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Approximate Bi-Additive Mappings in Intuitionistic Fuzzy Normed Spaces

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Abstract : In this paper, we determine some stability results concerning a 2-dimensional vector variable bi-additive functional equation in intuitionistic fuzzy normed spaces (IFNS). We generalize the intuitionistic fuzzy continuity to the bi-additive mappings and we prove that the existence of a solution for any approximately bi-additive mapping implies the completeness of IFNS.

Keywords : intuitionistic fuzzy normed spaces; generalized Ulam-Rassias stability; functional equations.
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1 Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. This new theory was introduced by Zadeh [1], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], fuzzy physics [6], fuzzy topology [7], fuzzy stability [8–12], nonlinear operators [13], statistical convergence [14, 15], etc. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [16]. In [17], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces (see [18–21]).

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- $(N1)$ $N(x, c) = 0$ for $c \leqslant 0$;
- $(N2)$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
- $(N3)$ $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- $(N4)$ $N(x + y, s + t) \geqslant \min\{N(x, s), N(y, t)\}$;
- $(N5)$ $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
- $(N6)$ For $x \neq 0, N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement the norm of x is less than or equal to the real number t.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [22] in 1940 and affirmatively solved by Hyers [23]. The result of Hyers was generalized by Aoki [24] for approximate additive function and by Rassias [25] for approximate linear functions by allowing the difference Cauchy equation $\|f(x_1 + x_2) - f(x_1) - f(x_2)\|$ to be controlled by $\varepsilon(\|x_1\|^p + \|x_2\|^p)$. Taking into consideration a lot of influencke of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the generalized Ulam-Rassias stability or Hyers-Ulam-Rassias stability (see [26–28]). In 1994, a generalization of Rassias theorem was obtained by Găvruta [29], who replaced $\varepsilon(\|x_1\|^p + \|x_2\|^p)$ by a general control function $\varphi(x_1, x_2)$.

The stability problem for the 2-dimensional vector variable bi-additive functional equation was proved by the authors [30] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banach space. In this paper, we determine some stability results concerning the 2-dimensional vector variable bi-additive functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w) \tag{1.1}$$

in intuitionistic fuzzy normed spaces. We apply the intuitionistic fuzzy continuity of the 2-dimensional vector variable bi-additive mappings and prove that the existence of a solution for any approximately 2-dimensional vector variable bi-additive mapping implies the completeness of intuitionistic fuzzy normed spaces (IFNS). It has shown that each mapping satisfies in (1.1) is $\mathbb{C}$-bilinear (see [31]).

In the following section, we recall some notations and basic definitions used in this paper.

2 Preliminaries

We use the definition of intuitionistic fuzzy normed spaces given in [16, 32, 33] to investigate some stability results for the functional equation (1.1) in the intuitionistic fuzzy normed vector space setting.

Definition 2.1 ([34]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:
(a) is commutative and associative;
(b) is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 ([34]). A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:
(a) is commutative and associative;
(b) is continuous;
(c) $a \circ 0 = a$ for all $a \in [0,1]$;
(d) $a \circ b \leqslant c \circ d$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [16], have introduced the concept of intuitionistic fuzzy normed space.

Definition 2.3 ([16,32]). The five-tuple $(X, \mu, \nu, *, \circ)$ is said to be an *intuitionistic fuzzy normed space* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions: For every $x, y \in X$ and $s, t > 0$,
(IF_1) $\mu(x,t) + \nu(x,t) \leqslant 1$;
(IF_2) $\mu(x,t) > 0$;
(IF_3) $\mu(x,t) = 1$ if and only if $x = 0$;
(IF_4) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(IF_5) $\mu(x,t) * \mu(y,s) \leqslant \mu(x+y, t+s)$;
(IF_6) $\mu(x, .) : (0,\infty) \longrightarrow [0,1]$ is continuous;
(IF_7) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to\infty} \mu(x,t) = 0$;
(IF_8) $\nu(x,t) < 1$;
(IF_9) $\nu(x,t) = 0$ if and only if $x = 0$;
(IF_{10}) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(IF_{11}) $\nu(x,t) \circ \nu(y,s) \geqslant \nu(x+y, t+s)$;
(IF_{12}) $\nu(x, .) : (0,1) \longrightarrow [0,1]$ is continuous;
(IF_{13}) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

Example 2.4. Let $(X, \|.\|)$ be a normed space, $a*b = ab$ and $a \circ b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leqslant 0 \end{cases} \quad \text{and} \quad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leqslant 0. \end{cases}$$

Then $(X, \mu, \nu, *, \circ)$ is an IFNS.

Remark 2.5. *In intuitionistic fuzzy normed space* $(X, \mu, \nu, *, \circ)$, $\mu(x, .)$ *is non-decreasing and* $\nu(x, .)$ *is non-increasing for all* $x \in X$ *(see* [16]*).*

Definition 2.6. Let $(X, \mu, \nu, *, \circ)$ be an IFNS. A sequence $\{x_n\}$ is said to be *intuitionistic fuzzy convergent* to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$. A sequence $\{x_n\}$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim_{k\to\infty} \mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_{k+p} - x_k, t) = 0$ for all $p \in \mathbb{N}$ and all $t > 0$. Then IFNS $(X, \mu, \nu, *, \circ)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \circ)$ intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \circ)$ and $(X, \mu, \nu, *, \circ)$ is also called an *intuitionistic fuzzy Banach space.*

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [16].

3 Intuitionistic Fuzzy Stability

For notational convenience, given a function $f : X \times X \to Y$, we define the difference operator

$$D_b f(x, y, z, w) = f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w).$$

We begin with a generalized Hyers-Ulam type theorem in IFNS for the functional equation (1.1).

