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Some Characterizations of Anti-Fuzzy (Generalized) Bi-Ideals of Semigroups

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Abstract : Our aim in this paper is to characterize anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of a semigroup S. We define certain subsets of S, $[0,1]$ and $S \times [0,1]$. The relationships between sets of anti-fuzzy points and the certain subsets of $S \times [0,1]$ are investigated. Some interesting characterizations of anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of semigroups are investigated by using the certain subsets of S, $[0,1]$ and $S \times [0,1]$.

Keywords : semigroups; anti-fuzzy points; anti-fuzzy subsemigroups; anti-fuzzy generalized bi-ideals; anti-fuzzy bi-ideals.
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1 Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [1], plays a major role in mathematics with wide applications in many other branches e.g. theoretical physics, computer science, control engineering, information science, measure theory. Rosenfeld [2] gave definitions of a fuzzy subgroupoid and a fuzzy subgroup, and obtained some properties of them. Since then, many fuzzy algebraic struc-

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tures have been rapidly introduced and discussed by many authors (for example, see [3–12]).

On anti-fuzzy algebraic structures, Biswas [13] introduced the concept of anti-fuzzy subgroups of groups and lower level sets of fuzzy subsets, and also showed that a fuzzy subset f of a group G is an anti-fuzzy subgroup of G if and only if for every $\alpha \in [0, 1]$, a lower level set $L(f : \alpha) = \{x \in G \mid f(x) \leq \alpha\}$ is either empty or a subgroup of G. The concept of lower level sets of fuzzy subsets is one of mathematical methods for studying anti-fuzzy algebraic structures, some papers used the concept of lower level sets seen in [14–23]. Modifying and applying Biswas' idea, concepts of many types of anti-fuzzy algebraic structures have been introduced and studied extensively by many authors. For example, Shabir and Nawas [22] in 2009 introduced the concept of an anti-fuzzy (generalized) bi-ideal of any semigroup S and characterized anti-fuzzy (generalized) bi-ideals by using lower level sets. Moreover, they characterized semigroups in terms of anti-fuzzy (generalized) bi-ideals. Khan and Asif [18], the continuation of the work carried out by Shabir and Nawas, introduced anti-fuzzy interior ideals of S and characterized semigroups by the properties of anti-fuzzy (generalized) bi-ideals and anti-fuzzy interior ideals. Khan et al. [24] gave relationships between anti-fuzzy (generalized) bi-ideals and anti-fuzzy right ideals on semilattice of left groups. Characterizations of semilattices of left (right) groups are investigated by using anti-fuzzy (generalized) bi-ideals and anti-fuzzy one-sided ideals [24]. Due to these possibilities of applications, semigroups and related structures are studied via anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals.

Our propose of this work is to promote and develop anti-fuzzy algebraic structures by studying anti-fuzzy semigroup theory. We define the certain subsets of S, $[0, 1]$ and $S \times [0, 1]$ and investigate their properties. In particular, we define a certain subset $L(\mathcal{R} : \alpha)$ of S where $\mathcal{R}$ is a subset of $S \times [0, 1]$ and this set is a general concept of the lower level set of a fuzzy set. We also describe relationship between sets of anti-fuzzy points and the certain subsets of $S \times [0, 1]$. Some interesting characterizations of anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of semigroups are investigated by using the certain subsets of S, $[0, 1]$ and $S \times [0, 1]$. Moreover, we show that any fuzzy subset of S is an anti-fuzzy (generalized) bi-ideal if and only if there exists the unique chain of (generalized) bi-ideals of S together with two special conditions.

2 Preliminaries

In this section, we give basic definitions and results, which will be used in the next sections. A semigroup is an algebraic system $(S, \cdot)$ consisting of a nonempty set S together with an associative binary operation "$\cdot$". Throughout this paper, S stands for a semigroup. For nonempty subsets A and B of S, we denote $AB = \{ab \mid a \in A, b \in B\}$. A nonempty subset A of S is called a **subsemigroup** of S if $AA \subseteq A$. A nonempty subset A of S is called a **generalized bi-ideal** of S if $ASA \subseteq A$. A subsemigroup A of S is called a **bi-ideal** of S if $ASA \subseteq A$. By the above definitions, it is obvious that every bi-ideal of S is a generalized bi-ideal, but the converse is not true in general.

