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# Levitin-Polyak Well-Posedness for Strong Vector Mixed Quasivariational Inequality Problems

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**Abstract :** In this paper, we first introduce the concepts of Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense for strong vector mixed quasivariational inequality problems of the Minty type and the Stampacchia type (for short, (MQVI) and (SQVI), respectively). Sufficient conditions for such problems to be Levitin-Polyak well-posedness are established. We also introduce the gap functions for (MQVI) and (SQVI) and study some properties which are used to study the Levitin-Polyak well-posedness for such problems.

**Keywords :** Levitin-Polyak well-posedness; strong vector mixed quasivariational inequality problems; Minty type; Stampacchia type.
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## 1 Introduction

Well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating

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solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [1], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [2] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences. Subsequently, some authors studied the Levitin-Polyak well-posedness for convex optimization problems with functional constraints (Konsulova and Revalski [3]), general constrained nonconvex optimization problems (Huang and Yang [4]), general constrained vector optimization problems (Huang and Yang [5]) and generalized variational inequality problems with functional constraints (Huang and Yang [6]).

The notion of well-posedness for variational inequality problems was introduced by Lucchetti and Patrone [7] based on the fact that an optimization problem of minimizing a function can be formulated as a variational inequality problem involving the derivative of the objective. After that several researchers [8–13] have explored the forms of well-posedness for various forms of variational inequality problems. Lignola [8] considered two concepts of well-posednesses for quasivariational inequalities having a unique solution. Some equivalent characterizations of these concepts and classes of well- posed quasivariational inequalities are presented. In 2010, Zhong and Huang [14] studied the stability analysis for a class of Minty mixed variational inequalities in reflexive Banach spaces, when both the mapping and the constraint set are perturbed. Fang, Huang and Yao [15] considered an extension of the notion of well-posedness by perturbations for a minimization problem, to a mixed variational inequality problem in a Banach space. Recently, Li and Xia [16] introduced the concept of Levitin-Polyak well-posedness for the generalized mixed variational inequality in Banach spaces and established some characterizations of its Levitin-Polyak well-posedness.

However, to the best of our knowledge, there is no a result concerning the Levitin-Polyak well-posedness for both (MQVI) and (SQVI), which include as a special case the classical the generalized mixed variational inequality in [16]. Motivated by the work reported in [8,14–16], we introduce the concepts of Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense of (MQVI) and (SQVI), respectively. Sufficient conditions for such problems to be Levitin-Polyak well-posedness are established. We also introduce gap functions for (MQVI) and (SQVI) and study their properties which are used to study the Levitin-Polyak well-posedness for such problems.

## 2 Preliminaries

Let $X, Y$ be normed spaces and $C \subset Y$ be a closed, convex, pointed, and solid cone. Let $L(X, Y)$ be the space of all linear continuous operators from $X$ into $Y$, and $A \subset X$ be a nonempty closed subset. Let $K : A \rightrightarrows A$ and $T : A \rightrightarrows L(X, Y)$ be set-valued mappings. Let $H : L(X, Y) \to L(X, Y), f : A \times A \to Y, \eta : A \times A \to A, \psi : A \to A$ be continuous single-valued mappings with $\eta(x, \psi(x)) = 0$ and

$f(x, \psi(x)) = 0$ for every $x \in A$. $\langle z, x\rangle$ denotes the value of a linear operator $z \in L(X;Y)$ at $x \in X$.

We consider the following two strong vector mixed quasivariational inequality problems of the Minty type and the Stampacchia type, respectively.
(MQVI) Find $x \in K(x)$ such that

$$\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) \in C, \forall y \in K(x), \forall z \in T(y).$$

(SQVI) Find $x \in K(x)$ such that, there exists $z \in T(x)$ and

$$\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) \in C, \forall y \in K(x).$$

The solution sets of (MQVI) and (SQVI) are denoted by $S_M$ and $S_S$, respectively. In this paper we only focus on Levitin-Polyak well-posedness for (MQVI) and (SQVI). We always assume that all kinds of solution sets of these problems are nonempty. To provide our motivations for these settings, we discuss some special cases of the problems.

(a) If $X$ is a real reflexive Banach space, $X^*$ is its dual space, and $H(z) = \{z\}, K(x) = K, \eta(y, \psi(x)) = y - x, f(y, \psi(x)) = f(y) - f(x), C = \mathbb{R}_+$, then (MQVI) reduces to the Minty mixed variational inequality studied in Zhong and Huang [14].

(b) If $H(z) = \{z\}, K(x) = K(x), \eta(y, \psi(x)) = y - x, f(y, \psi(x)) = 0, C = \mathbb{R}_+$, and $T$ is an operator from $X$ into the collection of all continuous linear $X$ into $Y$ then the problem (SQVI) reduces to the problem (QVI) studied in [8].

(c) If $X = \mathbb{R}^n, Y = \mathbb{R}^m, C = \mathbb{R}_+, K(x, \gamma) = A, T(x, \gamma) = T(x)$, $f(x, z, y, \gamma) = \langle z, y - x\rangle$, then the problem (SQVI) reduces to the Stampacchia variational Inequalities (in short, (VI(T,A))) studied in [17].

Firstly, we recall some basic definitions and some of their properties.
Let $X$ and $Z$ be two metric spaces and $G : X \rightrightarrows 2^Z$ be a multifunction.

(i) $G$ is said to be *lower semicontinuous (lsc) at* $x_0$ if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Z$ implies the existence of a neighborhood $V$ of $x_0$ such that $G(x) \cap U \neq \emptyset$ for all $xx \in V$.

(ii) $G$ is said to be *upper semicontinuous (usc) at* $x_0$ if for each open set $U \supseteq G(x_0)$, there is a neighborhood $V$ of $x_0$ such that $U \supseteq G(x)$ for all $x \in V$.