Theorem 3.1. *Let* X *be a linear space and let* $(Z, \mu', \nu,)$ *be an IFNS. Let* $\varphi : X \times X \times X \times X \to Z$ *be a mapping such that, for some* $0 < \alpha < 4$.

$$\begin{cases} \mu'(\varphi(2x, 2y, 2z, 2w), t) \geqslant \mu'(\alpha\varphi(x, y, z, w), t), \\ \nu'(\varphi(2x, 2y, 2z, 2w), t) \leqslant \nu'(\alpha\varphi(x, y, z, w), t), \end{cases} \tag{3.1}$$

for all $x, y, z, w \in X$ *and all* $t > 0$*. Let* (Y, μ, ν) *be an intuitionistic fuzzy Banach space and let* $f : X \times X \to Y$ *be a mapping such that*

$$\begin{cases} \mu(D_b f(x, y, z, w), t) \geqslant \mu'(\varphi(x, y, z, w), t), \\ \nu(D_b f(x, y, z, w), t) \leqslant \nu'(\varphi(x, y, z, w), t) \end{cases} \tag{3.2}$$

for all $x, y, z, w \in X$ *and all* $t > 0$*. Then there exists a unique mapping* F :

$X \times X \to Y$ satisfying (1.1) such that

$$\begin{cases} \mu\Big(F(x,y) - f(x,y) + \frac{1}{3}f(0,0), t\Big) \\ \quad \geqslant *^{\infty}\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \\ \quad\quad *^{\infty}\mu\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(F(x,y) - f(x,y) + \frac{1}{3}f(0,0), t\Big) \\ \quad \leqslant \circ^{\infty}\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \circ^{\infty} \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \\ \quad\quad \circ^{\infty}\nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{3.3}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^{\infty}a := a * a * \cdots$ and $\circ^{\infty}a := a \circ a \circ \cdots$ for all $a \in [0, 1]$.

Proof. Put $y = -x$ and $w = z$ in (3.2) to obtain

$$\begin{cases} \mu(f(2x, 2z) - 2f(x, z) - 2f(-x, z) + f(0, 0), t) \geqslant \mu'(\varphi(x, -x, z, z), t), \\ \nu(f(2x, 2z) - 2f(x, z) - 2f(-x, z) + f(0, 0), t) \leqslant \nu'(\varphi(x, -x, z, z), t) \end{cases} \tag{3.4}$$

for all $x, z \in X$ and all $t > 0$. Let $x = z = 0$ in (3.2), we get

$$\begin{cases} \mu(f(y, -w) + f(-y, w) + 2f(y, w) - 2f(0, 0), t) \geqslant \mu'(\varphi(0, y, 0, w), t), \\ \nu(f(y, -w) + f(-y, w) + 2f(y, w) - 2f(0, 0), t) \leqslant \nu'(\varphi(0, y, 0, w), t) \end{cases} \tag{3.5}$$

for all $y, w \in X$ and all $t > 0$. Replacing y by x and w by z in (3.5), we get

$$\begin{cases} \mu(f(x, -z) + f(-x, z) + 2f(x, z) - 2f(0, 0), t) \geqslant \mu'(\varphi(0, x, 0, z), t), \\ \nu(f(x, -z) + f(-x, z) + 2f(x, z) - 2f(0, 0), t) \leqslant \nu'(\varphi(0, x, 0, z), t) \end{cases} \tag{3.6}$$

for all $x, z \in X$ and all $t > 0$. Putting $x = y$ and $w = -z$ in (3.2), we obtain

$$\begin{cases} \mu(f(2x, 2z) - 2f(x, z) + 2f(x, -z) + f(0, 0), t) \geqslant \mu'(\varphi(x, x, z, -z), t), \\ \nu(f(2x, 2z) - 2f(x, z) + 2f(x, -z) + f(0, 0), t) \leqslant \nu'(\varphi(x, x, z, -z), t) \end{cases} \tag{3.7}$$

for all $x, z \in X$ and all $t > 0$. By inequalities (3.4) and (3.7), we get

$$\begin{cases} \mu(2f(-x,z) - 2f(x,-z) + 2f(0,0), t) \geqslant \mu'(\varphi(x,x,z,-z), \frac{t}{2}) * \mu'(\varphi(x,-x,z,z), \frac{t}{2}), \\ \nu(2f(-x,z) - 2f(x,-z) + 2f(0,0), t) \leqslant \nu'(\varphi(x,x,z,-z), \frac{t}{2}) \circ \nu'(\varphi(x,-x,z,z), \frac{t}{2}) \end{cases} \tag{3.8}$$

for all $x, z \in X$ and all $t > 0$. And from (3.8), we can write

$$\begin{cases} \mu(f(-x,z) - f(x,-z) + f(0,0), t) \geqslant \mu'(2\varphi(x, x, z, -z), \frac{t}{2}) * \mu'(2\varphi(x, -x, z, z), \frac{t}{2}), \\ \nu(f(-x,z) - f(x,-z) + f(0,0), t) \leqslant \nu'(2\varphi(x, x, z, -z), \frac{t}{2}) \circ \nu'(2\varphi(x, -x, z, z), \frac{t}{2}) \end{cases} \tag{3.9}$$

for all $x, z \in X$ and all $t > 0$. By (3.6) and (3.7), we have

$$\begin{cases} \mu(f(2x,2z) - 4f(x,z) + f(x,-z) - f(-x,z) + 3f(0,0), t) \\ \qquad \geqslant \mu'(\varphi(x,x,z,-z), \frac{t}{2}) * \mu'(\varphi(0,x,0,z), \frac{t}{2}), \\ \nu(f(2x,2z) - 4f(x,z) + f(x,-z) - f(-x,z) + 3f(0,0), t) \\ \qquad \leqslant \nu'(\varphi(x,x,z,-z), \frac{t}{2}) \circ \nu'(\varphi(0,x,0,z), \frac{t}{2}) \end{cases} \tag{3.10}$$

