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Common Fixed Point Theorems in Fuzzy Metric Spaces under Implicit Relations

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Abstract : In this paper using the concept of a pair (A, B) being weakly A-compatible, we prove a common fixed point theorem for self maps in fuzzy metric spaces which modifies and generalizes some known results. On the other hand a common fixed point theorem for self maps in sequentially compact fuzzy metric spaces is also proved.

Keywords : weakly A-compatible pair (A, B); sequentially compact; common fixed points.
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1 Introduction

The concept of fuzzy sets introduced by Zadeh [1] laid the foundation of fuzzy mathematics. The motivation of introducing fuzzy metric space is the fact that in many situations the distance between two points is inexact due to fuzziness

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rather than randomness. Kramosil and Michalek [2] introduced the concept of fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. Further George and Veeramani [3] modified this concept of fuzzy metric space introduced in [2]. Adding to same Grabiec [4] extended the well known fixed point theorems of Banach [5] and Edelstein [6] to fuzzy metric spaces in the sense of [2]. Many authors in [3, 4, 7, 8] proved fixed and common fixed point theorems in fuzzy metric spaces. Later on in [9–13] we can find the results by the applications of compatible mapping conditions in fuzzy metric spaces. In this paper using the definition of the pair (A, B) being weakly A-compatible or weakly B-compatible, given by [14] we obtained a common fixed point theorem for such pairs of maps under an [2] it relation with rational contractive condition which generalizes [13, Theorem 3.1], [11, Corollary 1], [9, Theorems 3.1 and 3.5], and [8, Corollary 2]. We also proved a common fixed point theorem for pairs of weakly compatible maps in a sequentially compact fuzzy metric space using an compact fuzzy metric space using an implicit relation. First of all we give some known definitions and lemmas.

2 Preliminaries

Definition 2.1 ([15])**.** A binary operation $* : [0,1]^2 \times [0,1] \to [0,1]$ is a *continuous t-norm*, if $[0,1], *$ is an abelian topological monoid with a unit 1 such that $a * b \leq c * d$, whenever $a \leq c$, $b \leq d$, $\forall a, b, c, d \in [0,1]$.
Two examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 ([2])**.** The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, * a continuous t-norm and M a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(a) $M(x, y, t) > 0$;

(b) $M(x, y, t) = 1$ if and only of $x = y$;

(c) $M(x, y, t) = M(y, x, t)$;

(d) $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$;

(e) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous,for all $x, y, z \in X$ and $t, s > 0$.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Now let $(X, M, *)$ be a fuzzy metric space and τ the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X induced by the fuzzy metric M.

Definition 2.3 ([4])**.** A sequence $\{x_n\}$ in a fuzzy metric $(X, M, *)$ is said to be *convergent* to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$. The sequence $\{x_n\}$ is said to be *Cauchy* if $\lim_{n\to\infty} M(x_n, x_m, t) = 1$. The space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in X is convergent in X.

Definition 2.4. $(X, M, *)$ is said to be a *sequentially compact fuzzy metric space* if every sequence in X has a convergent sub-sequence.

Lemma 2.5 ([4])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.6 ([16])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Throughout this paper, we now assume that $\lim_{t\to\infty} M(x, y, t) = 1$ and that N is the set of all natural numbers.

Lemma 2.7 ([8])**.** *Let $\{y_n\}$ be a sequence in $(X, M, *)$. If there exists a positive number $k < 1$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \;\; t > 0, n \in N,$$

then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.8 ([8])**.** *If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

Definition 2.9 ([8])**.** Let A and B be self maps on a fuzzy metric space $(X, M, *)$. The pair (A, B) is said to be *compatible* if $\lim_{n\to\infty} M(ABx_n, BAx_n, kt) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some $z \in X$.

Definition 2.10 ([17])**.** Let A and B be self mappings on a fuzzy metric space $(X, M, *)$. Then the mappings are said to be *weakly compatible* if they commute at their coincidence point, that is, $Ax = Bx$ implies that $ABx = BAx$.

Definition 2.11 ([14])**.** The pair (A, B) is said to be *weakly A-compatible* if either $\lim_{n\to\infty} BAx_n = Az$ or $\lim_{n\to\infty} BBx_n = Az$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ and $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} AAx_n = Az$, for some $z \in X$.

