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On Some Čebyšev Type Inequalities for Functions whose Second Derivatives are r-Convex on the Co-Ordinates

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Abstract : The aim of this paper is to establish some new Čebyšev type inequalities involving functions whose second derivatives are co-ordinated r-convex.

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1 Introduction

In 1882, Čebyšev [1] gave the following inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.1}$$

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for $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and $g'$are bounded, where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = \underset{t\in[a,b]}{ess\sup}|f(t)|.$

During the past few years, many researchers have given considerable attention to the inequality (1.1) and various generalizations, extensions and variants of this inequality have appeared in the literature, we can mention the works [2–8]. Also in [9], the authors obtained some new Čebyšev type inequalities involving functions whose mixed partial derivatives are convex on the co-ordinates. Motivate by the results from [9] and by a similar argument to that in the paper [4], we will establish some new Čebyšev type inequalities involving functions whose mixed partial derivatives are r-convex on the co-ordinates.

2 Preliminaries

In this section, we begin by giving some necessary materials for our results. Throughout this paper we denote by Δ the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b] \times [c,d]$ with $a < b$ and $c < d$, $k = (b-a)(d-c)$ and $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$ by $f_{\lambda\alpha}$.

Definition 2.1. [10] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} f(tx + (1-t)u, \lambda y + (1-\lambda)v) \quad &\leq \quad t\lambda f(x,y) + t(1-\lambda)f(x,v) \\ &\quad + (1-t)\lambda f(u,y) + (1-t)(1-\lambda)f(u,v) \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (x,v), (u,y), (u,v) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex see [11].

Definition 2.2. [12] A function $f : \Delta \to \mathbb{R}$ is called r-convex on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} & f(tx + (1-t)u, \lambda y + (1-\lambda)v) \\ \leq & \begin{cases} \quad [t\lambda f^r(x,y) + t(1-\lambda)f^r(x,v) + (1-t)\lambda f^r(u,y) & \\ +(1-t)(1-\lambda)f^r(u,v)]^{\frac{1}{r}} & \text{if } r \neq 0 \\ f^{t\lambda}(x,y) f^{t(1-\lambda)}(x,v) f^{(1-t)\lambda}(u,y) f^{(1-t)(1-\lambda)}(u,v) & \text{if } r = 0 \end{cases} \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (x,v), (u,y), (u,v) \in \Delta$.

Lemma 2.3. [13] *For $a \geq 0$ and $b \geq 0$, the following algebraic inequalities are true*

$$(a+b)^{\lambda} \leq 2^{\lambda-1}\left(a^{\lambda}+b^{\lambda}\right), \qquad \text{for } \lambda \geq 1 \tag{2.1}$$

and

$$(a+b)^{\lambda} \leq a^{\lambda}+b^{\lambda}, \qquad \text{for } 0 \leq \lambda \leq 1. \tag{2.2}$$

In order to prove our main theorems we need the following lemma.

Lemma 2.4. [8, Lemma 1] *Let $f : \Delta \to \mathbb{R}$ be a partially differentiable mapping on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha} \in L(\Delta)$, then for any $(x,y) \in \Delta$, we have the equality:*

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{k}\int_a^b\int_c^d f(t,s)dsdt \\ &+\frac{1}{k}\int_a^b\int_c^d (x-t)(y-s) \\ &\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt. \end{aligned}$$

3 Main Results

Theorem 3.1. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are r-convex on the co-ordinates, then*

$$|T(f,g)| \leq \begin{cases} \frac{49\times 2^{\frac{2}{r}-2}}{3600} MN\left(\frac{r}{r+1}\right)^4 k^2 & \text{if } 0<r<1 \\ \frac{49}{3600} MN\left(\frac{r}{r+1}\right)^4 k^2 & \text{if } r \geq 1, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} T(f,g) &= \frac{1}{k}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx - \frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)\,dt\right)dydx \\ &-\frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,s)\,ds\right)dydx \\ &+\frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)\,dydx\right)\left(\int_a^b\int_c^d g(t,s)\,dtds\right), \end{aligned} \tag{3.2}$$

$r > 0$, $k = (b-a)(d-c)$,
$M = \underset{x,t\in[a,b],y,s\in[c,d]}{ess\sup} [|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,s)| + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,s)|]$, *and*
$N = \underset{x,t\in[a,b],y,s\in[c,d]}{ess\sup} [|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,s)| + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,s)|]$.