for all $x, z \in X$ and all $t > 0$. From (3.9) and (3.10), we get

$$\begin{aligned} &\mu(f(2x,2z) - 4f(x,z) + 4f(0,0), t) \\ &\quad \geqslant \mu'(2\varphi(x,x,z,-z), \frac{t}{4}) * \mu'(\varphi(x,x,z,-z), \frac{t}{4}) \\ &\quad * \mu'(\varphi 2(x,-x,z,z), \frac{t}{4}) * \mu'(\varphi(0,x,0,z), t) \\ &\quad \geqslant * \mu'(\varphi(x,x,z,-z), \frac{t}{8}) * \mu'(\varphi(x,x,z,-z), \frac{t}{8}) \\ &\quad * \mu'(\varphi(x,-x,z,z), \frac{t}{8}) * \mu'(\varphi(0,x,0,z), \frac{t}{8}) \\ &\quad = *^3 \mu'(\varphi(x,x,z,-z), \frac{t}{8}) * \mu'(\varphi(x,-x,z,z), \frac{t}{8}) * \mu'(\varphi(0,x,0,z), \frac{t}{8}), \end{aligned}$$

and also

$$\begin{aligned} &\nu(f(2x,2z) - 4f(x,z) + 4f(0,0), t) \\ &\qquad \leqslant \circ^2 \nu'(\varphi(x,x,z,-z), \frac{t}{2}) \circ \nu'(\varphi(x,-x,z,z), \frac{t}{2}) \circ \nu'(\varphi(0,x,0,z), \frac{t}{2}) \end{aligned}$$

for all $x, z \in X$ and all $t > 0$. We can write above inequalities as following

$$\begin{cases} \mu\Big(\frac{f(2x,2z)+f(0,0)}{4} - f(x,z), \frac{t}{4}\Big) \\ \quad \geqslant *^2 \mu'(\varphi(x,x,z,-z), \frac{t}{8}) * \mu'(\varphi(x,-x,z,z), \frac{t}{8}) * \mu'(\varphi(0,x,0,z), \frac{t}{8}), \\ \nu\Big(\frac{f(2x,2z)+f(0,0)}{4} - f(x,z), \frac{t}{4}\Big) \\ \quad \leqslant \circ^2 \nu'(\varphi(x,x,z,-z), \frac{t}{8}) \circ \nu'(\varphi(x,-x,z,z), \frac{t}{8}) \circ \nu'(\varphi(0,x,0,z), \frac{t}{8}) \end{cases} \tag{3.11}$$

for all $x, z \in X$ and all $t > 0$. Replacing x by $2^n x$ and z by $2^n z$ in (3.11) and using (3.1), we get

$$\begin{cases} \mu\Big(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}} - \frac{f(2^n x,2^n z)}{4^n}, \frac{t}{4^{n+1}}\Big) \\ \quad \geqslant *^2\mu'(\varphi(2^n x, 2^n x, 2^n z, -2^n z), \frac{t}{8}) *^2 \mu'(\varphi(2^n x, -2^n x, 2^n z, 2^n z), \frac{t}{8}) \\ \qquad *\mu'(\varphi(0, 2^n x, 0, 2^n z), \frac{t}{8}), \\ \quad \geqslant *^2\mu'(\varphi(x,x,z,-z), \frac{t}{8\alpha^n}) * \mu'(\varphi(x,-x,z,z), \frac{t}{8\alpha^n}) * \mu'(\varphi(0,x,0,z), \frac{t}{8\alpha^n}), \\ \nu\Big(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}} - \frac{f(2^n x,2^n z)}{4^n}, \frac{t}{4^{n+1}}\Big) \\ \quad \leqslant \circ^2\nu'(\varphi(2^n x, 2^n x, 2^n z, -2^n z), \frac{t}{8}) \circ \nu'(\varphi(2^n x, -2^n x, 2^n z, 2^n z), \frac{t}{8}) \\ \qquad \circ\nu'(\varphi(0, 2^n x, 0, 2^n z), \frac{t}{8}) \\ \quad \leqslant \circ^2\nu'(\varphi(x,x,z,-z), \frac{t}{8\alpha^n}) \circ \nu'(\varphi(x,-x,z,z), \frac{t}{8\alpha^n}) \circ \nu'(\varphi(0,x,0,z), \frac{t}{8\alpha^n}) \end{cases}$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. By replacing t by $\alpha^n t$ in above inequalities, we have

$$\begin{cases} \mu\Big(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}} - \frac{f(2^n x,2^n z)}{4^n}, \frac{\alpha^n t}{4^{n+1}}\Big) \\ \quad \geqslant *^2\mu'(\varphi(x,x,z,-z), \frac{t}{8}) * \mu'(\varphi(x,-x,z,z), \frac{t}{8}) * \mu'(\varphi(0,x,0,z), \frac{t}{8}), \\ \nu\Big(\frac{f(2^{n+1}x,2^{n+1}z)+f(0,0)}{4^{n+1}} - \frac{f(2^n x,2^n z)}{4^n}, \frac{\alpha^n t}{4^{n+1}}\Big) \\ \quad \leqslant \circ^2\nu'(\varphi(x,x,z,-z), \frac{t}{8}) \circ \nu'(\varphi(x,-x,z,z), \frac{t}{8}) \circ \nu'(\varphi(0,x,0,z), \frac{t}{8}) \end{cases} \tag{3.12}$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. It follows from

$$\sum_{k=0}^{n-1}\Big[\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}} - \frac{f(2^k x,2^k z)}{4^k}\Big] = \frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\Big(1-\frac{1}{4^n}\Big)f(0,0)$$