Similarly, can define weak B-compatibility of the pair (A, B). Clearly, both Definition 2.9 and 2.11 imply that the pair (A, B) is coincidentally commuting or a weakly compatible pair.

We observe that Definition 2.9 implies Definition 2.11. Note that a weakly A-compatible pair (A, B) need not be compatible.

Definition 2.12 ([14])**.** The pair (A, B) is said to be *A-continuous* if

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

In our next section, we use some implicit relations and obtained results by employing rational contractive condition.

3 Implicit Relations

Let Φ_6 denote the set of all continuous functions $\phi : [0,1]^6 \to R$ satisfying the conditions
$(\phi_1) : \phi_1$ is decreasing in t_2, t_3, t_4, t_5, and t_6,
$(\phi_2) : \phi_2(u, v, v, v, w, v) \geq 0$ implies $u \geq v$ and $u \geq w$ for all $u, v, w \in [0,1]$.

Example 3.1. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$.

Example 3.2. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \min\{t_i.t_j : i, j \in \{2, 3, 4, 5, 6\}\}$.

Example 3.3. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \min\{t_i.t_j.t_k : i, j, k \in \{2, 3, 4, 5, 6\}\}$.

Example 3.4. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\frac{t_2.t_4}{t_2+t_4} - b\frac{t_3.t_6}{t_5+t_4+1}, t_2 + t_4 \neq 0$ and $a, b > 0$.

3.1 Main Result

Theorem 3.5. *Let A, B, S and T be self maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geq t$, $\forall t \in [0,1]$ such that*

(3.1.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.1.2)

$$\phi\left(\begin{array}{c} M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Sx, (2-\alpha)t), \\ \frac{2M(Sx,Ty,t)}{M(Sx,Tyt)+M(Ax,Ty,t)}, \frac{M(Ax,Sx,t)+M(By,Ty,t)}{2} \end{array}\right) \geq 0$$

for all $x, y \in X$, $\forall t > 0$ and $\alpha \in (0,2)$, where $k \in (0,1)$ and $\phi \in \Phi_6$;
Further assume that

(3.1.3) (A, S) is weakly S-compatible, (B, T) is weakly T-compatible and either (A, S) is S-continuous or (B, T) is T-continuous;

(3.1.4) (A, S) is weakly A-compatible, (B, T) is weakly B-compatible and either (A, S) is f-continuous or (B, T) is B-continuous.
Then A, B, S and T have a unique common fixed point $z \in X$, and z is the unique common fixed point of A and S and of B and T.

Proof. Let $x_0 \in X$ be an arbitrary point. By (3.1.1), we can choose a sequence $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, ...,$. Let

$$d_m(t) = M(y_m, y_{m+1}, t), \ \forall t > 0.$$

Step 1. Putting $x = x_{2n}$, $y = x_{2n+1}$, $\alpha = 1 - q_1$ in (3.1.2), where $q_1 \in (k, 1)$, we have

$$0 \leq \phi \left(\begin{array}{c} M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n-1}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n-1}, \\ (1+q_1)t), \frac{2M(y_{2n}, y_{2n-1}, t)}{M(y_{2n}, y_{2n-1}, t) + M(y_{2n}, y_{2n-1}, t)}, \frac{M(y_{2n}, y_{2n-1}, t) + M(y_{2n}, y_{2n+1}, t)}{2} \end{array} \right)$$

$$0 \leq \phi \left(\begin{array}{c} M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n-1}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n-1}, y_{2n}, t) \\ *M(y_{2n+1}, y_{2n}, q_1 t), 1, \frac{M(y_{2n}, y_{2n-1}, t) + M(y_{2n}, y_{2n+1}, t)}{2} \end{array} \right).$$