Proof. Let F, G, $\widetilde{F}$ and $\widetilde{G}$ defined as follows

$$\begin{aligned}
F &= f(x,y) - \frac{1}{b-a}\int_a^b f(t,y)dt - \frac{1}{d-c}\int_c^d f(x,s)ds + \frac{1}{k}\int_a^b\int_c^d f(t,s)dsdt, \\
G &= g(x,y) - \frac{1}{b-a}\int_a^b g(t,y)dt - \frac{1}{d-c}\int_c^d g(x,s)ds + \frac{1}{k}\int_a^b\int_c^d g(t,s)dsdt, \\
\widetilde{F} &= \frac{1}{k}\int_a^b\int_c^d (x-t)(y-s)\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s)\, d\alpha d\lambda\right)dsdt, \\
\widetilde{G} &= \frac{1}{k}\int_a^b\int_c^d (x-t)(y-s)\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s)\, d\alpha d\lambda\right)dsdt.
\end{aligned}$$

From Lemma 2.4, we have

$$F = \widetilde{F} \ \text{ and } \ G = \widetilde{G},$$

which implies

$$FG = \widetilde{F}\widetilde{G}. \tag{3.3}$$

Integrating (3.3) with respect to x and y over Δ, multiplying the resultant equality by $\frac{1}{k}$, using Fubini's Theorem and the modulus, we obtain

$$\begin{aligned}
|T(f,g)| \leq \frac{1}{k^3}\Bigg| \int_a^b\int_c^d & \Bigg[\int_a^b\int_c^d (x-t)(y-s) \\
&\times \left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s)\, d\alpha d\lambda\right) dsdt\Bigg] \\
&\times \Bigg[\int_a^b\int_c^d (x-t)(y-s) \\
&\times \left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s)\, d\alpha d\lambda\right) dsdt\Bigg] dydx\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda\right)dsdt\right]\\
&\times\left[\int_a^b\int_c^d |x-t|\,|y-s|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)s)|\,d\alpha d\lambda\right)dsdt\right]dydx.
\end{aligned}
\tag{3.4}
$$

Using the r-convexity, we have

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda\right)dsdt\right]\\
&\times\left[\int_a^b\int_c^d |x-t|\,|y-s|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda\right)dsdt\right]dydx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[\int_0^1\int_0^1 [\alpha\lambda\,|f_{\lambda\alpha}(x,y)|^r+\alpha(1-\lambda)\,|f_{\lambda\alpha}(x,s)|^r\right.\right.\\
&+\left.\left.(1-\alpha)\,\lambda\,|f_{\lambda\alpha}(t,y)|^r+(1-\alpha)(1-\lambda)\,|f_{\lambda\alpha}(t,s)|^r]^{\frac{1}{r}}\,d\alpha d\lambda\right]dsdt\right]dydx\\
&\times\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[\int_0^1\int_0^1 [\alpha\lambda\,|g_{\lambda\alpha}(x,y)|^r+\alpha(1-\lambda)\,|g_{\lambda\alpha}(x,s)|^r\right.\right.\\
&+\left.\left.(1-\alpha)\,\lambda\,|g_{\lambda\alpha}(t,y)|^r+(1-\alpha)(1-\lambda)\,|g_{\lambda\alpha}(t,s)|^r]^{\frac{1}{r}}\,d\alpha d\lambda\right]dsdt\right]dydx.
\end{aligned}
\tag{3.5}
$$