and (3.12),

$$\begin{cases} \mu\Big(\frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \geqslant \prod_{k=0}^{n-1}\mu\Big(\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}} - \frac{f(2^k x,2^k z)}{4^k}, \frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \geqslant *^{2n}\mu'(\varphi(x,x,z,-z), \frac{t}{8}) *^n \mu'(\varphi(x,-x,z,z), \frac{t}{8}) *^n \mu'(\varphi(0,x,0,z), \frac{t}{8}), \\ \nu\Big(\frac{f(2^n x,2^n z)}{4^n} - f(x,z) + \frac{1}{3}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \leqslant \coprod_{k=0}^{n-1}\nu\Big(\frac{f(2^{k+1}x,2^{k+1}z)+f(0,0)}{4^{k+1}} - \frac{f(2^k x,2^k z)}{4^k}, \frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \leqslant \circ^{2n}\nu'(\varphi(x,x,z,-z), \frac{t}{8}) \circ^n \nu'(\varphi(x,-x,z,z), \frac{t}{8}) \circ^n \nu'(\varphi(0,x,0,z), \frac{t}{8}) \end{cases} \tag{3.13}$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$, where $\prod_{j=1}^n a_j := a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^n a_j := a_1 \circ a_2 \circ \cdots \circ a_n$, $*^n a := \prod_{j=1}^n a = \underbrace{a * \cdots * a}_{n\ times}$ and $\circ^n a := \coprod_{j=1}^n a =$

$\underbrace{a \circ \cdots \circ a}_{n\ times}$ for all $a, a_1, a_2, \cdots, a_n \in [0,1]$. By replacing x with $2^m x$ and z with $2^m z$ in (3.13), we have

$$\begin{cases} \mu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+m+1}}\Big) \\ \quad \geqslant *^{2n}\mu'(\varphi(2^m x, 2^m x, 2^m z, -2^m z), \frac{t}{8}) *^n \mu'(\varphi(2^m x, -2^m x, 2^m z, 2^m z), \frac{t}{8}) \\ \qquad *^n \mu'(\varphi(0, 2^m x, 0, 2^m z), \frac{t}{8}), \\ \quad \geqslant *^{2n}\mu'(\varphi(x,x,z,-z), \frac{t}{8\alpha^m}) *^n \mu'(\varphi(x,-x,z,z), \frac{t}{8\alpha^m}) * 6n\mu'(\varphi(0,x,0,z), \frac{t}{8\alpha^m}), \\ \nu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^{k+m+1}}\Big) \\ \quad \leqslant \circ^{2n}\nu'(\varphi(2^m x, 2^m x, 2^m z, -2^m z), \frac{t}{8}) \circ^n \nu'(\varphi(2^m x, -2^m x, 2^m z, 2^m z), \frac{t}{8}) \\ \qquad \circ^n \nu'(\varphi(0, 2^m x, 0, 2^m z), \frac{t}{8}) \\ \quad \leqslant \circ^{2n}\nu'(\varphi(x,x,z,-z), \frac{t}{8\alpha^m}) \circ^n \nu'(\varphi(x,-x,z,z), \frac{t}{8\alpha^m}) \circ^n \nu'(\varphi(0,x,0,z), \frac{t}{8\alpha^m}) \end{cases}$$

for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. So we have gotten that

$$\begin{cases} \mu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=m}^{n+m-1}\frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \geqslant *^{2n}\mu'(\varphi(x,x,z,-z), \frac{t}{8}) *^n \mu'(\varphi(x,-x,z,z), \frac{t}{8}) *^n \mu'(\varphi(0,x,0,z), \frac{t}{8}), \\ \nu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), \sum_{k=m}^{n+m-1}\frac{\alpha^k t}{4^{k+1}}\Big) \\ \quad \leqslant \circ^{2n}\nu'(\varphi(x,x,z,-z), \frac{t}{8}) \circ^n \nu'(\varphi(x,-x,z,z), \frac{t}{8}) \circ^n \nu'(\varphi(0,x,0,z), \frac{t}{8}) \end{cases}$$

for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. Replacing t by $\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^k}}$, we obtain

$$\begin{cases} \mu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), t\Big) \\ \quad \geqslant *^{2n}\mu'(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}) *^n \mu'(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}) \\ \qquad *^n \mu'(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^{k+1}}{4^k}}), \\ \nu\Big(\frac{f(2^{n+m}x,2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x,2^m z)}{4^m} + \frac{1}{3.4^m}\Big(1-\frac{1}{4^n}\Big)f(0,0), t\Big) \\ \quad \leqslant \circ^{2n}\nu'(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}) \circ^n \nu'(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}) \\ \qquad \circ^n \nu'(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}) \end{cases} \tag{3.14}$$

for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. Since $0 < \alpha < 4$, $\sum_{k=0}^{\infty}(\frac{\alpha}{4})^k < \infty$ and $\sum_{k=m}^{n+m-1}(\frac{\alpha^k}{4^k}) \to 0$ as $m \to \infty$ for all $n \in \mathbb{N}$. Thus $\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^k}} \to \infty$ and

$$*^2\mu'\Big(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{\alpha^k}{4^{k+1}}}\Big) * \mu'\Big(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}(\frac{\alpha^k}{4^k})}\Big)$$
$$* \mu'\Big(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}(\frac{\alpha^k}{4^{k+1}})}\Big) \longrightarrow 1$$

as $m \to \infty$ for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. Hence the Cauchy criterion for convergence in IFNS shows that $\left(\frac{f(2^n x, 2^n z)}{4^n}\right)$ is a Cauchy sequence in (Y, μ, ν) for all $x, z \in X$. Since (Y, μ, ν) is complete, then this sequence converges to some point $F(x, z) \in Y$ defined by $F(x, y) = \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n}$ for all $x, z \in X$. Now by putting $m = 0$ in (3.14), we obtain

$$\begin{cases} \mu\Big(\frac{f(2^n x, 2^n z)}{4^n} - f(x, z) + \frac{1}{3}\Big(1 - \frac{1}{4^n}\Big) f(0, 0), t\Big) \\ \quad \geqslant *^{2n} \mu'(\varphi(x, x, z, -z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}) *^n \mu'(\varphi(x, -x, z, z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}) \\ \qquad *^n \mu'(\varphi(0, x, 0, z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}), \\ \nu\Big(\frac{f(2^n x, 2^n z)}{4^n} - f(x, z) + \frac{1}{3}\Big(1 - \frac{1}{4^n}\Big) f(0, 0), t\Big) \\ \quad \leqslant \circ^{2n} \nu'(\varphi(x, x, z, -z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}) \circ^n \nu'(\varphi(x, -x, z, z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^{k+1}}{4^k}}) \\ \qquad \circ^n \nu'(\varphi(0, x, 0, z), \frac{t}{8\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}) \end{cases}$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. By taking limit from above inequalities as $n \to \infty$ and using the definition of IFNS, we get