And so
(i) $\phi(d_{2n}(kt), d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t) * d_{2n}(q_1 t), 1, \frac{d_{2n-1}(t) + d_{2n+1}(t)}{2} \geq 0$
If $d_{2n}(t) < d_{2n-1}(t)$, then
$d_{2n}(q_1 t) * d_{2n-1}(t) \geq d_{2n}(q_1 t) * d_{2n}(q_1 t) \geq d_{2n}(q_1 t)$,
and from (ϕ_1), we have
$\phi(d_{2n}(kt), d_{2n}(q_1 t), d_{2n}(q_1 t).d_{2n}(q_1 t) d_{2n}(q_1 t) d_{2n}(q_1 t)) \geq 0$.
Then again from (ϕ_2), we have
$d_{2n}(kt) > d_{2n}(q_1 t)$
a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(t)$, for every $n \in N$ and $\forall t > 0$.
Now from (i) and (ϕ_1) we have
$\phi(d_{2n}(kt), d_{2n-1}(q_1 t), d_{2n-1}(q_1 t) d_{2n-1}(q_1 t) d_{2n-1}(q_1 t) d_{2n-1}(q_1 t)) \geq 0$
and from (ϕ_2), we have
(ii) $d_{2n}(kt) > d_{2n-1}(q_1 t)$.
Step 2. Similarly, putting $x = x_{2n}$, $y = x_{2n-1}$, $\alpha = 1 - q_2$ in (3.1.2), where $q_2 \in (k, 1)$, we can show that (iii) $d_{2n-1}(kt) > d_{2n-2}(q_2 t)$,
Now let $q = \min\{q_1, q_2\}$ so that $q \in (k, 1)$. Then from (ii) and (iii) we have
$d_n(kt) \geq d_{2n-1}(qt)$.
For every $n \in N$, and so

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_{n-1}, y_n, (q/k)t) \\ &\geq M(y_{n-2}, y_{n-1}, (q/k)^2 t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq M(y_0, y_1, (q/k)^n t). \end{aligned}$$

Hence, by Lemma 2.7, $\{y_n\}$ is a Cauchy sequence and from the completeness of X, $\{y_n\}$ converges to some point z in X.
Now suppose that the conditions in (3.1.3) are true.
Step 3. Suppose that (A, S) is S-continuous. Then $SAx_{2n} \to Sz$ and $SSx_{2n} \to Sz$ as $n \to \infty$. Since (A, S) is weakly S-compatible, we have either $ASx_{2n} \to Sz$ or $AAx_{2n} \to Sz$ as $n \to \infty$.

Case 1. Suppose that $ASx_{2n} \to Sz$ as $n \to \infty$. Then putting $x = Sx_{2n}$, $y = x_{2n+1}$, $\alpha = 1$ in (3.1.2), we get

$$\phi\left(\begin{array}{c} M(ASx_{2n}, Bx_{2n+1}, kt), M(SSx_{2n}, Tx_{2n+1}, t), M(ASx_{2n}, SSx_{2n}, t), \\ M(Bx_{2n+1}, SSx_{2n}, (2-\alpha)t), \frac{2M(SSx_{2n}, Tx_{2n+1}, t)}{M(SSx_{2n}, Tx_{2n+1}t)+M(ASx_{2n}, Tx_{2n+1}, t)}, \\ \frac{M(ASx_{2n}, SSx_{2n}, t)+M(Bx_{2n+1}, Tx_{2n+1}, t)}{2} \end{array}\right)$$
$$\geq 0.$$

Letting $n \to \infty$, we have

$$0 \leq \phi(M(Sz, z, kt), M(Sz, z, t), 1, M(Sz, z, t), 1, 1),$$

$$0 \leq \phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), M(z, Sz, t)).$$

From (ϕ_2), we have $M(Sz, z, kt) \geq M(Sz, z, t)$, which implies by Lemma 2.8 that $Sz = z$.

Case 2. Suppose $AAx_{2n} \to Sz$ as $n \to \infty$. Putting $x = Ax_{2n}$, $y = x_{2n+1}$, $\alpha = 1$ in (3.1.2), we get

$$\phi\left(\begin{array}{c} M(AAx_{2n}, Bx_{2n+1}, kt), M(SAx_{2n}, Tx_{2n+1}, t), M(AAx_{2n}, SAx_{2n}, t), \\ M(Bx_{2n+1}, SAx_{2n}, (2-\alpha)t), \frac{2M(SAx_{2n}, Tx_{2n+1}, t)}{M(SAx_{2n}, Tx_{2n+1}t)+M(AAx_{2n}, Tx_{2n+1}, t)}, \\ \frac{M(AAx_{2n}, SAx_{2n}, t)+M(Bx_{2n+1}, Tx_{2n+1}, t)}{2} \end{array}\right)$$
$$\geq 0.$$

Letting $n \to \infty$, we have

$$0 \leq \phi(M(Sz, z, kt), M(Sz, z, t), 1, M(Sz, z, t), 1, 1),$$

$$0 \leq \phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), M(z, Sz, t)).$$

From (ϕ_2), we have $M(Sz, z, kt) \geq M(Sz, z, t)$, which implies that $Sz = z$.