(iii) $G$ is said to be *closed at* $x_0$ if for each sequence $\{(x_n, y_n)\} \subseteq \text{graph}G := \{(x, y)|y \in G(x)\}, (x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \text{graph}G$.

**Lemma 2.1.** (See e.g., [18, 19])

(i) *If $Z$ is compact and $G$ is closed at $x_0$, then $G$ is usc at $x_0$.*

(ii) *If $G$ is usc at $x_0$ and $G(x_0)$ is closed, then $G$ is closed at $x_0$.*

(iii) *If $G(x_0)$ is compact then $G$ is usc at $x_0$ if and only if, for any sequence $\{x_n\}$ converging to $x_0$ and for any sequence $\{y_n\} \subseteq G(x_n)$, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to some $y_0 \in G(x_0)$.*

**Lemma 2.2.** (See e.g., [20–22]) *For any fixed each $e \in \mathrm{intC}, y \in Y$ and the nonlinear scalarization function $\xi_e : Y \to R$ defined by $\xi_e(y) := \min\{r \in R : y \in re - C\}$:*

(i) *$\xi_e$ is a continuous and convex function in $Y$;*

(ii) *$\xi_e(y) \leq r \Leftrightarrow y \in re - C$;*

(iii) *$\xi_e(y) > r \Leftrightarrow y \notin re - C$.*

# 3 Levitin-Polyak Well-Posedness of (MQVI) and (SQVI)

In this section we introduce the concepts of Levitin-Polyak well-posedness and Levitin-Polyak well-posedness in the generalized sense for (MQVI) and (SQVI). Sufficient conditions for such problems to be Levitin-Polyak well-posedness are established. Since the study of the existence conditions for the class of these problems has been intensively studied, we always assume that all kinds of solution sets of these problems are nonempty.

**Definition 3.1.** A sequence $\{x_n\} \subset A$ is said to be a Levitin-Polyak approximating sequence for (MQVI)/or (SQVI), if there exists a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ converging to 0 such that $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), z \in T(y), \forall n \in \mathbb{N},$$

or

$$\exists z \in T(x_n) : \langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall n \in \mathbb{N},$$

respectively.

**Definition 3.2.** The problem (MQVI)/or (SQVI) is said to be Levitin-Polyak well-posed, if

(i) the problem (MQVI)/or (SQVI) has a unique solution;

(ii) for every Levitin-Polyak approximating sequence $\{x_n\}$ for (MQVI)/or (SQVI) converging to the unique solution of (MQVI)/or (SQVI), respectively.

**Definition 3.3.** The problem (MQVI)/or (SQVI) is said to be Levitin-Polyak well-posed in the generalized sense, if

(i) the problem (MQVI)/or (SQVI) has solutions;

(ii) for every Levitin-Polyak approximating sequence $\{x_n\}$ for (MQVI)/or (SQVI) must have a subsequence converging to an element in $S_M$/or $S_S$, respectively.

**Remark 3.4.** *When $X$ is a real reflexive Banach space with its dual $X^*$, $K(x) \equiv K, H(z) = \{z\}, \eta(y, \psi(x)) = y - x, f(y, \psi(x)) = \phi(y) - \phi(x)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ and $e = 1$. Definitions* 3.2 *and* 3.3 *reduce to the Definitions* 3.2 *and* 3.3 *of* [16]*, respectively.*

For $\varepsilon > 0$, we denote the $\varepsilon$-solution sets of (MQVI) and (SQVI), denoted by $\widetilde{S}_M(\varepsilon)$ and $\widetilde{S}_S(\varepsilon)$, are defined as

$$\begin{aligned}\widetilde{S}_M(\varepsilon) = &\{x \in A : d(x, K(x)) \leq \varepsilon \text{ and} \\ &\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \in C, \forall y \in K(x), \forall z \in T(y)\},\end{aligned}$$

and

$$\begin{aligned}\widetilde{S}_S(\varepsilon) = &\{x \in A : d(x, K(x)) \leq \varepsilon \text{ and } \exists z \in T(x), \\ &\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \in C, \forall y \in K(x)\}, \text{respectively.}\end{aligned}$$

**Theorem 3.5.** *Assume that $K$ is continuous with compact values on $A$. Then,*

(i) *for each $\varepsilon \geq 0$, $\widetilde{S}_M(\varepsilon)$ is compact, if $T$ is lower semicontinuous on $A$;*

(ii) *for each $\varepsilon \geq 0$, $\widetilde{S}_S(\varepsilon)$ is compact, if $T$ is upper semicontinuous on $A$.*

*Proof.* As an example we present only the proof for (i). Let $\{x_n\} \subset \widetilde{S}_M(\varepsilon)$, $x_n \to x$, we will prove that $x \in \widetilde{S}_M(\varepsilon)$. For each $n \in N$, we have $d(x_n, K(x_n)) \leq \varepsilon$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon e \in C, \forall y \in K(x_n), \forall z \in T(y).$$

For each $n$, there is $\bar{x}_n \in K(x_n)$ such that

$$d(x_n, \bar{x}_n) \leq d(x_n, K(x_n)) + \frac{1}{n} \leq \varepsilon + \frac{1}{n}.$$

Since $K$ is usc with compact values, we can assume that $\{\bar{x}_n\}$ converges to an element $\bar{x}$ in $K(x)$. So,

$$d(x, K(x)) \leq d(x, \bar{x}) = \lim_{n\to\infty} d(x_n, \bar{x}_n) \leq \varepsilon.$$