$$\begin{cases} \mu\Big(F(x, y) - f(x, y) + \frac{1}{3} f(0, 0), t\Big) \geqslant *^\infty \mu'\Big(\varphi(x, x, z, -z), \frac{(4-\alpha)}{8} t\Big) \\ \qquad *^\infty \mu'\Big(\varphi(x, -x, z, z), \frac{(4-\alpha)}{8} t\Big) *^\infty \mu'\Big(\varphi(0, x, 0, z), \frac{(4-\alpha)}{8} t\Big), \\ \nu\Big(F(x, y) - f(x, y) + \frac{1}{3} f(0, 0), t\Big) \leqslant \circ^\infty \nu'\Big(\varphi(x, x, z, -z), \frac{(4-\alpha)}{8} t\Big) \\ \qquad \circ^\infty \nu'\Big(\varphi(x, -x, z, z), \frac{(4-\alpha)}{} t\Big) \circ^\infty \nu'\Big(\varphi(0, x, 0, z), \frac{(4-\alpha)}{8} t\Big) \end{cases}$$

for all $x, z \in X$ and all $t > 0$, which are the desired inequalities (3.3).

Now we show that F satisfies in (1.1). Replacing x, y, z, w and t in (3.2) respectively by $2^n x, 2^n y, 2^n z, 2^n w$ and $4^n t$, we get

$$\begin{cases} \mu\Big(\frac{f(2^n x + 2^n y, 2^n z - 2^n w)}{4^n} + \frac{f(2^n x - 2^n y, 2^n z + 2^n w)}{4^n} - 2\frac{f(2^n x, 2^n z)}{4^n} + 2\frac{f(2^n y, 2^n w)}{4^n}, t\Big) \\ \geqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) \geqslant \mu'(\varphi(x, y, z, w), \frac{4^n t}{\alpha^n}) \\ \nu\Big(\frac{f(2^n x + 2^n y, 2^n z - 2^n w)}{4^n} + \frac{f(2^n x - 2^n y, 2^n z + 2^n w)}{4^n} - 2\frac{f(2^n x, 2^n z)}{4^n} + 2\frac{f(2^n y, 2^n w)}{4^n}, t\Big) \\ \leqslant \nu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) \leqslant \nu'(\varphi(x, y, z, w), \frac{4^n t}{\alpha^n}) \end{cases}$$

for all $x, y, z, w \in X$ all $n \in \mathbb{N}$ and all $t > 0$. Since $\frac{4^n t}{\alpha^n} \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mu'(\varphi(x, ny, z, nw), \frac{4^n t}{\alpha^n}) = 1$$

and

$$\lim_{n\to\infty} \nu'(\varphi(x, ny, z, nw), \frac{4^n t}{\alpha^n}) = 0$$

for all $x, y, z, w \in X$ and all $t > 0$.

To prove the uniqueness of the mapping F, assume that there exists a mapping $G : X \times X \to Y$ which satisfies (1.1) and (3.3). For fix $x, y \in X$, we know that $F(2^n x, 2^n y) = 4^n F(x, y)$ and $G(2^n x, 2^n y) = 4^n G(x, y)$ for all $n \in \mathbb{N}$. It follows from (3.3) that

$$\begin{aligned}
\mu(F(x,y) - G(x,y), t) &= \mu\Big(\frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n y)}{4^n}, t\Big) \\
&\geqslant \mu\Big(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{3.4^n} f(0,0), \frac{t}{2}\Big) \\
&\quad * \mu\Big(- \frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3.4^n} f(0,0), \frac{t}{2}\Big) \\
&\geqslant *^2 *^\infty \mu'\Big(\varphi(2^n x, 2^n x, 2^n y, -2^n y), \frac{4^n(4-\alpha)t}{16}\Big) \\
&\quad *^2 *^\infty \mu'\Big(\varphi(2^n x, -2^n x, 2^n y, 2^n y), \frac{4^n(4-\alpha)t}{16}\Big) \\
&\quad *^2 *^\infty \mu'\Big(\varphi(0, 2^n x, 0, 2^n y), \frac{4^n(4-\alpha)t}{16}\Big) \\
&\geqslant *^2 *^\infty \mu'\Big(\varphi(x, x, y, -y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
&\quad *^2 *^\infty \mu'\Big(\varphi(x, -x, y, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
&\quad *^2 *^\infty \mu'\Big(\varphi(0, x, 0, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big)
\end{aligned}$$

for all $x, y \in X$, all $n \in \mathbb{N}$ and all $t > 0$, and similarly

$$\begin{aligned}
\nu(F(x,y) - G(x,y), t) \leqslant \circ^2 \circ^\infty \nu'\Big(\varphi(x, x, y, -y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
\circ^2 \circ^\infty \nu'\Big(\varphi(x, -x, y, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
\circ^2 \circ^\infty \nu'\Big(\varphi(0, x, 0, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big)
\end{aligned}$$

for all $x, y \in X$, all $n \in \mathbb{N}$ and all $t > 0$. Since $\lim_{n\to\infty} \frac{4^n(4-\alpha)t}{4\alpha^n} = \infty$ for all $t > 0$, we get

$$\begin{aligned}
\lim_{n\to\infty} \mu'\Big(\varphi(x, x, y, -y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) * \mu'\Big(\varphi(x, -x, y, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
* \mu'\Big(\varphi(0, x, 0, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) = 1
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \nu'\Big(\varphi(x, x, y, -y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \circ \nu'\Big(\varphi(x, -x, y, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) \\
\circ \nu'\Big(\varphi(0, x, 0, y), \frac{4^n(4-\alpha)t}{16\alpha^n}\Big) = 0
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Therefore $\mu(F(x,y)-G(x,y),t) = 1$ and $\nu(F(x,y)-G(x,y),t) = 0$ for all $t > 0$. Thus it is concluded that $F(x,y) = G(x,y)$. $\square$

