the exact solution of this problem.

Example 4.2.

Let us consider the integro - differential equation with variable coefficients [3], [8]

$$\begin{aligned} y'''(x) - xy'(x-1) + y''(x-1) - xy(x-1) &= -(x+1)(\sin(x-1) + \cos x) \\ &\quad - \cos 2 + 1 + \int_{-1}^{1} y(t-1)dt \end{aligned}$$

with conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.$$

The exact solution of this problem is $y = \sin x$. Here $p_{00}(x) = p_{10}(x) = -x$, $K_{00}(x,t) = 1$ and $f(x) = -(x+1)(\sin(x-1) + \cos x) - \cos 2 + 1$. From equation (3.12), the fundamental matrix equation for this problem is

$$\begin{aligned} \Big(\mathbf{P}_{00}\mathbf{X}\mathbf{D}(-1)(\mathbf{T}^T)^{-1} + \mathbf{P}_{10}\mathbf{X}(\mathbf{T}^T)^{-1}\mathbf{M} &+ \mathbf{X}\mathbf{D}(-1)(\mathbf{T}^T)^{-1}\mathbf{M}^2 + \mathbf{X}(\mathbf{T}^T)^{-1}\mathbf{M}^3 \\ &- \mathbf{X}\mathbf{G}_{00}\mathbf{D}(-1)(\mathbf{T}^T)^{-1}\Big)\mathbf{A} = \mathbf{F}. \end{aligned}$$

Thus, we obtain the approximate solution by the Euler polynomials of the problem for $N = 4, 6, 8$ respectively,

$$\begin{aligned} y_4(x) &= -0.03059x^4 - 0.18882x^3 + x \\ y_6(x) &= 0.00258x^6 + 0.01097x^5 - 0.01298x^4 - 0.149368x^3 + x \\ y_8(x) &= -(3.17317e-6)x^8 - (1.65243e-4)x^7 - (1.31593e-4)x^5 \\ &\quad +0.00826x^5 + 0.00187x^4 - 0.16898x^3 + x. \end{aligned}$$

Table 1 shows numerical results of the exact solutions and approximate solutions for Example 4.2 for $N = 4, 6, 8$ by presented method. From Table 1, the results of the solutions obtained by present method for $N = 8$ are more accurate with the same number of the collocation points. Table 2 shows the L_∞ errors of the present method, Fibonacci collocation method (FCM) [3], and Legendre polynomials [8]. As can be seen from Table 2, the presented method is more accurate than the method given in [3] and [8].

Table 1: Numerical results along with exact results for Example 4.2.

x	Exact	Present method $N=4$	$N=6$	$N=8$
-1	-0.841471	-0.841800	-0.872002	-0.837377
-0.8	-0.717356	-0.715868	-0.731758	-0.715423
-0.6	-0.564642	-0.563185	-0.570151	-0.563902
-0.4	-0.389418	-0.388700	-0.390874	-0.389222
-0.2	-0.198669	-0.198538	-0.198829	-0.198648
0	0	0	0	0
0.2	0.198669	0.198441	0.198788	0.198654
0.4	0.389418	0.387133	0.390231	0.389317
0.6	0.564642	0.555253	0.567028	0.564375
0.8	0.717356	0.690801	0.722478	0.716887
1	0.841471	0.780600	0.851202	0.840855

Table 2: L_∞ error for Example 4.2.

Error	Present method		
	$N=4$	$N=6$	$N=8$
L_∞	$6.088441e-2$	$3.053102e-2$	$4.094052e-3$

Error	FCM [3]		Legendre	polynomials [8]
	$N=8$	$N=9$	$m=6$	$m=7$
L_∞	$3.937393e-1$	$2.23705e-0$	$3.84e-2$	$5.05e-3$

Example 4.3.

As the next example, consider the following second-order pantograph Volterra integro-differential equation of the neutral type

$$y''(x)-(x+1)y'(x)+y(x)=\int_{-1}^{x}(xy(t)+y'(t)+ty''(t))dt+(x+1)(\sin x-\sin 1),$$

and the boundary conditions

$$y(-1)=\cos 1, \quad y(1)=\cos 1,$$

correspond to the exact solution $y(x)=\cos x$. Here $p_{10}(x)=-(x+1)$, $\hat{K}_{00}(x,t)=x$, $\hat{K}_{10}=1$, $\hat{K}_{20}(x,t)=t$ and $f(x)=(x+1)(\sin x-\sin 1)$. From equation (3.12), the fundamental matrix equation for this problem is

$$\begin{aligned}\Big(\mathbf{X}(\mathbf{T}^T)^{-1}+\mathbf{P}_{10}\mathbf{X}(\mathbf{T}^T)^{-1}\mathbf{M}+\mathbf{X}(\mathbf{T}^T)^{-1}\mathbf{M}^2 &- \mathbf{W}_{00}(\mathbf{T}^T)^{-1}-\mathbf{W}_{10}(\mathbf{T}^T)^{-1}\mathbf{M}\\ &- \mathbf{W}_{20}(\mathbf{T}^T)^{-1}\mathbf{M}^2\Big)\mathbf{A}=\mathbf{F}.\end{aligned}$$

Thereby, taking $N=4$ and $N=9$ respectively, we have the approximated solution by the Euler polynomials of this problem

$$\begin{aligned} y_8(x) &= -0.07973x^8 - 0.10876x^7 - 0.04048x^6 - 0.07949x^5 - 0.12887x^4 \\ &\quad -0.30957x^3 - 0.66488x^2 + 0.49782x + 1.45427 \\ y_9(x) &= 0.05614x^9 + 0.02674x^8 - 0.10715x^7 - 0.12512x^6 - 0.10855x^5 \\ &\quad -0.11913x^4 - 0.31186x^3 - 0.67749x^2 + 0.47142x + 1.43530. \end{aligned}$$

Table 3 shows the numerical results of the exact solution and the approximate solutions for $N = 8, 9$ by the present method. The L_∞ errors are shown in Table 4.

Table 3: Numerical results along with exact results for Example 4.3.

		Present method	
x	Exact	$N=8$	$N=9$
-1	0.54030	0.54031	0.54030
-0.8	0.69671	0.76107	0.75764
-0.6	0.82534	0.97238	0.96596
-0.4	0.92106	1.16605	1.15602
-0.2	0.98007	1.33040	1.31625
0	1.00000	1.45427	1.43530
0.2	0.98007	1.52452	1.49975
0.4	0.92106	1.52270	1.49069
0.6	0.82534	1.41758	1.37519
0.8	0.69671	1.14287	1.09156
1	0.54030	0.54031	0.54030

Table 4: L_∞ error for Example 4.3.

Error	Present method	
	$N=8$	$N=9$
L_∞	0.60163	0.56963

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