If $x \notin \widetilde{S}_M(\varepsilon)$ then there are $y \in K(x)$ and $z \in T(y)$ such that

$$\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \notin C.$$

Since $K$ and $T$ are lsc, for each $n$, we can pick $y_n \in K(x_n)$, $z_n \in T(y_n)$ such that $\{y_n\} \to y$ and $\{z_n\} \to z$. As $x_n \in \widetilde{S}_M(\varepsilon)$ and $\eta, \psi, f, H$ are continuous, we have,

$$\langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \in C,$$

which is impossible. Hence, $x \in \widetilde{S}_M(\varepsilon)$, i.e., $\widetilde{S}_M(\varepsilon)$ is compact. □

**Theorem 3.6.**

(i) *The problem* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense, if and only if $S_M$/or $S_S$ is a nonempty compact subset and $\widetilde{S}_M$/or $\widetilde{S}_S$ is upper semicontinuous at* 0*, respectively.*

(ii) *Suppose that $S_M$ and $S_S$ are singletons. Then, the problem* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed, if and only if $\widetilde{S}_M$/or $\widetilde{S}_S$ is upper semicontinuous at* 0*, respectively.*

*Proof.* Since the proof techniques are similar, we discuss only the Levitin-Polyak well-posedness in the generalized sense for (MQVI). Suppose that $S_M$ is a nonempty compact subset and $\widetilde{S}_M$ is usc at 0. Let $\{x_n\}$ be a Levitin-Polyak approximating sequence for (MQVI). Then, there exists a sequence $\{\varepsilon_n\} \subset (0, +\infty)$, $\varepsilon_n \to 0$ such that, for any $n$, $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall z \in T(y).$$

So, $x_n \in \widetilde{S}_M(\varepsilon_n)$. Since $S_M = \widetilde{S}_M(0)$ is compact, $\widetilde{\Phi}$ is upper semicontinuous with compact values at 0. Therefore, we can get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $x_{n_k}$ converges to $x_0$, for some point $x_0 \in \widetilde{S}_M(0) = S_M$. Hence, (MQVI) is Levitin-Polyak well-posed in the generalized sense.

Conversely, if (MQVI) is well-posed in the generalized sense then $S_M$ is a nonempty compact subset. Let $\{\varepsilon_n\} \subseteq (0, +\infty)$ be an arbitrary sequence with $\varepsilon_n \to 0$ and $x_n \in \widetilde{S}_M(\varepsilon_n)$. For any $n$, we have $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall z \in T(y).$$

So, $\{x_n\}$ is a Levitin-Polyak approximating sequence for (MQVI). By the Levitin-Polyak well-posedness in the generalized sense for (MQVI), $\{x_n\}$ much have a subsequence converging to an element in $S_M = \widetilde{S}_M(0)$. Lemma 2.1(iii) implies the fact that $\widetilde{S}_M$ is usc at 0. □

**Theorem 3.7.** *Assume that $K$ is continuous with compact values on $A$. Then, the following statements are true:*

(i) *If $T$ is lower semicontinuous on $A$ then* (MQVI) *is Levitin-Polyak well-posed if and only if*

$$\widetilde{S}_M(\varepsilon) \neq \emptyset, \forall \varepsilon > 0, \;\; \textit{and} \;\; \mathrm{diam}\widetilde{S}_M(\varepsilon) \to 0 \;\; \textit{as} \;\; \varepsilon \to 0.$$

(ii) *If $T$ is upper semicontinuous with compact values on $A$ then* (SQVI) *is Levitin-Polyak well-posed if and only if*

$$\widetilde{S}_S(\varepsilon) \neq \emptyset, \forall \varepsilon > 0, \;\; \textit{and} \;\; \mathrm{diam}\widetilde{S}_S(\varepsilon) \to 0 \;\; \textit{as} \;\; \varepsilon \to 0.$$

*Proof.* As an example we demonstrate only (ii). If (SQVI) is Levitin-Polyak well-posed then $S_S = \{x_0\}$, and hence $\widetilde{S}_S(\varepsilon) \neq \emptyset$, for any $\varepsilon > 0$. Suppose that $\mathrm{diam}\widetilde{S}_S(\varepsilon) \not\to 0$ as $\varepsilon \to 0$, i.e., then exist $\rho > 0$, $\{\varepsilon_n\} \subset (0, +\infty)$, $\varepsilon_n \to 0$ and

$$\mathrm{diam}\widetilde{S}_S(\varepsilon_n) > \rho.$$

So, there exist $x_n^1, x_n^2 \in \widetilde{S}_S(\varepsilon_n)$ such that $d(x_n^1, x_n^2) > \frac{q}{2} > 0$. Since $x_n^1, x_n^2 \in \widetilde{S}_S(\varepsilon_n)$, we have $d(x_n^1, K(x_n^1)) \leq \varepsilon_n, d(x_n^2, K(x_n^2)) \leq \varepsilon_n$ and there are $z_n^1 \in T(x_n^1)$ and $z_n^2 \in T(x_n^2)$ such that

$$\langle H(z_n^1), \eta(y, \psi(x_n^1))\rangle + f(y, \psi(x_n^1)) + \varepsilon_n e \in C, \forall y \in K(x_n^1),$$

$$\langle H(z_n^2), \eta(y, \psi(x_n^2))\rangle + f(y, \psi(x_n^2)) + \varepsilon_n e \in C, \forall y \in K(x_n^2),$$

i.e., $\{x_n^1\}$ and $\{x_n^2\}$ are Levitin-Polyak approximating sequences for (SQVI), and hence $\{x_n^1\}$ and $\{x_n^2\}$ converge to the unique solution $x_0$ of (SQVI), contradicting the fact that $d(x_n^1, x_n^2) > \frac{\rho}{2} > 0$, for all $n$.