Example 3.2. Let X be a Hilbert space with inner product $\langle .,. \rangle$ and Z be a normed spaced. Denote by (μ,ν) and (μ',ν') the intuitionistic fuzzy norms given as in Example 2.4 on X and Z, respectively. Let $\|.\|$ be induced norm on X by the inner product $\langle .,. \rangle$ on X. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping defined by $\varphi(x,y,z,w) = 2(\|x\|+\|y\|+\|z\|+\|w\|)z_0$ for all $x,y,z,w \in X$, where z_0 is a fixed unit vector in Z. Define a mapping $f : X \times X \to X$ by $f(x,y) := \langle x, y+x_0\rangle x_0$ for all $x, y \in X$, where x_0 is a fixed unit vector in X. Then

$$\mu(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w),t) = \mu(2\langle y,x_0\rangle x_0,t)$$
$$= \frac{t}{t+2|\langle y,x_0\rangle|} \geqslant \frac{t}{t+2\|y\|} \geqslant \frac{t}{t+2(\|x\|+\|y\|+\|z\|+\|w\|)} = \mu'(\varphi(x,y,z,w),t)$$

and

$$\nu(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w),t) = \nu(2\langle y,x_0\rangle x_0,t)$$
$$= \frac{2|\langle y,x_0\rangle|}{t+2|\langle y,x_0\rangle|} \leqslant \frac{2\|y\|}{t+\|y\|} \leqslant \frac{2(\|x\|+\|y\|+\|z\|+\|w\|)}{t+2(\|x\|+\|y\|+\|z\|+\|w\|)} = \nu'(\varphi(x,y,z,w),t)$$

for all $x,y,z,w \in X$ and all $t > 0$. Also we can get

$$\mu'(\varphi(2x,2y,2z,2w),t) = \frac{t}{t+4(\|x\|+\|y\|+\|z\|+\|w\|)} = \mu'(2\varphi(x,y,z,w),t)$$

and

$$\nu'(\varphi(2x,2y,2z,2w),t) = \frac{4(\|x\|+\|y\|+\|z\|+\|w\|)}{t+4(\|x\|+\|y\|+\|z\|+\|w\|)} = \nu'(2\varphi(x,y,z,w),t)$$

for all $x,y,z,w \in X$ and all $t > 0$. Therefore

$$\lim_{n\to\infty} \mu'(\varphi(2x,2y,2z,2w),4^n t) = \lim_{n\to\infty} \frac{4^n t}{4^n t + 2^{n+1}(\|x\|+\|y\|+\|z\|+\|w\|)} = 1$$

and

$$\lim_{n\to\infty} \nu'(\varphi(2x,2y,2z,2w),4^n t) = \lim_{n\to\infty} \frac{2^{n+1}(\|x\|+\|y\|+\|z\|+\|w\|)}{4^n t + 2^{n+1}(\|x\|+\|y\|+\|z\|+\|w\|)} = 0$$

for all $x,y,z,w \in X$ and all $t > 0$. Hence the assumptions of Theorem 3.1 for $\alpha = 2$ are fulfilled. Therefore, there exist a unique bi-additive mapping $F : X \times X \to X$ such that

$$\mu(F(x,y)-f(x,y),t) \geqslant *^2\mu'(4(\|x\|+\|y\|)z_0,t) * \mu'(2(\|x\|+\|y\|)z_0,t)$$

and

$$\nu(F(x,y)-f(x,y),t) \leqslant \circ^2\nu'(4(\|x\|+\|y\|)z_0,t) \circ \nu'(2(\|x\|+\|y\|)z_0,t)$$

for all $x, y \in X$ and all $t > 0$.

The following theorem will be proved the case $\alpha > 4$.

Theorem 3.3. *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times \to Z$ be a maping such that, for some $\alpha > 4$,*

$$\mu'\Big(\varphi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\Big), t\Big) \geqslant \mu'(\varphi(x, y, z, w), \alpha t),$$

$$\nu'\Big(\varphi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\Big), t\Big) \leqslant \nu'(\varphi(x, y, z, w), \alpha t),$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (3.2) with $f(0, 0) = 0$. Then there exists a unique mapping $F : X \times X \to Y$ such that

$$\begin{aligned}\mu(F(x,y) - f(x,y), t) \geqslant {*}^{\infty}\mu'\Big(\varphi(x, x, y, -y), \frac{(\alpha-4)}{8}t\Big) \\ {*}^{\infty}\, \mu'\Big(\varphi(x, -x, y, y), \frac{(\alpha-4)}{8}t\Big) {*}^{\infty}\, \mu\Big(\varphi(0, x, 0, y), \frac{(\alpha-4)}{8}t\Big)\end{aligned}$$

and

$$\begin{aligned}\mu(F(x,y) - f(x,y), t) \leqslant \circ^{\infty}\nu'\Big(\varphi(x, x, y, -y), \frac{(\alpha-4)}{8}t\Big) \\ \circ^{\infty}\, \nu'\Big(\varphi(x, -x, y, y), \frac{(\alpha-4)}{8}t\Big) \circ^{\infty}\, \nu'\Big(\varphi(0, x, 0, y), \frac{(\alpha-4)}{8}t\Big)\end{aligned}$$

for all $x, y \in X$ and all $t > 0$.