A function f from S to the real closed interval $[0,1]$ is called a **fuzzy subset** (or **fuzzy set**) [1] of S. For $x \in S$, define $F_x = \{(y,z) \in S \times S \mid x = yz\}$. Let f and g be fuzzy subsets of S, then their anti-product $f \bullet g$ [18] is defined by for all $x \in S$

$$(f \bullet g)(x) = \begin{cases} \inf\{\max\{f(y), g(z)\} \mid (y,z) \in F_x\}, & \text{if } F_x \neq \emptyset; \\ 1, & \text{otherwise.} \end{cases}$$

For $x \in S$, a fuzzy subset f of S of the form

$$f(y) = \begin{cases} \alpha \in [0,1), & \text{if } x = y; \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in S$ is called an **anti-fuzzy point** [17] with support x and value α and is denoted by x^α. We denote by $AFP(S)$ the set of all anti-fuzzy points of S, that is,

$$AFP(S) = \{x^\alpha \mid x \in S, \alpha \in [0,1)\}.$$

Then $(AFP(S), \bullet)$ is a semigroup and we conveniently denote it by $AFP(S)$. Indeed, we see that for all $x^\alpha, y^\beta, z^\gamma \in AFP(S)$ $x^\alpha \bullet y^\beta = (xy)^{\max\{\alpha,\beta\}}$ and $(x^\alpha \bullet y^\beta) \bullet z^\gamma = (xyz)^{\max\{\alpha,\beta,\gamma\}} = x^\alpha \bullet (y^\beta \bullet z^\gamma)$. For all $A, B \subseteq AFP(S)$, we define the product of two sets A and B as $A \bullet B = \{x^\alpha \bullet y^\beta \mid x^\alpha \in A, y^\beta \in B\}$. For every fuzzy subset f of S, let $\bar{f} = \{x^\alpha \in AFP(S) \mid f(x) \leq \alpha\}$. Note that $\bar{f}$ is empty if and only if $f(x) = 1$ for all $x \in S$.

Definition 2.1. [18] A fuzzy subset f of a semigroup S is called an **anti-fuzzy subsemigroup** of S if $f(ab) \leq \max\{f(a), f(b)\}$ for all $a, b \in S$.

Definition 2.2. [18] A fuzzy subset f of a semigroup S is called an **anti-fuzzy generalized bi-ideal** of S if $f(axb) \leq \max\{f(a), f(b)\}$ for all $a, b, x \in S$.

Definition 2.3. [18] An anti-fuzzy subsemigroup f of a semigroup S is called an **anti-fuzzy bi-ideal** of S if $f(axb) \leq \max\{f(a), f(b)\}$ for all $a, b, x \in S$.

Define a binary operation "$\diamond$" on $S \times [0,1]$ as follows: for all $(x,\alpha), (y,\beta) \in S \times [0,1]$

$$(x,\alpha) \diamond (y,\beta) = (xy, \max\{\alpha,\beta\}). \tag{2.1}$$

Then $(S \times [0,1], \diamond)$ is a semigroup. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be subsets of $S \times [0,1]$. Define the multiplication $\mathcal{R}_1 \diamond \mathcal{R}_1$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ as follows:

$$\mathcal{R}_1 \diamond \mathcal{R}_2 \quad = \quad \{(a,\alpha) \diamond (b,\beta) \mid (a,\alpha) \in \mathcal{R}_1 \text{ and } (b,\beta) \in \mathcal{R}_2\}. \tag{2.2}$$