Step 4. Putting $x = z$, $y = x_{2n+1}$, $\alpha = 1$ in (3.1.2), we have

$$\phi\left(\begin{array}{c} M(Az, Bx_{2n+1}, kt), M(Sz, Tx_{2n+1}, t), M(Az, Sz, t), M(Bx_{2n+1}, Sz, t), \\ \frac{2M(Sz, Tx_{2n+1}, t)}{M(Sz, Tx_{2n+1}t)+M(Az, Tx_{2n+1}, t)}, \frac{M(Az, Sz, t)+M(Bx_{2n+1}, Tx_{2n+1}, t)}{2} \end{array}\right) \geq 0.$$

Letting $n \to \infty$, we have

$$0 \leq \phi\left(M(Az, z, kt), 1, M(Az, z, t), 1, \frac{2}{1+M(Az, z, t)}, \frac{M(Az, z, t)+1}{2}\right).$$

From (ϕ_1) and (ϕ_2), we have $M(Az, z, kt) \geq M(Az, z, t)$, which implies that $Az = z$.

Step 5. Since $A(X) \subseteq T(X)$, there exists $w \in X$ such that $z = Az = Tw$. Putting $x = x_{2n}$, $y = w$, $\alpha = 1$ in (3.1.2), we have

$$\phi\left(\begin{array}{c} M(Ax_{2n}, Bw, kt), M(Sx_{2n}, Tw, t), M(Ax_{2n}, Sx_{2n}, t), M(Bw, Sx_{2n}, t), \\ \frac{2M(Sx_{2n}, Tw, t)}{M(Sx_{2n}, Tw, t)+M(Ax_{2n}, Tw, t)}, \frac{M(Ax_{2n}, Sx_{2n}, t)+M(Bw, Tw, t)}{2} \end{array}\right) \geq 0.$$

Letting $n \to \infty$, we have

$$\phi\left(M(z, Bw, kt), 1, 1, M(Bw, z, t), 1, \frac{1 + M(Bw, z, t)}{2}\right) \geq 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z, Bw, kt) \geq M(z, Bw, t)$, which implies that $Bw = z$.
Thus $Tw = Bw$.
Since (B, T) is weakly T-compatible it follows that (B, T) is a weakly compatible pair. Hence $TBw = BTw$, so that $Tz = Bz$.
Step 6. Putting $x = x_{2n}$, $y = z$, $\alpha = 1$ in (3.1.2) we have

$$\phi\left(\begin{array}{c} M(Ax_{2n}, Bz, kt), M(Sx_{2n}, Tz, t), M(Ax_{2n}, Sx_{2n}, t), M(Bz, Sx_{2n}, t), \\ \frac{2M(Sx_{2n}, Tz, t)}{M(Sx_{2n}, Tz, t) + M(Ax_{2n}, Tz, t)}, \frac{M(Ax_{2n}, Sx_{2n}, t) + M(Bz, Tz, t)}{2} \end{array}\right) \geq 0.$$

Letting $n \to \infty$, we have

$$\phi(M(z, Tz, kt), M(z, Tz, t), 1, M(z, Tz, t), 1, 1) \geq 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z, Tz, kt) \geq M(Tz, z, t)$, which implies that $Tz = z$.
Hence $Bz = Tz = z$ and so z is a common fixed point of A, B, S and T.
Step 7. Suppose that z_0 is another common fixed point of A, B, S and T. Putting $x = z$, $y = z_0$, $\alpha = 1$ in (3.1.2), we have

$$\phi\left(\begin{array}{c} M(Az, Bz_0, kt), M(Sz, Tz_0, t), M(Az, Sz, t), M(Bz_0, Sz, t), \\ \frac{2M(Sz, Tz_0, t)}{M(Sz, Tz_0 t) + M(Az, Tz_0, t)}, \frac{M(Az, Sz, t) + M(Bz_0, Tz_0, t)}{2} \end{array}\right) \geq 0.$$