Conversely, let $\{x_n\}$ be a Levitin-Polyak approximating sequence for (SQVI). For each $n$, there are $\{\varepsilon_n\} \to 0$ and $z_n \in T(x_n)$ such that, $d(x_n, K(x_n)) \leq \varepsilon_n$ and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall z \in T(y),$$

and so $x_n \in \widetilde{S}_S(\varepsilon_n)$. Since $\mathrm{diam}\widetilde{S}_S(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$, we conclude that $\{x_n\}$ is a Cauchy sequence and converges to an element $x_0$ in $A$. As $K$ is usc with compact values, we have $x_0 \in K(x_0)$. By using the same argument as for Theorem 3.5, we also deduce that $x_0$ belongs to $S_S$. To accomplish the proof we prove that (SQVI) has a unique solution. Suppose that $S_S$ has two distinct solution $x_1$ and $x_2$. Since, for all $\varepsilon > 0$, $x_1, x_2 \in \widetilde{S}_S(\varepsilon)$, we have

$$0 < d(x_1, x_2) \leq \mathrm{diam}\widetilde{S}_S(\varepsilon) \to 0 \ \ as \ \ \varepsilon \to 0,$$

which is impossible. Hence, (SQVI) is Levitin-Polyak well-posed. □

**Remark 3.8.** *If $K(x) \equiv K, H(z) = \{z\}, \eta(y, \psi(x)) = y - x, f(y, \psi(x)) = \phi(y) - \phi(x)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ the problem (SQVI) reduces to the generalized mixed variational inequality (in short, (GMVI)) studied in* [16]*. Hence, Theorem* 3.1 *in* [16] *is particular case of Theorem* 3.7*. However, the assumptions and our proof methods are very different from Theorem* 3.1 *in* [16]*.*

The following example shows that we can not replace the assumed Levitin-Polyak well-posed in Theorem 3.7 by the Levitin-Polyak well-posedness in the generalized sense.

**Example 3.9.** Let $X = Y = \mathbb{R}, e(x) = 1, A = [-1, 1], C = \mathbb{R}_+, \varepsilon \in \ \mathrm{int}\, C$. Let $\psi, H$ be identity maps, and let $K : A \rightrightarrows A, f : A \times A \to Y, \eta : A \times A \to A$, and $T : A \rightrightarrows L(X, Y)$ be defined by

$$K(x) = [0, 1], f(y, x) = 1,$$

$$\eta(y,x) = x(x-y),$$
$$T(y) = \{0\}.$$

We show that the assumptions of Theorem 3.7 are easily seen to be fulfilled and

$$\begin{aligned}\widetilde{S}_M(\varepsilon) &= \{x \in A : d(x, K(x)) \leq \varepsilon \ \text{ and} \\ &\quad \langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \in C, \forall y \in K(x), \forall z \in T(y)\} \\ &= \{x \in [-1,1] : d(x, [0,1]) \leq \varepsilon \ \text{ and} \\ &\quad \langle z, \eta(y,x)\rangle + f(y,x) + \varepsilon e \in C, \forall y \in [0,1], \forall z \in \{0\}\} \\ &= \{x \in [-1,1] : d(x, [0,1]) \leq \varepsilon \ \text{ and} \\ &\quad \langle 0, x(x-y)\rangle + 1 + \varepsilon \in C, \forall y \in [0,1]\} \\ &= [-1,1].\end{aligned}$$

Hence, the family $\{(\text{MQVI})\}$ is Levitin-Polyak well-posed at 0. But diam $\widetilde{S}_M(\varepsilon) \nrightarrow 0$ *as* $\varepsilon \to 0$.

Next, we recall the Kurastowski measures of noncompactness.

**Definition 3.10.** ([23]) Let $M$ be a nonempty subset of $X$. The Kurastowki measure of $M$ is

$$\zeta(M) = \inf\left\{\vartheta > 0 : M \subseteq \bigcup_{i=1}^{n} L_i, L_i \subseteq X, \operatorname{diam} L_i < \vartheta, i = 1, 2, ..., n, \ \exists\, n \in N\right\},$$

where diam(.) denotes the diameter.

**Remark 3.11.** ([24,25]) *The function $\zeta$ is a regular measure of noncompactness, i.e., it satisfies the following conditions:*

(i) $\zeta(M) = +\infty$ *if and only if the set $M$ is unbounded;*

(ii) $\zeta(M) = \zeta(\text{cl}(M))$*;*

(iii) *from $\zeta(M) = 0$ it follows that $M$ is a totally bounded set;*

(iv) *from $N \subset M$ it follows that $\zeta(N) \leq \zeta(M)$;*

(v) *if $X$ is a complete space, and $\{M_n\}$ is a sequence of closed subsets of $X$ such that $M_{n+1} \subset M_n$ for each $n \in N$ and $\lim_{n\to+\infty} \zeta(M_n) = 0$, then $M = \bigcap_{n\in N} M_n$ is a nonempty compact subset of $X$ and $\lim_{n\to+\infty} H(M_n, M) = 0$, where $H$ is Hausdorff metric.*

Employing the Kuratowski measure of noncompactness of approximate solution sets, we establish a metric characterization of the Levitin-Polyak well-posedness in the generalized sense for (MQVI) and (SQVI).