For every subsemigroup A of S and nonempty subset Δ of $[0,1]$, we have $(A \times \Delta, \diamond)$ is a subsemigroup of $(S \times [0,1], \diamond)$. In what follows, let $S \times \Delta$ denote the semigroup $(S \times \Delta, \diamond)$. Let f be a fuzzy subset of S, $A \subseteq S$, $\alpha \in [0,1]$, $\Delta \subseteq [0,1]$ and

$\mathcal{R} \subseteq S \times [0,1]$. We give the certain subsets of S, $[0,1]$ and $S \times [0,1]$ as the following.

$$[A \times \Delta]_f = \{(x,\alpha) \in A \times \Delta \mid f(x) \leq \alpha\}. \tag{2.3}$$
$$L(\mathcal{R} : \alpha) = \{x \in S \mid (x,\beta) \in \mathcal{R} \text{ and } \beta \leq \alpha \text{ for some } \beta \in [0,1]\}. \tag{2.4}$$
$$(Imf)^{\alpha} = \{\beta \in Imf \mid \beta \leq \alpha\}. \tag{2.5}$$

In particular, if $\mathcal{R}$ is a fuzzy subset of S, then

$$L(\mathcal{R} : \alpha) = \{x \in S \mid \mathcal{R}(x) \leq \alpha\}.$$

If $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$, then $L(\mathcal{R} : \alpha) \subseteq L(\mathcal{R} : \beta)$ and hence the set $\{L(\mathcal{R} : \alpha) \mid \alpha \in [0,1]\}$ is a chain of subsets of S under the inclusion relation "$\subseteq$".

Proposition 2.4. *Let f be a fuzzy subset of a semigroup S. Then the following statements are true.*

(i) $(Imf)^{\alpha} \subseteq Imf$ *for all* $\alpha \in [0,1]$.

(ii) $L(f : \alpha) = \bigcup_{\gamma \in (Imf)^{\alpha}} f^{-1}(\gamma) = f^{-1}((Imf)^{\alpha})$ *for all* $\alpha \in [0,1]$.

(iii) $[S \times \Delta]_f = \bigcup_{\gamma \in \Delta} (L(f : \gamma) \times \{\gamma\})$ *for all* $\Delta \subseteq [0,1]$.

(iv) *If* $\Delta \subseteq [0,1]$ *and* $\mathcal{R} = [S \times \Delta]_f$, *then* $L(\mathcal{R} : \alpha) = L(f : \alpha)$ *for all* $\alpha \in \Delta$.

Proposition 2.5. *Let S be a semigroup, Δ be a nonempty subset of $[0,1]$ and $\mathcal{R}$ be a subsemigroup of $S \times \Delta$. Then $L(\mathcal{R} : \alpha)$ is either empty or a subsemigroup of S for all $\alpha \in \Delta$.*

Proposition 2.6. *Let S be a semigroup, Δ be a nonempty subset of $[0,1]$ and $\mathcal{R}$ be a generalized bi-ideal of $S \times \Delta$. Then $L(\mathcal{R} : \alpha)$ is either empty or a generalized bi-ideal of S for all $\alpha \in \Delta$.*

Proposition 2.7. *Let S be a semigroup, Δ be a nonempty subset of $[0,1]$ and $\mathcal{R}$ be a bi-ideal of $S \times \Delta$. Then $L(\mathcal{R} : \alpha)$ is either empty or a bi-ideal of S for all $\alpha \in \Delta$.*

3 Anti-Fuzzy Subsemigroups of Semigroups

In this section, we characterize anti-fuzzy subsemigroups of a semigroup S by using the certain subsets of S, $[0,1]$, $AFP(S)$ and $S \times [0,1]$.

For the following theorem, we discuss characterizations of anti-fuzzy subsemigroups of S via the certain subsets of $[0,1]$ and $S \times [0,1]$.