This implies that

$$\phi(M(z, z_0, kt), M(z, z_0, t), 1, M(z_0, z, t), 1, 1) \geq 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z, z_0, kt) \geq M(z, z_0, t)$, which implies that $z = z_0$.
Hence z is the unique common fixed point of A, B, S and T.
Step 8. Suppose that z_1 is another common fixed point of A and S. Putting $x = z_1$, $y = z$, $\alpha = 1$ in (3.1.2), we have

$$\phi\left(\begin{array}{c} M(Az_1, Bz, kt), M(Sz_1, Tz, t), M(Az_1, Sz_1, t), M(Bz, Sz_1, t), \\ \frac{2M(Sz_1, Tz, t)}{M(Sz_1, Tz, t) + M(Az_1, Tz, t)}, \frac{M(Az_1, Sz_1, t) + M(Bz, Tz, t)}{2} \end{array}\right) \geq 0.$$

This implies that

$$\phi(M(z_1, z, kt), M(z_1, z, t), 1, M(z, z_1, t), 1, 1) \geq 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z_1, z, kt) \geq M(z_1, z, t)$, which implies that $z_1 = z$.
Hence z is the unique common fixed point of A and S.
Similarly we can show that z is the unique common fixed point of B and T.
Similarly we can prove the theorem if (B, T) is T-continuous. Also we can prove the theorem if the conditions in (3.1.4) are true. □

Example 3.6. Let $X = [0,1]$, $a * b = \min\{a,b\}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, define $Ax = Bx = 1$ and

$$Sx = Tx = \begin{cases} \frac{2+x}{3} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Then all the conditions of Theorem 3.5 are satisfied with $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4,t_5,t_6\}$. Clearly 1 is the unique common fixed point of A, B, S and T.

In our next section, we give another [2] it relations and obtained results in sequentially compact fuzzy metric space.

4 Implicit Relations

Let ψ_6 be the set of all functions $\psi : [0,1]^6 \to R$ such that
(ψ_1) : $\psi(v,u,u,v,w,1) > 0$ or $\psi(v,u,v,u,1,w) > 0$ implies $u < v$ for all $u, v \in [0,1)$ and $w \leq 1$,
(ψ_2) : $\psi(v,1,1,v,v,1) \leq 0$, $\psi(v,v,1,1,v,v) \leq 0$ and $\psi(v,1,v,1,1,v) \leq 0$ for all $v \in [0,1)$.

Example 4.1. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \{t_2,t_3,t_4\} - b(t_5,t_6)$, where $b \geq 0$.

Example 4.2. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \{t_2^2,t_3,t_4\} - b(t_5,t_6)$, where $b \geq 0$.

Example 4.3. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \{t_2,t_3,t_4\} - b(t_5^2 t_6 + t_5 t_6^2)$, where $b \geq 0$.

Example 4.4. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2.t_4}{t_2+t_4} - b\frac{t_5.t_6}{t_5+t_4+1}, t_2 + t_4 \neq 0$ and $a, b > 0$.

Theorem 4.5. *Let A, B, S and T be self mappings of a sequentially compact fuzzy metric space $(X, M, *)$ such that*

(1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(2)

$$\psi\begin{pmatrix} M(Ax,By,kt), \frac{M(Ax,Sx,t)+M(Sx,Ty,t)}{2}, M(Ty,By,t), M(Sx,By,t), \\ M(Ax,Ty,t), \frac{2M(Sx,Ty,t)}{M(Sx,Tyt)+M(Ax,Ty,t)} \end{pmatrix} > 0$$

for every $x, y \in X$, with one of $Ax \neq By$, $Ax \neq Sx$ and $By \neq Ty$ and $\forall t > 0$ where $\psi \in \psi_6$;

(3) (A,S) and (B,T) are weakly compatible;

(4) Either A and S are continuous or B and T are continuous.
Then A, B, S and T have a unique common fixed point in X, Further p is the unique common fixed point of A and S and of B and T.