**Theorem 3.12.** *Assume that $K$ is continuous with compact values on $A$. Then, the following statements hold:*

(i) *If $T$ is lower semicontinuous on $A$ then* (MQVI) *is Levitin-Polyak well-posed in the generalized sense if and only if*

$$\widetilde{S}_M(\varepsilon) \neq \emptyset, \forall \varepsilon > 0, \text{ and } \zeta(\widetilde{S}_M(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0.$$

(ii) *If $T$ is upper semicontinuous with compact values on $A$ then* (SQVI) *is Levitin-Polyak well-posed in the generalized sense if and only if*

$$\widetilde{S}_S(\varepsilon) \neq \emptyset, \forall \varepsilon > 0, \text{ and } \zeta(\widetilde{S}_S(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0.$$

*Proof.* As an example we present only the proof for (i). Suppose that (MQVI) is Levitin-Polyak well-posed in the generalized sense. Let $\{x_n\}$ be any sequence in $S_M$. Since $\{x_n\}$ is a Levitin-Polyak approximating sequence for (MQVI), there exists a subsequence convergent to some point of $S_M$. So, $S_M$ is compact. From the Levitin-Polyak well-posedness of (MQVI), we conclude that $S_M \neq \emptyset$. Hence, $\widetilde{S}_M(\varepsilon) \neq \emptyset$ for any $\varepsilon \geq 0$. Next, we will show that $\zeta(\widetilde{S}_M(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $S_M \subseteq \widetilde{S}_M(\varepsilon)$, for every $\varepsilon \geq 0$, we have

$$\begin{aligned} H(\widetilde{S}_M(\varepsilon), S_M) &= \max\{H^*(\widetilde{S}_M(\varepsilon), S_M), H^*(S_M, \widetilde{S}_M(\varepsilon))\} \\ &= H^*(\widetilde{S}_M(\varepsilon), S_M), \end{aligned}$$

where $H^*(A, B) = \sup\{d(x, B) : x \in A\}$. Suppose that $S_M \subseteq \bigcup_{i=1}^n L_i$, $\mathrm{diam} L_i < \vartheta, i = 1, 2, ..., n,$ for some $n \in \mathbb{N}$.

Setting $\Omega_i = \{x \in A | d(x, L_i) \leq H(\widetilde{S}_M(\varepsilon), S_M)\}$. We claim that $\widetilde{S}_M(\varepsilon) \subseteq \bigcup_{i=1}^n \Omega_i$. Let $x \in \widetilde{S}_M(\varepsilon)$, then $d(x, S_M) \leq H(\widetilde{S}_M(\varepsilon), S_M)$. Since $S_M \subseteq \bigcup_{i=1}^n L_i$, we have

$$d(x, \bigcup_{i=1}^n L_i) \leq H(\widetilde{S}_M(\varepsilon), S_M).$$

Hence, there is $i_0$ such that

$$d(x, L_{i_0}) \leq H(\widetilde{S}_M(\varepsilon), S_M),$$

i,e., $x \in \Omega_{i_0}$. Hence, $\widetilde{S}_M(\varepsilon) \subseteq \bigcup_{i=1}^n \Omega_i$. It is not hard to see that

$$\begin{aligned} \mathrm{diam}\Omega_i &= \mathrm{diam} L_i + 2H(\widetilde{S}_M(\varepsilon), S_M) \\ &\leq \vartheta + 2H(\widetilde{S}_M(\varepsilon), S_M). \end{aligned}$$

Therefore,

$$\zeta(\widetilde{S}_M(\varepsilon)) \leq \zeta(S_M) + 2H(\widetilde{S}_M(\varepsilon), S_M).$$

Since $S_M$ is compact, we have $\zeta(S_M) = 0$. So, $\zeta(\widetilde{S}_M(\varepsilon)) \leq 2H^*(\widetilde{S}_M(\varepsilon), S_M)$. Now we prove that $H^*(\widetilde{S}_M(\varepsilon), S_M) \to 0$ as $\varepsilon \to 0$. Suppose to the contrary that $H^*(\widetilde{S}_M(\varepsilon), S_M) \not\to 0$ as $\varepsilon \to 0$. Then, there are $\rho > 0$, $\{\varepsilon_n\} \subset (0, +\infty)$, $\varepsilon_n \to 0$, and $x_n \in \widetilde{S}_M(\varepsilon_n)$ such that, for each $n$, we have $d(x_n, S_M) \geq \rho > 0$. Since $\{x_n\}$ is a Levitin-Polyak approximating sequence for (MQVI), it must have a subsequence converging to a point of $S_M$, which is impossible as $d(x_n, S_M) \geq \rho > 0$, for all $n$. Hence, $\zeta(\widetilde{S}_M(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

Conversely, suppose that $\zeta(\widetilde{S}_M) \neq \emptyset$ for all $\varepsilon \geq 0$ and $\zeta(\widetilde{S}_M(\varepsilon)) \to 0$ as $\varepsilon \to 0$. By Theorem 3.5, $\widetilde{S}_M(\varepsilon)$ is closed, for all $\varepsilon > 0$. Since $\zeta(\widetilde{S}_M(\varepsilon)) \to 0$ as $\varepsilon \to 0$, and $S_M = \bigcap_{\varepsilon>0} \widetilde{S}_M(\varepsilon)$, the regular measure properties of $\zeta$ imply that $S_M$ is compact and $H(\widetilde{S}_M(\varepsilon), S_M) \to 0$ as $\varepsilon \to 0$. Let $\{x_n\}$ be a Levitin-Polyak approximating sequence for (MQVI), then there is a sequence $\{\varepsilon_n\} \subset (0, +\infty), \varepsilon_n \to 0$ such that, for each $n$, $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall z \in T(y),$$

i.e., $x_n \in \widetilde{S}_M(\varepsilon_n)$. Hence, $d(x_n, S_M) \leq H(\widetilde{S}_M(\varepsilon_n), S_M) \to 0$ as $n \to +\infty$. So, there is $\bar{x}_n \in S_M$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to +\infty$. Since $S_M$ is compact, we can assume that $\{\bar{x}_n\}$ converges to an element $\bar{x}$ in $S_M$, and so $\{x_n\}$ also converges to $\bar{x}$. Thus, (MQVI) is Levitin-Polyak well-posed in the generalized sense. □

**Remark 3.13.** *If $K(x) \equiv X, \psi(x) = x, \eta(y, \psi(x)) = y - x, f(y, \psi(x)) = \phi(y) - \phi(x)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ the problem (SQVI) reduces to the generalized mixed variational inequality (in short, (GMVI)) studied in* [16]*. Hence, Theorem* 3.2 *in* [16] *is particular cases of Theorem* 3.12*. However, the assumptions and our proof methods are very different from Theorem* 3.2 *in* [16].