Theorem 3.1. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is an anti-fuzzy subsemigroup of S.*

(ii) *For every subsemigroup A of S and $\Delta \subseteq [0,1]$, we have $[A \times \Delta]_f$ is either empty or a subsemigroup of $S \times \Delta$.*

(iii) *$[S \times \Delta]_f$ is a subsemigroup of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0,1]$.*

(iv) *For all $a, b \in S$, $(Imf)^{f(ab)} \subseteq (Imf)^{f(a)} \cup (Imf)^{f(b)}$.*

Proof. $(i \Rightarrow ii)$ Let A be a subsemigroup of S, $\Delta \subseteq [0,1]$ and $(a, \alpha), (b, \beta) \in [A \times \Delta]_f$. Then $f(a) \le \alpha$, $f(b) \le \beta$ and $\max\{\alpha, \beta\} \in \Delta$. Since f is an anti-fuzzy subsemigroup of S and A is a subsemigroup of S, we have $ab \in A$ and

$$f(ab) \le \max\{f(a), f(b)\} \le \max\{\alpha, \beta\}.$$

Thus $(a, \alpha) \diamond (b, \beta) \in [A \times \Delta]_f$. Hence $[A \times \Delta]_f$ is a subsemigroup of $S \times \Delta$.
$(ii \Rightarrow iii)$ It is obvious.
$(iii \Rightarrow iv)$ Suppose that $\alpha \in (Imf)^{f(ab)}$ and $\alpha \notin (Imf)^{f(a)} \cup (Imf)^{f(b)}$ for some $a, b \in S$, $\alpha \in [0,1]$. Then $\max\{f(a), f(b)\} < \alpha \le f(ab)$. By the statement (iii) and $(a, f(a)), (b, f(b)) \in [S \times Imf]_f$, we have $(a, f(a)) \diamond (b, f(b)) \in [S \times Imf]_f$. Hence $f(ab) \le \max\{f(a), f(b)\}$. It is a contradiction. Therefore $(Imf)^{f(ab)} \subseteq (Imf)^{f(a)} \cup (Imf)^{f(b)}$ for all $a, b \in S$.
$(iv \Rightarrow i)$ It is straightforward. □

By using and applying Theorem 3.1, we have Corollary 3.2.

Corollary 3.2. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is an anti-fuzzy subsemigroup of S.*

(ii) *$[S \times [0,1)]_f$ is either empty or a subsemigroup of $S \times [0,1)$.*

(iii) *$[S \times Imf]_f$ is a subsemigroup of $S \times Imf$.*

(iv) *$[S \times [0,1]]_f$ is a subsemigroup of $S \times [0,1]$.*

Example 3.3. Let $S = \{a, b, c, d\}$ and define a binary operation "$\cdot$" on S as follows :

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then $(S, \cdot)$ is a semigroup. Let f be a fuzzy subset of S such that

$$f(a) = f(b) = 0.1, \quad f(c) = 0.5, \quad f(d) = 0.7.$$

Thus, by routine calculations, we can check that $[S \times Imf]_f$=$\{(a, 0.1), (a, 0.5), (a, 0.7), (b, 0.1), (b, 0.5), (b, 0.7), (c, 0.5), (c, 0.7), (d, 0.7)\}$ is a subsemigroup of $S \times Imf$. By Corollary 3.2$(iii \Rightarrow i)$, we have f is an anti-fuzzy subsemigroup of S.

Proposition 3.4. *Let f be a fuzzy subset of a semigroup S. Then $[S \times [0,1)]_f$ is a subsemigroup of $S \times [0,1)$ if and only if $\bar{f}$ is a subsemigroup of $AFP(S)$.*

Proof. It is straightforward. □

Theorem 3.5. *Let f be a fuzzy subset of a semigroup S. Then f is an anti-fuzzy subsemigroup of S if and only if $\bar{f}$ is either empty or a subsemigroup of $AFP(S)$.*

Proof. It follows from Corollary 3.2($i \Leftrightarrow ii$) and Proposition 3.4. □

In the following theorem, we characterize anti-fuzzy subsemigroups of a semigroup S by chain of subsemigroups of S.