Proof. The proof is similar to the proof of Theorem 3.1. Then we present a summary proof. From (3.11), we have

$$\begin{cases} \mu(f(2x, 2z) - 4f(x, z), t) \geqslant *^2\mu'(\varphi(x, x, z, -z), \frac{t}{8}) * \mu'(\varphi(x, -x, z, z), \frac{t}{8}) \\ \qquad\qquad *\mu'(\varphi(0, x, 0, z), \frac{t}{8}), \\ \nu(f(2x, 2z) - 4f(x, z), t) \leqslant \circ^2\nu'(\varphi(x, x, z, -z), \frac{t}{8}) \circ \nu'(\varphi(x, -x, z, z), \frac{t}{8}) \\ \qquad\qquad \circ\nu'(\varphi(0, x, 0, z), \frac{t}{8}) \end{cases}$$

for all $x, z \in X$ and all $t > 0$. Thus we get

$$\begin{cases} \mu\Big(f(x, z) - 4f\Big(\frac{x}{2}, \frac{z}{2}\Big), t\Big) \geqslant *^2\mu'(\varphi(x, x, z, -z), \frac{\alpha t}{8}) \\ \qquad\qquad *\mu'(\varphi(x, -x, z, z), \frac{\alpha t}{8}) * \mu'(\varphi(0, x, 0, z), \frac{\alpha t}{8}), \\ \nu\Big(f(x, z) - 4f\Big(\frac{x}{2}, \frac{z}{2}\Big), t\Big) \leqslant \circ^2\nu'(\varphi(x, x, z, -z), \frac{\alpha t}{8}) \circ \nu'(\varphi(x, -x, z, z), \frac{\alpha t}{8}) \\ \qquad\qquad \circ\nu'(\varphi(0, x, 0, z), \frac{\alpha t}{8}) \end{cases}$$

for all $x, z \in X$ and all $t > 0$. Similar in (3.13), for all $x, z \in X$, all $m, n \in \mathbb{N}$ and $t > 0$, we can conclude

$$\begin{cases} \mu\Big(4^m f(\frac{x}{2^m}, \frac{z}{2^m}) - 4^{n+m} f(\frac{x}{2^{n+m}}, \frac{z}{2^{n+m}}), t\Big) \\ \quad \geqslant *^{2n}\mu'\Big(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) *^n \mu'\Big(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \\ \quad\quad *^n\mu'\Big(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big), \\ \nu\Big(4^m f(\frac{x}{2^m}, \frac{z}{2^m}) - 4^{n+m} f(\frac{x}{2^{n+m}}, \frac{z}{2^{n+m}}), t\Big) \\ \quad \leqslant \circ^{2n}\nu'\Big(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \circ^n \nu'\Big(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \\ \quad\quad \circ^n\nu'\Big(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \end{cases} \tag{3.15}$$

for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. Since $\alpha > 4$, $\sum_{k=0}^{\infty}(\frac{4}{\alpha})^k < \infty$ and $\sum_{k=m}^{n+m-1}(\frac{4}{\alpha})^k \to 0$ as $m \to \infty$ for all $n \in \mathbb{N}$. Thus $\frac{t}{\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^k}} \to \infty$, then we have

$$*^2\mu'\Big(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) * \mu'\Big(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) * \mu'\Big(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \longrightarrow 0$$

and

$$\circ^2\nu'\Big(\varphi(x,x,z,-z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \circ \nu'\Big(\varphi(x,-x,z,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \circ \nu'\Big(\varphi(0,x,0,z), \frac{t}{8\sum_{k=m}^{n+m-1}\frac{4^k}{\alpha^{k+1}}}\Big) \longrightarrow 0$$

as $m \to \infty$ for all $x, z \in X$, all $m, n \in \mathbb{N}$ and all $t > 0$. Hence the Cauchy criterion for convergence in IFNS shows that $4^n f(\frac{x}{2^n}, \frac{z}{2^n})$ is a Cauchy sequence in (Y, μ, ν) for all $x, z \in X$. Since (Y, μ, ν) is complete, then this sequence converges to some point $F(x, z) \in Y$ defined by $F(x, y) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ for all $x, z \in X$. By putting $m = 0$ in (3.15), we can deduce

$$\mu(F(x,y) - f(x,y), t) \geqslant *^\infty\mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) *^\infty \mu\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

and

$$\nu(F(x,y) - f(x,y), t) \leqslant \circ^\infty\nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \circ^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) \circ^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

for all $x, y \in X$ and all $t > 0$. The remainder of the proof is similar to the proof of Theorem 3.1. $\square$

4 Intuitionistic Fuzzy Continuity

In this section we apply the instuitionistic fuzzy continuity, which is discussed in [13], to study continuous mapping satisfying (1.1) approximately.

Definition 4.1. Let $g : \mathbb{R} \to X$ be a mapping, where $\mathbb{R}$ is endowed with the Euclidean topology and X is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm (μ, ν). Then $L \in X$ is said to be *intuitionistic fuzzy limit* of g at some $r_0 \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ and $\alpha, \beta \in (0, 1)$ there exists some $\delta = \delta(\varepsilon, \alpha, \beta) > 0$ such that $\mu(g(r) - L, \varepsilon) \geqslant \alpha$ and $\mu(g(r) - L, \varepsilon) \leqslant 1 - \beta$ whenever $0 < |r - r_0| < \delta$. In this case, we write $\lim_{n\to\infty} g(r) = L$, which also means that $\lim_{r\to r_0} \mu(g(r) - L, t) = 1$ and $\lim_{r\to r_0} \nu(g(r) - L, t) = 0$ or $\mu(g(r) - L, t) = 1$ and $\nu(g(r) - L, t) = 0$ as $r \to r_0$ for all $t > 0$.