The following example shows that all the assumptions of Theorem 3.12 are fulfilled.

**Example 3.14.** Let $X = Y = \mathbb{R}, e(x) = 1, A = [0, 2], C = \mathbb{R}_+, \varepsilon \in \text{ int } C$. Let $\psi, H$ be identity maps, and let $K : A \rightrightarrows A, f : A \times A \to Y, \eta : A \times A \to A$, and $T : A \rightrightarrows L(X, Y)$ be defined by

$$K(x) = [0, 1], f(y, x) = 1,$$
$$\eta(y, x) = 2x(x - y),$$
$$T(y) = \{0\}.$$

We show that the assumptions of Theorem 3.12 are easily seen to be fulfilled and

$$\begin{aligned}
\widetilde{S}_M(\varepsilon) &= \{x \in A : d(x, K(x)) \leq \varepsilon \text{ and} \\
&\quad \langle H(z), \eta(y, \psi(x))\rangle + f(y, \psi(x)) + \varepsilon e \in C, \forall y \in K(x), \forall z \in T(y)\} \\
&= \{x \in [0, 2] : d(x, [0, 1]) \leq \varepsilon \text{ and} \\
&\quad \langle z, \eta(y, x)\rangle + f(y, x) + \varepsilon e \in C, \forall y \in [0, 1], \forall z \in \{0\}\} \\
&= \{x \in [0, 2] : d(x, [0, 1]) \leq \varepsilon \text{ and} \\
&\quad \langle 0, 2x(x - y)\rangle + [0, 1] + \varepsilon \in C, \forall y \in [0, 1]\} \\
&= [0, 1 + \varepsilon].
\end{aligned}$$

Hence, $\zeta(\widetilde{S}_M(\varepsilon)) \to 0$ as $\varepsilon \to 0$, and the family $\{$(MQVI)$\}$ is Levitin Polyak well-posed in generalized sense at 0.

# 4 Levitin-Polyak Well-Posedness via the Gap Functions of (MQVI) and (SQVI)

In this section, we introduce the gap functions for (MQVI) and (SQVI), then we study some their properties which are used to study the Levitin-Polyak well-posedness for such problems.

Firstly, motivated and inspired by the statements studied in [4,26], we consider the following assumptions:

(M): $S_M \neq \emptyset$ and for any Levitin-Polyak approximating sequence for (MQVI), we have

$$d(x_n, S_M) \to 0.$$

(S): $S_S \neq \emptyset$ and for any Levitin-Polyak approximating sequence for (SQVI), we have

$$d(x_n, S_S) \to 0.$$

**Proposition 4.1.** *If* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense, then assumption* (M)*/or* (S) *holds, respectively. Conversely, if assumption* (M)*/or* (S) *is satisfied and the solution* $S_M$*/or* $S_S$ *is compact, then* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense, respectively.*

*Proof.* As an example we discuss only the proof for (MQVI). Firstly, we prove that, if (MQVI) is Levitin-Polyak well-posed in the generalized sense then (M) holds. Since (MQVI) is Levitin-Polyak well-posed in the generalized sense, $S_M$ is nonempty and compact. If (M) is not true then there exists a Levitin-Polyak approximating sequence $\{x_n\}$ for (MQVI) and $\rho > 0$ such that $d(x_n, S_M) \geq \rho$ for $n$ sufficiently large. Since $\{x_n\}$ is a Levitin-Polyak approximating sequence for (MQVI), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of $S_M$, which contradicts the fact that $d(x_n, S_M) \geq \rho$ for $n$ sufficiently large.

Conversely, suppose that (M) holds and $S_M$ is compact. We will show that (MQVI) is Levitin-Polyak well-posed in the generalized sense. Let $\{x_n\}$ be a Levitin-Polyak approximating sequence for (MQVI). Since (M) holds, we conclude that $d(x_n, S_M) \to 0$ as $n \to \infty$. As $S_M$ is compact, for each $n$, there is $\bar{x}_n \in S_M$ such that $d(x_n, \bar{x}_n) = d(x_n, S_M)$, and hence $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. Since $S_M$ is compact, we can assume that $t_n \to t_0$ for some $t_0 \in S_M$. By the compactness of $S_M$, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $x_0$ in $S_M$. Then, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ will converge to $\{x_0\}$. So, (MQVI) is Levitin-Polyak well-posed in the generalized sense. □

**Definition 4.2.** A function $g : A \to \mathbb{R}$ is said to be a gap function for (MQVI) ((SQVI), respectively), if:

(i) $g(x) \geq 0$, for all $x \in K(x)$;

(ii) $g(x) = 0$ if and only if $x \in S_M$ ($x \in S_S$, respectively).

In the sequel, we suppose that $K$ and $T$ have compact values in a neighborhood of the reference point. We define two functions $h, k : A \to R$ as follows

$$h(x) = \max_{z\in T(y)} \max_{y\in K(x)} \xi_e(\langle H(z), \eta(\psi(x), y)\rangle - f(y, \psi(x))), \tag{4.1}$$

and

$$k(x) = \min_{z\in T(x)} \max_{y\in K(x)} \xi_e(\langle H(z), \eta(\psi(x), y)\rangle - f(y, \psi(x))). \tag{4.2}$$

Combining the compact values of $K$ and $T$ with the continuity of $\xi_e$ and $f$, we conclude that $h$ and $k$ are well-defined.