Theorem 3.6. *Let f be a fuzzy subset of a semigroup S. Then f is an anti-fuzzy subsemigroup of S if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of subsemigroups of S such that*

i) *$f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in Imf$ and*

ii) *for all $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\alpha \subset A_\beta$ and $A_\alpha \cap f^{-1}(\beta) = \emptyset$.*

Proof. ($\Rightarrow$) For each $\alpha \in Imf$, we choose $A_\alpha = L(f : \alpha)$. By Proposition 2.4(iv), Proposition 2.5 and Theorem 3.1($i \Rightarrow iii$), we get $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S. By Proposition 2.4(ii), we have the conditions i) and ii). Suppose that $\{B_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S with the conditions i) and ii). Let $\alpha \in Imf$ and $a \in B_\alpha$. If $\alpha < f(a)$ then by the condition ii), we have $B_\alpha \cap f^{-1}(f(a)) = \emptyset$. Since $a \in f^{-1}(f(a))$, we get $a \in B_\alpha \cap f^{-1}(f(a))$. It is a contradiction. Thus $f(a) \leq \alpha$, so $a \in L(f : \alpha) = A_\alpha$. Hence $B_\alpha \subseteq A_\alpha$. Let $a \in A_\alpha$. Then $f(a) \leq \alpha$. By the conditions i) and ii), we get

$$a \in f^{-1}(f(a)) \subseteq B_{f(a)} \subseteq B_\alpha.$$

Hence $A_\alpha \subseteq B_\alpha$. Therefore $A_\alpha = B_\alpha$.
($\Leftarrow$) Let $(a,\alpha),(b,\beta) \in [S \times Imf]_f$. Then $f(a) \leq \alpha$, $f(b) \leq \beta$ and $\max\{\alpha,\beta\} \in Imf$. Suppose that $\max\{\alpha,\beta\} < f(ab)$. By the condition ii), we have $A_{\max\{\alpha,\beta\}} \cap f^{-1}(f(ab)) = \emptyset$. Since $f(a) \leq \max\{\alpha,\beta\}$ and by the conditions i) and ii), we have

$$a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\max\{\alpha,\beta\}}.$$

In the same way, we have $b \in A_{\max\{\alpha,\beta\}}$. Since $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S, we get $ab \in A_{\max\{\alpha,\beta\}}$. Then $ab \in A_{\max\{\alpha,\beta\}} \cap f^{-1}(f(ab)) = \emptyset$. It is a contradiction. Thus $f(ab) \leq \max\{\alpha,\beta\}$. Hence $(a,\alpha) \diamond (b,\beta) \in [S \times Imf]_f$. Therefore $[S \times Imf]_f$ is a subsemigroup of $S \times Imf$. By Corollary 3.2($iii \Rightarrow i$), we have f is an anti-fuzzy subsemigroup of S. □

In the proof of Theorem 3.6, the unique chain of subsemigroups of S, satisfying conditions i) and ii), is the set $\{L(f : \alpha) \mid \alpha \in Imf\}$. Next, we consider one formula of an anti-fuzzy subsemigroup f of a semigroup where Imf is finite.

Corollary 3.7. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 < \alpha_2 < ... < \alpha_n$. Then f is an anti-fuzzy subsemigroup of S if and only if $\{L(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of subsemigroups of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in L(f : \alpha_n)\backslash L(f : \alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1})\backslash L(f : \alpha_{n-2}) \\ ... & \\ \alpha_2 & \text{if } x \in L(f : \alpha_2)\backslash L(f : \alpha_1) \\ \alpha_1 & \text{if } x \in L(f : \alpha_1) \end{cases}$$

for all $x \in S$.