We distinguish two cases

Case 1 : $0 < r < 1$. Using (2.1) in (3.5), we obtain

$$\begin{aligned}
|T(f,g)| \leq\ & \frac{2^{\frac{2}{r}-2}}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s| \left[|f_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\,d\alpha d\lambda\right.\right.\\
&+|f_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\,d\alpha d\lambda+|f_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\,d\alpha d\lambda\\
&\left.+\,|f_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\,d\alpha d\lambda\right] dsdt\\
&\times\int_a^b\int_c^d |x-t|\,|y-s|\times\left[|g_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\,d\alpha d\lambda\right.\\
&+|g_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\,d\alpha d\lambda+|g_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\,d\alpha d\lambda\\
&\left.\left.+\,|g_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\,d\alpha d\lambda\right] dsdt\right] dydx\\
\leq\ & \frac{2^{\frac{2}{r}-2}}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s|\,M\left(\frac{r}{r+1}\right)^2 dsdt\right]\\
&\times\left[\int_a^b\int_c^d |x-t|\,|y-s|\,N\left(\frac{r}{r+1}\right)^2 dsdt\right] dydx\\
=\ & \frac{2^{\frac{2}{r}-2}}{k^3}MN\left(\frac{r}{r+1}\right)^4\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt\right]^2 dydx. \qquad (3.6)
\end{aligned}$$

Note that

$$\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt\right]^2 dydx=\frac{49}{3600}k^5. \qquad (3.7)$$

Substituting (3.7) in (3.6), we obtain the first inequality of (3.1).

Case 2 : $r \geq 1$. Using (2.2) in (3.5), we get

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s| \left[|f_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda \right.\right.\\
&\quad + |f_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda + |f_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda\\
&\quad \left. + |f_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda \right] dsdt\\
&\quad \times \int_a^b\int_c^d |x-t|\,|y-s|\\
&\quad \times \left[|g_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda + |g_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda \right.\\
&\quad + |g_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda\\
&\quad \left.\left. + |g_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda \right] dsdt \right] dydx\\
&\leq \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s|\, M\left(\frac{r}{r+1}\right) dsdt\right]\\
&\quad \times \left[\int_a^b\int_c^d |x-t|\,|y-s|\, N\left(\frac{r}{r+1}\right) dsdt\right] dydx\\
&= \frac{1}{k^3}MN\left(\frac{r}{r+1}\right)^4 \int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-s|\, dsdt\right]^2 dydx. \qquad (3.8)
\end{aligned}$$

Substituting (3.7) in (3.8), we obtain the second inequality of (3.1). This completes the proof of Theorem 3.1. □

Remark 3.2. *Theorem* 3.1 *will be reduced to the inequality* (6) *of Theorem* 2.1 *from* [9], *if we choose* $r = 1$.

Corollary 3.3. *Under the same assumptions of Theorem* 3.1*, if we take* $M = 4\left\Vert f_{\lambda\alpha}\right\Vert_{\infty}$ *and* $N = 4\left\Vert g_{\lambda\alpha}\right\Vert_{\infty}$ *we have the following estimates*

$$\left\vert T(f,g)\right\vert \leq \begin{cases} \frac{49\times 2^{\frac{2}{r}+2}}{3600}\left(\frac{r}{r+1}\right)^{4}k^{2}\left\Vert f_{\lambda\alpha}\right\Vert_{\infty}\left\Vert g_{\lambda\alpha}\right\Vert_{\infty} & \text{if } 0<r<1 \\ \frac{49}{225}\left(\frac{r}{r+1}\right)^{4}k^{2}\left\Vert f_{\lambda\alpha}\right\Vert_{\infty}\left\Vert g_{\lambda\alpha}\right\Vert_{\infty} & \text{if } r\geq 1. \end{cases} \tag{3.9}$$

Remark 3.4. *Corollary* 3.3 *will be reduced to Theorem* 1 *from* [4]*, in the case* $r = 1$.

Theorem 3.5. *Let* $f, g : \Delta \to \mathbb{R}$ *be partially differentiable functions, such that their second derivatives* $f_{\lambda\alpha}$ *and* $g_{\lambda\alpha}$ *are integrable on* Δ*. If* $|f_{\lambda\alpha}|$ *and* $|g_{\lambda\alpha}|$ *are* r*-convex on the co-ordinates, then we have*