**Theorem 4.3.** *The functions $h$ and $k$ defined as above are the gap functions for* (MQVI) *and* (SQVI)*, respectively.*

*Proof.* Firstly, we define a function $\varphi : A \times B \to R$ as follows

$$\varphi(x, z) = \max_{y\in K(x)} \xi_e(\langle H(z), \eta(\psi(x), y)\rangle - f(y, \psi(x))).$$

(i) We first prove that $\varphi(x, z) \geq 0$ for every $x \in E$, where $E = \{x \in A : x \in K(x)\}$. Indeed, suppose to the contrary that there is $(x_0, z_0) \in E \times B$, such that $\varphi(x_0, z_0) < 0$, then

$$\begin{aligned} 0 > \varphi(x_0, z_0) &= \max_{y\in K(x_0)} \xi_e(\langle H(z_0), \eta(\psi(x_0), y)\rangle - f(y, \psi(x_0))) \\ &\geq \xi_e(\langle H(z_0), \eta(\psi(x_0), y)\rangle - f(y, \psi(x_0))), \forall y \in K(x_0). \end{aligned}$$

When $y = x_0$, we have

$$\begin{aligned} \xi_e(\langle H(z_0), \eta(x_0, \xi(x_0))\rangle - f(x_0, \xi(x_0))) &= \xi_e(0) \\ &= \min\{r \in R : 0 \in re - C\} \\ &= \min\{r \in R : -re \in -C\} \\ &= \min\{r \in R : r \geq 0\} = 0, \end{aligned}$$

which is a contradiction. Hence,

$$h(x) = \max_{z\in T(y)} \max_{y\in K(x)} \xi_e(\langle H(z), \eta(y, \psi(x))\rangle - f(y, \psi(x))) \geq 0.$$

(ii) From the definition, it is clear that $h(\bar{x}) = 0$ if and only if, for any $y \in K(\bar{x})$ and $z \in T(y)$,

$$\xi_e(\langle H(z), \eta(y, \psi(\bar{x}))\rangle - f(y, \psi(\bar{x}))) \leq 0.$$

By Lemma 2.2 (ii), the above inequality holds if and only if, for any $y \in K(\bar{x})$ and $z \in T(y)$,

$$\langle H(z), \eta(y, \psi(\bar{x}))\rangle - f(y, \psi(\bar{x})) \in -C,$$

or

$$\langle H(z), \eta(y, \psi(\bar{x}))\rangle + f(y, \psi(\bar{x})) \in C,$$

i.e., $\bar{x} \in S_M$. Hence, $h$ is a gap function for (MQVI).

In the turn of $k$, we consider a similar function $\tau : A \times B \to R$ given by

$$\tau(x, z) = \max_{y \in K(x)} \xi_e(\langle H(z), \eta(y, \psi(x_0))\rangle - f(y, \psi(x_0))).$$

By employing the same argument as above, we also conclude that $k$ is a gap function for (SQVI). □

Now, we consider the following quasioptimization problems:

$$\textbf{(QOP}_1\textbf{)} \qquad \begin{cases} \text{minimize } \ h(x) \\ \text{subject to } \ x \in K(x) \end{cases}$$

and

$$\textbf{(QOP}_2\textbf{)} \qquad \begin{cases} \text{minimize } \ k(x) \\ \text{subject to } \ x \in K(x), \end{cases}$$

where $h(x)$ and $k(x)$ are given by (4.1) and (4.2), respectively. We denote the solution sets of $(\mathrm{QOP}_1)$ and $(\mathrm{QOP}_2)$ by $S_1$ and $S_2$, respectively.

**Definition 4.4.** A sequence $\{x_n\} \subset A$ is said to be a Levitin-Polyak minimizing sequence for $(\mathrm{QOP}_1)$/or $(\mathrm{QOP}_2)$, if there exists a sequence $\{\varepsilon_n\} \subset (0, +\infty)$ converging to 0 such that $d(x_n, K(x_n)) \le \varepsilon_n$ and $\limsup_{n\to+\infty} h(x_n) \le 0$ or $\limsup_{n\to+\infty} k(x_n) \le 0$, respectively.

**Definition 4.5.** The problem $(\mathrm{QOP}_1)$/or $(\mathrm{QOP}_2)$ is said to be Levitin-Polyak well-posed in the generalized sense, if

(i) $(\mathrm{QOP}_1)$/or $(\mathrm{QOP}_2)$ has solution;

(ii) for every Levitin-Polyak minimizing sequence $\{x_n\}$ for $(\mathrm{QOP}_1)$/or $(\mathrm{QOP}_2)$ much have a subsequence converging to an element in $S_1$/or $S_2$, respectively.

**Theorem 4.6.** *The problem* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense if and only if* $(\mathrm{QOP}_1)$*/or* $(\mathrm{QOP}_2)$ *is Levitin-Polyak well-posed in the generalized sense, respectively.*

*Proof.* As an example, we demonstrate only for the problem (SQVI). To obtain the conclusions in the theorem, we need to show that a sequence $\{x_n\}$ is a Levitin-Polyak approximating sequence for (SQVI) if and only if it is a Levitin-Polyak approximating sequence for $(\mathrm{QOP}_2)$. Then, there are a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $d(x_n, K(x_n)) \le \varepsilon_n$ and $z \in T(x_n)$ such that

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall n \in \mathbb{N},$$

i.e.,

$$\langle H(z), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n)) \in \varepsilon_n e - C, \forall y \in K(x_n), \forall n \in \mathbb{N}. \tag{4.3}$$

From Lemma 2.2 (ii) and (4.3), we have

$$\xi_e(\langle H(z), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n, \forall n.$$

Since $y \in K(x_n)$ is arbitrary, we conclude that

$$\max_{y\in K(x_n)} \xi_e(\langle H(z), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n.$$

So,

$$k(x_n) = \min_{z\in T(x_n)} \max_{y\in K(x_n)} \xi_e(\langle H(z), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n.$$

As $x_n \in K(x_n, \gamma_n)$ is arbitrary, we have

$$\limsup_{n\to+\infty} k(x_n) \leq 0, \text{ as } \varepsilon_n \to 0,$$

and hence, $\{x_n\}$ is a Levitin-Polyak minimizing sequence for ($\text{QOP}_2$).