Proof. Apply Theorem 3.6. □

Corollary 3.8. *Let f be a fuzzy subset of a semigroup S and $Imf \subseteq \Delta \subseteq [0, 1]$. The following statements are equivalent.*

(i) *f is an anti-fuzzy subsemigroup of S.*

(ii) *There exists a subsemigroup $\mathcal{R}$ of $S \times \Delta$ such that $L(\mathcal{R} : \alpha) = L(f : \alpha)$ for all $\alpha \in \Delta$.*

(iii) *$L(f : \alpha)$ is either empty or a subsemigroup of S for all $\alpha \in \Delta$.*

Proof. $(i \Rightarrow ii)$ Choose $\mathcal{R} = [S\times\Delta]_f$ and use Theorem 3.1$(i \Rightarrow iii)$ and Proposition 2.4(iv).
$(ii \Rightarrow iii)$ It follows from Proposition 2.5.
$(iii \Rightarrow i)$ Apply Theorem 3.6. □

4 Anti-Fuzzy (Generalized) Bi-Ideals of Semigroups

In this section, characterizations of anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of a semigroup S are investigated by using the certain subsets of S, $[0, 1]$, $AFP(S)$ and $S \times [0, 1]$.

In the following theorem, we characterize anti-fuzzy generalized bi-ideals of a semigroup S by the certain subsets of $[0, 1]$ and $S \times [0, 1]$.

Theorem 4.1. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is an anti-fuzzy generalized bi-ideal of S.*

(ii) *For every generalized bi-ideal A of S and $\Delta \subseteq [0, 1]$, we have $[A \times \Delta]_f$ is either empty or a generalized bi-ideal of $S \times \Delta$.*

(iii) *$[S \times \Delta]_f$ is a generalized bi-ideal of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.*

(iv) *For all $a, b, x \in S$, $(Imf)^{f(axb)} \subseteq (Imf)^{f(a)} \cup (Imf)^{f(b)}$.*

Proof. ($i \Rightarrow ii$) Let A be a generalized bi-ideal of S, $\Delta \subseteq [0,1]$, $(x,\gamma) \in S \times \Delta$ and $(a,\alpha),(b,\beta) \in [A \times \Delta]_f$. Then $f(a) \leq \alpha$, $f(b) \leq \beta$ and $\max\{\alpha,\beta,\gamma\} \in \Delta$. Since f is a fuzzy generalized bi-ideal of S and A is a generalized bi-ideal of S, we get $axb \in A$ and

$$f(axb) \leq \max\{f(a), f(b)\} \leq \max\{\alpha,\beta\} \leq \max\{\alpha,\beta,\gamma\}.$$

Thus $(a,\alpha) \diamond (x,\gamma) \diamond (b,\beta) \in [A \times \Delta]_f$. Hence $[A \times \Delta]_f$ is a generalized bi-ideal of $S \times \Delta$.
($ii \Rightarrow iii$) It is obvious.
($iii \Rightarrow iv$) Suppose that $\alpha \in (Imf)^{f(axb)}$ and $\alpha \notin (Imf)^{f(a)} \cup (Imf)^{f(b)}$ for some $a,b,x \in S$ and $\alpha \in [0,1]$. Then $\max\{f(a), f(b)\} < \alpha \leq f(axb)$. Since $(a,f(a)),(b,f(b)) \in [S \times Imf]_f$, $(x,f(a)) \in S \times Imf$ and the statement (iii), we have $(a,f(a)) \diamond (x,f(a)) \diamond (b,f(b)) \in [S \times Imf]_f$. Thus $f(axb) \leq \max\{f(a), f(b)\}$. It is a contradiction. Hence $(Imf)^{f(axb)} \subseteq (Imf)^{f(a)} \cup (Imf)^{f(b)}$ for all $a,b,x \in S$.
($iv \Rightarrow i$) It is straightforward. □

By using and applying Theorem 4.1, we get Corollary 4.2.