Proof. Suppose that A and S are continuous and for any $t > 0$, let

$$m = \sup\{M(Ax, Sx, t) : x \in X\}.$$

Since A and S are continuous on a sequentially compact fuzzy metric space, there exists $u \in X$ such that $m = M(Au, Su, t)$. Since $S(X) \subseteq B(X)$, there exists $v \in X$ such that
(5) $Su = Bv$.
Since $T(X) \subseteq A(X)$, there exists $w \in X$ such that
(6) $Tv = Aw$.
Suppose neither A and S nor A and T have a coincidence point in X. Then
$m = M(Au, Su, t) < 1$,
$M(Bv, Tv, t) < 1$ and $M(Aw, Sw, t) < 1$. We have

$$\psi\left(\begin{array}{c} M(Ax, By, kt), \frac{M(Ax,Sx,t)+M(Sx,Ty,t)}{2}, M(Ty, By, t), M(Sx, By, t), \\ M(Ax, Ty, t), \frac{2M(Sx,Ty,t)}{M(Sx,Ty,t)+M(Ax,Ty,t)} \end{array}\right) > 0,$$

$$\begin{aligned} 0 <&\psi\left(\begin{array}{c} M(Au, Bv, kt), \frac{M(Au,Su,t)+M(Su,Tv,t)}{2}, M(Tv, Bv, t), M(Su, Bv, t), \\ M(Au, Tv, t), \frac{2M(Su,Tv,t)}{M(Su,Tv,t)+M(Au,Tv,t)} \end{array}\right) \\ &= \psi\left(M(Bv, Tv, kt), m, M(Tv, Bv, t), M(Tv, Bv, t), m, \frac{2m}{1+m}\right) \end{aligned}$$

and by (ψ_1), we have
(7) $m < M(Bv, Tv, t)$.
Now from (ψ_2), we have

$$\begin{aligned} 0 <&\psi\left(\begin{array}{c} M(Aw, Bv, kt), \frac{M(Aw,Sw,t)+M(Sw,Tv,t)}{2}, M(Tv, Bv, t), M(Sw, Bv, t), \\ M(Aw, Tv, t), \frac{2M(Sw,Tv,t)}{M(Sw,Tv,t)+M(Aw,Tv,t)} \end{array}\right) \\ &= \psi\left(\begin{array}{c} M(Aw, Sw, kt), \frac{M(Aw,Sw,t)+M(Sw,Tv,t)}{2}, M(Tv, Sw, t), 1, \\ M(Aw, Tv, t), \frac{2M(Sw,Tv,t)}{M(Sw,Tv,t)+M(Aw,Tv,t)} \end{array}\right). \end{aligned}$$

By (ψ_1), we have
(8) $M(Bv, Tv, t) < M(Aw, Sw, t)$.
Now from the definition of m and the inequalities (7) and (8), we have

$$m \geq M(Aw, Sw, t) > M(Bv, Tv, t) > m,$$

a contradiction. Hence there exists $\alpha \in X$ such that $A\alpha = S\alpha$ or $B\alpha = T\alpha$.
Case (a): Suppose that $S\alpha = A\alpha$. Since $A(X) \subseteq T(X)$, there exists $\alpha \in X$ such that $A\alpha = T\alpha$. Suppose that $M(T\beta, B\beta, t) < 1$. then from (2) we have

$$0 < \psi\left(\begin{array}{c} M(A\alpha, B\beta, kt), \frac{M(A\alpha,S\alpha,t)+M(S\alpha,T\beta,t)}{2}, M(T\beta, B\beta, t), M(S\alpha, B\beta, t), \\ M(A\alpha, T\beta, t), \frac{2M(S\alpha,T\beta,t)}{M(S\alpha,T\beta,t)+M(A\alpha,T\beta,t)} \end{array}\right),$$

$$0 < \psi\left(\begin{array}{c} M(T\alpha, B\beta, kt), \frac{M(A\alpha,A\alpha,t)+M(A\alpha,B\beta,t)}{2}, M(B\beta, B\beta, t), M(A\alpha, B\beta, t), \\ M(A\alpha, A\beta, t), \frac{2M(A\alpha,A\alpha,t)}{M(A\alpha,A\alpha,t)+M(A\alpha,A\alpha,t)} \end{array}\right),$$

$$0 < \psi\left(M(T\beta, B\beta, kt), \frac{M(T\alpha, B\alpha, t)+1}{2}, 1, 1, M(T\alpha, B\beta, t), 1\right).$$