$$\left\vert T(f,g)\right\vert \leq \begin{cases} \frac{2^{\frac{2}{r}-5}}{k^{2}}\left(\frac{r}{1+r}\right)^{2}\int\limits_{a}^{b}\int\limits_{c}^{d}\left[M\left\vert g(x,y)\right\vert + N\left\vert f(x,y)\right\vert\right] \\ \times\left[(x-a)^{2}+(b-x)^{2}\right]\left[(y-c)^{2}+(d-y)^{2}\right]dydx & \text{if } 0<r<1, \\ \frac{1}{8k^{2}}\left(\frac{r}{1+r}\right)^{2}\int\limits_{a}^{b}\int\limits_{c}^{d}\left[M\left\vert g(x,y)\right\vert + N\left\vert f(x,y)\right\vert\right] \\ \times\left[(x-a)^{2}+(b-x)^{2}\right]\left[(y-c)^{2}+(d-y)^{2}\right]dydx & \text{if } r\geq 1, \end{cases} \tag{3.10}$$

where $T(f,g)$, M, N, k *and* r *are defined as in Theorem* 3.1.

Proof. From Lemma 2.4, we have

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int\limits_{a}^{b}f(t,y)dt + \frac{1}{d-c}\int\limits_{c}^{d}f(x,s)ds - \frac{1}{k}\int\limits_{a}^{b}\int\limits_{c}^{d}f(t,s)dsdt \\ &\quad + \frac{1}{k}\int\limits_{a}^{b}\int\limits_{c}^{d}(x-t)(y-s) \\ &\quad \times\left(\int\limits_{0}^{1}\int\limits_{0}^{1}f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt, \end{aligned} \tag{3.11}$$

and

$$g(x,y) = \frac{1}{b-a}\int\limits_{a}^{b}g(t,y)dt + \frac{1}{d-c}\int\limits_{c}^{d}g(x,s)ds - \frac{1}{k}\int\limits_{a}^{b}\int\limits_{c}^{d}g(t,s)dsdt$$

$$+\frac{1}{k}\int_a^b\int_c^d (x-t)(y-s) \times \left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt. \quad (3.12)$$

Multiplying (3.11) by $\frac{1}{2k}g(x,y)$ and (3.12) by $\frac{1}{2k}f(x,y)$, summing the resultant equalities and then integrating these with respect to x and y over Δ, we get

$$\begin{aligned} T(f,g) &= \frac{1}{2k^2}\left[\int_a^b\int_c^d g(x,y)\left[\int_a^b\int_c^d (x-t)(y-s)\right.\right. \\ &\times \left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt\right]dydx \\ &+\int_a^b\int_c^d f(x,y)\left[\int_a^b\int_c^d (x-t)(y-s)\right. \\ &\times \left.\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt\right]dydx\right]. \end{aligned} \quad (3.13)$$

Using the modulus and the r-convexity, we get

$$\begin{aligned} |T(f,g)| \le\ & \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-s|\right.\right. \\ &\times \left[\int_0^1\int_0^1 \left[\alpha\lambda\,|f_{\lambda\alpha}(x,y)|^r+\alpha(1-\lambda)\,|f_{\lambda\alpha}(x,s)|^r+(1-\alpha)\lambda\,|f_{\lambda\alpha}(t,y)|^r\right.\right. \\ &+\ \left.\left.\left.(1-\alpha)(1-\lambda)\,|f_{\lambda\alpha}(t,s)|^r\right]^{\frac{1}{r}}d\alpha d\lambda\right]dsdt\right]dydx \\ &+\int_a^b\int_c^d |f(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-s|\right. \\ &\times \left[\int_0^1\int_0^1 \left[\alpha\lambda\,|g_{\lambda\alpha}(x,y)|^r+\alpha(1-\lambda)\,|g_{\lambda\alpha}(x,s)|^r+(1-\alpha)\lambda\,|g_{\lambda\alpha}(t,y)|^r\right.\right. \\ &+\ \left.\left.\left.\left.(1-\alpha)(1-\lambda)\,|g_{\lambda\alpha}(t,s)|^r\right]^{\frac{1}{r}}d\alpha d\lambda\right]dsdt\right]dydx\right]. \end{aligned} \quad (3.14)$$