Corollary 4.2. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is an anti-fuzzy generalized bi-ideal of S.*

(ii) *$[S \times [0,1)]_f$ is either empty or a generalized bi-ideal of $S \times [0,1)$.*

(iii) *$[S \times Imf]_f$ is a generalized bi-ideal of $S \times Imf$.*

(iv) *$[S \times [0,1]]_f$ is a generalized bi-ideal of $S \times [0,1]$.*

Example 4.3. Let $S = \{a,b,c,d\}$ be the semigroup under the same binary operation in Example 3.3. Let f be a fuzzy subset of S such that $f(a) = 0.3$, $f(b) = 0.5$, $f(c) = 0.4$, $f(d) = 0.6$. Then $[S \times Imf]_f = \{(a,0.3), (a,0.4), (a,0.5), (a,0.6), (b,0.5), (b,0.6), (c,0.4), (c,0.5), (c,0.6), (d,0.6)\}$ is a generalized bi-ideal of $S \times Imf$. By Corollary 4.2($i \Rightarrow iii$), we get f is an anti-fuzzy generalized bi-ideal of S.

Proposition 4.4. *Let f be a fuzzy subset of a semigroup S. Then $[S \times [0,1)]_f$ is a generalized bi-ideal of $S \times [0,1)$ if and only if $\bar{f}$ is a generalized bi-ideal of $AFP(S)$.*

Proof. It is straightforward. □

Theorem 4.5. *Let f be a fuzzy subset of a semigroup S. Then f is an anti-fuzzy generalized bi-ideal of S if and only if $\bar{f}$ is either empty or a generalized bi-ideal of $AFP(S)$.*

Proof. It follows from Corollary 4.2($i \Leftrightarrow ii$) and Proposition 4.4. □

In the following theorem, we characterize anti-fuzzy generalized bi-ideal of a semigroup S by chain of generalized bi-ideals of S.

Theorem 4.6. *Let f be a fuzzy subset of a semigroup S. Then f is an anti-fuzzy generalized bi-ideal of S if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of generalized bi-ideals of S such that*

i) *$f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in Imf$ and*

ii) *for all $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\alpha \subset A_\beta$ and $A_\alpha \cap f^{-1}(\beta) = \emptyset$.*

Proof. ($\Rightarrow$) Choose $A_\alpha = L(f : \alpha)$ for all $\alpha \in Imf$. By Proposition 2.4(iv), Proposition 2.6 and Theorem 4.1($i \Rightarrow iii$), we get $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of generalized bi-ideals of S satisfying the conditions i) and ii). For the proof of uniqueness, it is similar to the proof of Theorem 3.6.
($\Leftarrow$) Let $(a, \alpha), (b, \beta) \in [S \times Imf]_f$ and $(x, \gamma) \in S \times Imf$. Then $\max\{\alpha, \beta, \gamma\} \in Imf$ and

$$\max\{f(a), f(b)\} \leq \max\{\alpha, \beta\} \leq \max\{\alpha, \beta, \gamma\}.$$

Suppose that $\max\{\alpha, \beta, \gamma\} < f(axb)$. By the condition ii), we get $A_{\max\{\alpha,\beta,\gamma\}} \cap f^{-1}(f(axb)) = \emptyset$. Since $f(a) \leq \max\{\alpha, \beta, \gamma\}$ and by the conditions i) and ii), we have

$$a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\max\{\alpha,\beta,\gamma\}}.$$

Similarly, we have $b \in A_{\max\{\alpha,\beta,\gamma\}}$. Since $A_{\max\{\alpha,\beta,\gamma\}}$ is a generalized bi-ideal of S, we have $axb \in A_{\max\{\alpha,\beta,\gamma\}}$. Then $axb \in A_{\max\{\alpha,\beta,\gamma\}} \cap f^{-1}(f(axb)) = \emptyset$. It is a contradiction. Thus $f(axb) \leq \max\{\alpha, \beta, \gamma\}$. Hence $(a, \alpha) \diamond (x, \gamma) \diamond (b, \beta) \in [S \times Imf]_f$. Therefore $[S \times Imf]_f$ is a generalized bi-ideal of $S \times Imf$. By Corollary 4.2($iii \Rightarrow i$), we get f is an anti-fuzzy generalized bi-ideal of S. □

In the proof of Theorem 4.6, the unique chain of generalized bi-ideals of S, satisfying conditions i) and ii), is the set $\{L(f : \alpha) \mid \alpha \in Imf\}$. Next, we consider one formula of an anti-fuzzy generalized bi-ideal f of S where Imf is finite.