Conversely, assume that $\{x_n\}$ is a Levitin-Polyak approximating sequence for ($\text{QOP}_2$). Then, $d(x_n, K(x_n)) \to 0$, and $\limsup_{n\to+\infty} h(x_n) \leq 0$. Thus, there exists $\varepsilon_n \in \mathbb{R}_+$ convergent to 0 such that $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$k(x_n) = \min_{z\in T(x_n)} \max_{y\in K(x_n)} \xi_e(\langle H(z), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n.$$

By the compactness of $T(x_n)$, there exists $z_n \in T(x_n)$ such that

$$k(x_n) = \max_{y\in K(x_n)} \xi_e(\langle H(z_n), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n,$$

and hence,

$$\xi_e(\langle H(z_n), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n))) \leq \varepsilon_n, \forall y \in K(x_n).$$

By using Lemma 2.2 (ii), for each $y \in K(x_n)$, we have

$$\langle H(z_n), \eta(y, \psi(x_n))\rangle - f(y, \psi(x_n)) \in \varepsilon_n e - C,$$

i.e.,

$$\langle H(z_n), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C.$$

Hence, $\{x_n\}$ is a Levitin-Polyak approximating sequence for (SQVI). □

Now we derive some criterion and characterizations for the Levitin-Polyak well-posedness for (MQVI) and (SQVI). Firstly, we consider a real-valued function $p : [0, \infty) \times [0, \infty) \to R$ satisfying

$$p(0, 0) = 0, p(t, s) \geq 0, \forall (t, s) \in [0, \infty) \times [0, \infty). \tag{4.4}$$

$$[s_n \to 0, t_n \geq 0, p(t_n, s_n) \to 0] \Rightarrow [t_n \to 0]. \tag{4.5}$$

**Theorem 4.7.** *If* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense then there exists a function $p$ satisfying* (4.4) *and* (4.5) *such that, for each $x \in A$,*

$$|h(x)| \geq p(d(x, S_M), d(x, K(x))), \tag{4.6}$$

*or*

$$|k(x)| \geq p(d(x, S_S), d(x, K(x))), \tag{4.7}$$

*respectively.*

*Proof.* As an example, we discuss only for (MQVI). We consider a real function $p : [0, \infty) \times [0, \infty) \to R$ given by

$$p(t, s) = \inf\{|h(x)| : d(x, S_M) = t, d(x, K(x)) = s\}.$$

Then, $p$ is satisfied (4.6), $p(t, s) \geq 0$, for all $(t, s) \in [0, \infty) \times [0, \infty)$ and $p(0, 0) = 0$ as $h$ is a gap function for (MQVI). Now let $s_n \to 0, t_n \geq 0$ and $p(t_n, s_n) \to 0$. Then there exists a sequence $\{x_n\} \subset A$ such that

$$d(x_n, S_M) = t_n, d(x_n, K(x_n)) = s_n, h(x_n) \to 0,$$

which implies that $\{x_n\}$ is a Levitin-Polyak minimizing sequence for (QOP$_1$). Using the same argument as in the proof for Theorem 4.6, $\{x_n\}$ is also a Levitin-Polyak approximating sequence for (MQVI). Since (MQVI) is Levitin-Polyak well-posed in the generalized sense, by applying Proposition 4.1, we conclude that $t_n = d(x_n, S_M) \to 0$. □

**Theorem 4.8.** *Suppose that $S_M$/or $S_S$ is nonempty and compact and assume further that* (4.6)*/or* (4.7) *holds for some $p$ satisfied* (4.4) *and* (4.5)*. Then* (MQVI)*/or* (SQVI) *is Levitin-Polyak well-posed in the generalized sense, respectively.*

*Proof.* (a) Let $\{x_n\}$ be a Levitin-Polyak approximating sequence for (MQVI), then there exists sequence $\{\varepsilon_n\} \subset (0, +\infty)$ converging to 0 such that $d(x_n, K(x_n)) \leq \varepsilon_n$, and

$$\langle H(z), \eta(y, \psi(x_n))\rangle + f(y, \psi(x_n)) + \varepsilon_n e \in C, \forall y \in K(x_n), \forall z \in T(y).$$

From (4.6), we have

$$|h(x_n)| \geq p(d(x_n, S_M), d(x_n, K(x_n))). \tag{4.8}$$

Let $t_n = d(x_n, S_M)$ and $s_n = d(x_n, K(x_n))$. It is easy to see that $s_n \to 0$. By using the same argument as in the proof of Theorem 4.6, we conclude that $\{x_n\}$ is a Levitin-Polyak minimizing sequence for (QOP$_1$), and hence $h(x_n) \to 0$. Employing (4.5) and (4.8), we have $t_n = d(x_n, S_M) \to 0$. Since $S_M$ is nonempty and compact, Proposition 4.1 implies the fact that (MQVI) is Levitin-Polyak well-posed in the generalized sense.

(b) The Levitin-Polyak well-posedness in the generalized sense for (SQVI) is also established by using the same argument as above. □

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