By (2), we have $M(T\beta, B\beta, t) = 1$, so that $T\beta = B\beta$. Thus
(9) $S\alpha = A\alpha = T\beta = B\beta = p$, say.
Since the pair (A, S) is weakly compatible, we have
(10) $Sp = SA\alpha = AS\alpha = Ap$.
Suppose that $M(Ap, p, t) < 1$. From (2), we have

$$\begin{aligned}
0 < &\psi\left(\begin{array}{c} M(Ap, B\beta, kt), \frac{M(Ap,Sp,t)+M(Sp,T\beta,t)}{2}, M(T\beta, B\beta, t), M(Sp, B\beta, t), \\ M(Ap, T\beta, t), \frac{2M(Sp,T\beta,t)}{M(Sp,T\beta,t)+M(Ap,T\beta,t)} \end{array}\right) \\
&= \psi\left(\begin{array}{c} M(Ap, p, kt), \frac{M(Ap,Sp,t)+M(Ap,p,t)}{2}, M(p, p, t), M(Ap, p, t), \\ M(Ap, p, t), \frac{2M(Ap,p,t)}{M(Ap,p,t)+M(Ap,p,t)} \end{array}\right) \\
&= \psi\left(M(Ap, p, kt), \frac{M(Ap, p, t)+1}{2}, 1, M(Ap, p, t), M(Ap, p, t), 1\right).
\end{aligned}$$

Hence from (ψ_2), we have $Ap = p$. Thus (11) $Tp = Bp = p$.
Since the pair (B, T) is weakly compatible, we have
$Tp = TB\beta = BT\beta = Tp$.

Using (2) with $x = \alpha$ and $y = p$ and (ψ_2) we can show that $Bp = p$. Thus,
(12) $Tp = Bp = p$.
Hence p is a common fixed point of A, B, S and T.
Case (b): Suppose that $T\alpha = B\beta$. Since $B(X) \subseteq T(X)$, there exists $\alpha \in X$ such that $B\alpha = S\beta$. Suppose that $M(S\beta, A\beta, t) < 1$. From (2), we have

$$0 < \psi\left(\begin{array}{c} M(A\beta, B\alpha, kt), \frac{M(S\beta,T\alpha,t)+M(A\beta,S\beta,t)}{2}, M(T\alpha, B\alpha, t), M(S\beta, B\alpha, t), \\ M(A\beta, T\alpha, t), \frac{2M(S\beta,T\alpha,t)}{M(S\beta,T\alpha,t)+M(A\beta,T\alpha,t)} \end{array}\right),$$

$$0 < \psi\left(\begin{array}{c} M(A\beta, T\beta, kt), \frac{M(S\beta,S\beta,t)+M(A\beta,S\beta,t)}{2}, M(B\alpha, B\alpha, t), M(B\alpha, B\alpha, t), \\ M(A\beta, B\alpha, t), \frac{2M(B\alpha,B\alpha,t)}{M(B\alpha,B\alpha,t)+M(A\beta,B\alpha,t)} \end{array}\right).$$

Hence from (ψ_2),We have $A\beta = T\beta$. Thus $A\beta = T\beta = B\alpha = S\alpha = p$, say. Now as in case (a), we can show that p is a common fixed point of A, B, S and T. Suppose that p_0 is another common fixed point of A, B, S and T. Using (2) with $x = p$, $y = p_0$ and (ψ_2), we can show that $p_0 = p$. Thus p is the unique common fixed point of A, B, S and T.

Now suppose that p_1 is another common fixed point of A and S. Using (2) with $x = p_1$, $y = p$ and (ψ_2), we can show that $p_1 = p$. Thus p is the unique common fixed point of A and S. Similarly we can show that p is the unique common fixed point of B and T. Similarly the theorem holds when B and T are continuous. □

Remark. Theorem 4.5 holds if the inequality (2) is replaced by one of the following inequalities:

(a) $M(Ax, By, t) > \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\}$,
(b) $M^2(Ax, By, t) > \min\{M^2(Sx, Ty, t), M(Sx, Ax, t)M(Ty, By, t)\}$,
(c) $M^3(Ax, By, t) > M(Sx, Ty, t)M(Sx, Ax, t)M(Ty, By, t)$,
(d) $M(Ax, By, t) > \min\Big\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ax, Ty, t), M(Ty, By, t), \frac{(M(Ax,By,t)+M(By,Ty,t)}{2})\Big\}$.

Example 4.6. Let $X = [0, 1]$, $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, define $Ax = Bx = 1$, $Sx = \frac{x+2}{3}$ and $Tx = \frac{3+x}{4}$ for all $x \in X$. Then all the conditions of Theorem 4.5 are satisfied with $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4\}$. Clearly 1 is the unique common fixed point of A, B, S and T.

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