Corollary 4.7. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 < \alpha_2 < ... < \alpha_n$. Then f is an anti-fuzzy generalized bi-ideal of S if and only if $\{L(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of generalized bi-ideals of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in L(f : \alpha_n) \backslash L(f : \alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1}) \backslash L(f : \alpha_{n-2}) \\ ... & \\ \alpha_2 & \text{if } x \in L(f : \alpha_2) \backslash L(f : \alpha_1) \\ \alpha_1 & \text{if } x \in L(f : \alpha_1) \end{cases}$$

for all $x \in S$.

Proof. Apply Theorem 4.6. □

Corollary 4.8. *Let f be a fuzzy subset of a semigroup S and $Imf \subseteq \Delta \subseteq [0,1]$. The following statements are equivalent.*

(i) *f is an anti-fuzzy generalized bi-ideal of S.*

(ii) *There exists a generalized bi-ideal $\mathcal{R}$ of $S \times \Delta$ such that $L(\mathcal{R} : \alpha) = L(f : \alpha)$ for all $\alpha \in \Delta$.*

(iii) *$L(f : \alpha)$ is either empty or a generalized bi-ideal of S for all $\alpha \in \Delta$.*

Proof. $(i \Rightarrow ii)$ Choose $\mathcal{R} = [S\times\Delta]_f$ and use Theorem 4.1$(i \Rightarrow iii)$ and Proposition 2.4(iv).
$(ii \Rightarrow iii)$ It follows from Proposition 2.6.
$(iii \Rightarrow i)$ Apply Theorem 4.6. □

In the following two results, we characterize anti-fuzzy bi-ideal of a semigroup S by using the certain subsets of S, $[0,1]$, $AFP(S)$ and $S \times [0,1]$.

Theorem 4.9. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is an anti-fuzzy bi-ideal of S.*

(ii) *$[A \times \Delta]_f$ is either empty or a bi-ideal of $S \times \Delta$ for every bi-ideal A of S and every subset Δ of $[0,1]$.*

(iii) *$[S \times \Delta]_f$ is a bi-ideal of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0,1]$.*

(iv) *$\bar{f}$ is either empty or a bi-ideal of $AFP(S)$.*

(v) *There exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of bi-ideals of S such that*

  a) *$f^{-1}(\alpha) \subseteq A_\alpha$ for every $\alpha \in Imf$ and*

  b) *for every $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\alpha \subset A_\beta$ and $A_\alpha \cap f^{-1}(\beta) = \emptyset$.*

(vi) *Choosing $Imf \subseteq \Delta \subseteq [0,1]$, we have $L(f : \alpha)$ is either empty or a bi-ideal of S for every $\alpha \in \Delta$.*

(vii) *Choosing $Imf \subseteq \Delta \subseteq [0,1]$, there exists a bi-ideal $\mathcal{R}$ of $S \times \Delta$ such that $L(\mathcal{R} : \alpha) = L(f : \alpha)$ for every $\alpha \in \Delta$.*

(viii) *$(Imf)^{f(axb)} \cup (Imf)^{f(ab)} \subseteq (Imf)^{f(a)} \cup (Imf)^{f(b)}$ for every $a, b, x \in S$.*

Corollary 4.10. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 < \alpha_2 < ... < \alpha_n$. Then f is an anti-fuzzy bi-ideal of S if and only if $\{L(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of bi-ideals of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in L(f : \alpha_n)\backslash L(f : \alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1})\backslash L(f : \alpha_{n-2}) \\ ... & \\ \alpha_2 & \text{if } x \in L(f : \alpha_2)\backslash L(f : \alpha_1) \\ \alpha_1 & \text{if } x \in L(f : \alpha_1) \end{cases}$$

for all $x \in S$.

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