Case 1 : $0 < r < 1$. Using (2.1) in (3.14), we get

$$\begin{aligned}
|T(f,g)| \leq \; & \frac{2^{\frac{2}{r}-3}}{k^2}\left[\int_a^b\int_c^d |g(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[|f_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda\right.\right.\right.\\
& + |f_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda + |f_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda\\
& \left.\left. + |f_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda\right] dsdt\right.\\
& + \int_a^b\int_c^d |f(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[|g_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda\right.\right.\\
& + |g_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda + |g_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda\\
& \left.\left.\left. + |g_{\lambda\alpha}(t,s)|\int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda\right] dsdt\right] dydx\\
\leq \; & \frac{2^{\frac{2}{r}-3}}{k^2}\left(\frac{r}{1+r}\right)^2 \int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|]\\
& \times \left[\int_a^b\int_c^d |x-t|\,|y-s|\, dsdt\right] dydx. \qquad (3.15)
\end{aligned}$$

Note that

$$\int_a^b\int_c^d |x-t|\,|y-s|\, dsdt = \tfrac{1}{4}\left[(x-a)^2 + (b-x)^2\right]\left[(y-c)^2 + (d-y)^2\right]. \qquad (3.16)$$

Substituting (3.16) in (3.15), we obtain the first inequality of (3.10).

Case 2 : $r \geq 1$. Using (2.2) in (3.14), we obtain

$$\begin{aligned}
|T(f,g)| \leq \; & \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[|f_{\lambda\alpha}(x,y)|\int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda\right.\right.\right.\\
& + |f_{\lambda\alpha}(x,s)|\int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda + |f_{\lambda\alpha}(t,y)|\int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda
\end{aligned}$$

$$
\begin{aligned}
&+ |f_{\lambda\alpha}(t,s)| \int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda \Bigg] dsdt \\
&+\int_a^b\int_c^d |f(x,y)| \Bigg[\int_a^b\int_c^d |x-t|\,|y-s| \Bigg[|g_{\lambda\alpha}(x,y)| \int_0^1\int_0^1 (\alpha\lambda)^{\frac{1}{r}}\, d\alpha d\lambda \\
&+ |g_{\lambda\alpha}(x,s)| \int_0^1\int_0^1 [\alpha(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda + |g_{\lambda\alpha}(t,y)| \int_0^1\int_0^1 [(1-\alpha)\lambda]^{\frac{1}{r}}\, d\alpha d\lambda \\
&+ |g_{\lambda\alpha}(t,s)| \int_0^1\int_0^1 [(1-\alpha)(1-\lambda)]^{\frac{1}{r}}\, d\alpha d\lambda \Bigg] dsdt \Bigg] dydx \\
\leq\ & \frac{1}{2k^2}\left(\frac{r}{1+r}\right)^2 \int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|] \\
&\times \left[\int_a^b\int_c^d |x-t|\,|y-s|\, dsdt\right] dydx.
\end{aligned} \tag{3.17}
$$

Substituting (3.16) in (3.17), we obtain the second inequality of (3.10). This completes the proof of Theorem 3.5. □

Remark 3.6. *Theorem* 3.5 *will be reduced to the inequality* (7) *of Theorem* 2.1 *in* [9], *if we choose* $r = 1$.

Corollary 3.7. *Under the same assumptions of Theorem* 3.5, *if we take* $M = 4\,\|f_{\lambda\alpha}\|_\infty$ *and* $N = 4\,\|g_{\lambda\alpha}\|_\infty$ *we have the following estimates*

$$
|T(f,g)| \leq \begin{cases}
\frac{2^{\frac{2}{r}-3}}{k^2}\left(\frac{r}{1+r}\right)^2 \int_a^b\int_c^d [|g(x,y)|\,\|f_{\lambda\alpha}\|_\infty + |f(x,y)|\,\|g_{\lambda\alpha}\|_\infty] & \\
\times \left[(x-a)^2 + (b-x)^2\right]\left[(y-c)^2 + (d-y)^2\right] dydx & \text{if } 0 < r < 1, \\
\frac{1}{2k^2}\left(\frac{r}{1+r}\right)^2 \int_a^b\int_c^d [|g(x,y)|\,\|f_{\lambda\alpha}\|_\infty + |f(x,y)|\,\|g_{\lambda\alpha}\|_\infty] & \\
\times \left[(x-a)^2 + (b-x)^2\right]\left[(y-c)^2 + (d-y)^2\right] dydx & \text{if } r \geq 1.
\end{cases}
$$

Remark 3.8. *Corollary* 3.7 *will be reduced to Theorem* 2 *in* [4], *in the case* $r = 1$.

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