Theorem 4.2. *Let X be a normed space and (Y, μ, ν) be an intuitionistic fuzzy Banach space. Let (Z, μ', ν') be an IFNS and let $0 < p < 2$ and $z_0 \in Z$. Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{cases} \mu(D_b f(x, y, z, w), t) \geqslant \mu'((\|x\| + \|y\| + \|z\| + \|w\|)z_0, t), \\ \nu(D_b f(x, y, z, w), t) \leqslant \nu'((\|x\| + \|y\| + \|z\| + \|w\|)z_0, t) \end{cases} \tag{4.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique mapping $F : X \times X \to Y$ satisfies (1.1) *such that*

$$\begin{cases} \mu(F(x, y) - f(x, y), t) \geqslant *^{\infty}\mu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad *^{\infty}\mu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad *^{\infty}\mu\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \nu(F(x, y) - f(x, y), t) \leqslant \circ^{\infty}\nu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad \circ^{\infty}\nu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad \circ^{\infty}\nu'\Big((\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \end{cases} \tag{4.2}$$

for all $x, y, z, w \in X$ and all $t > 0$. Furthermore, if the mapping $g : \mathbb{R} \to Y$ defined by $g(r) := \frac{f(2^n rx, 2^n ry)}{4^n}$ is intuitionistic fuzzy continuous for some $x, y \in X$ and all $n \in \mathbb{N}$, then the mapping $r \to F(rx, ry)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous; in this case, $F(rx, ry) = r^2 F(x, y)$ for all $r \in \mathbb{R}$.

Proof. Define $\varphi : X \times X \times X \times X \to Z$ by $\varphi(x, y, z, w) = (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0$ for all $x, y, z, w \in X$. Existence and uniqueness of the mapping F satisfying (1.1) and (4.1) are deduced from Theorem 3.1. Note that, for all $x, y \in X$, all $n \in \mathbb{N}$ and all $t > 0$, we get

$$\begin{cases} \mu\Big(F(x,y) - \frac{f(2^n x, 2^n y)}{4^n}, t\Big) = \mu\Big(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n}, t\Big) \\ \qquad = \mu\Big(F(2^n x, 2^n y) - f(2^n x, 2^y), 4^n t\Big) \\ \qquad \geqslant *^{\infty} \mu'\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big) \\ \qquad\quad *^{\infty} \mu'\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big) \\ \qquad\quad *^{\infty} \mu\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big), \\ \nu\Big(F(x,y) - \frac{f(2^n x, 2^n y)}{4^n}, t\Big) = \nu\Big(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n}, t\Big) \\ \qquad = \nu\Big(F(2^n x, 2^n y) - f(2^n x, 2^y), 4^n t\Big) \\ \qquad \leqslant \circ^{\infty} \nu'\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big) \\ \qquad\quad \circ^{\infty} \nu'\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big) \\ \qquad\quad \circ^{\infty} \nu\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{8} t\Big). \end{cases} \tag{4.3}$$

By putting $x = y = 0$ in (4.3), we have

$$\begin{cases} \mu\Big(F(0,0) - \frac{1}{4^n} f(0,0), t\Big) \geqslant 1, \\ \nu\Big(F(0,0) - \frac{1}{4^n} f(0,0), t\Big) \leqslant 0 \end{cases}$$

for all $n \in \mathbb{N}$ and $t > 0$.

Consider fix $x, y \in X$. From (4.3), we obtain

$$\begin{cases} \mu\Big(F(rx,ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t\Big) \geqslant *^{\infty} \mu'\Big((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big) \\ \quad *^{\infty} \mu'\Big((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big) *^{\infty} \mu\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big), \\ \nu\Big(F(rx,ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t\Big) \leqslant \circ^{\infty} \nu'\Big((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big) \\ \quad \circ^{\infty} \nu'\Big((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big) \circ^{\infty} \nu\Big(2^{np+1}(\|x\|^p + \|y\|^p) z_0, \frac{4^n(4-2^p)}{2^{np+4}|r|^p} t\Big) \end{cases}$$

for all $r \in \mathbb{R} \backslash \{0\}$. Since $\lim_{n\to\infty} \frac{4^n(4-2^p)t}{2^{np}|r|^p} = \infty$ for all $t > 0$, then we get

$$\begin{cases} \lim_{n\to\infty} \mu\Big(F(rx,ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, \frac{t}{3}\Big) = 1, \\ \lim_{n\to\infty} \nu\Big(F(rx,ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, \frac{t}{3}\Big) = 0 \end{cases}$$

for all $r \in \mathbb{R} \backslash \{0\}$. Consider fix $r_0 \in \mathbb{R}$, from the intuitionistic fuzzy continuity of the mapping $t \to \frac{f(2^n x, 2^n y)}{4^n}$, we have

$$\begin{cases} \lim_{n\to\infty} \mu\Big(\frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^{n_0}}, \frac{t}{3}\Big) = 1, \\ \lim_{n\to\infty} \nu\Big(\frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^{n_0}}, \frac{t}{3}\Big) = 0. \end{cases}$$

It is concluded that

$$\begin{aligned}
&\mu(F(rx,ry)-F(r_0x,r_0y),t)\\
&\quad\geqslant \mu\Big(F(rx,ry)-\frac{f(2^nrx,2^nry)}{4^n},\frac{t}{3}\Big)*\mu\Big(\frac{f(2^nrx,2^nry)}{4^n}-\frac{f(2^nr_0x,2^nr_0y)}{4^n},\frac{t}{3}\Big)\\
&\quad\quad*\mu\Big(\frac{f(2^nr_0x,2^nr_0y)}{4^n}-F(r_0x,r_0y),\frac{t}{3}\Big)\geqslant 1
\end{aligned}$$

and

$$\nu(F(rx,ry)-F(r_0x,r_0y),t)\leqslant 0$$

as $r\to r_0$ for all $t>0$. Therefore it is concluded that mapping $r\to F(rx,ry)$ is intuitionistic fuzzy continuous.

By using the intuitionistic fuzzy continuity of the mapping $r\to F(rx,ry)$ we show that $f(sx,sy)=s^2F(x,y)$ for all $s\in\mathbb{R}$. By considering fix $s\in\mathbb{R}$ and $t>0$, then for each $0<\alpha<1$, there exists $\delta>0$ such that

$$\mu\Big(F(rx,ry)-F(sx,sy),\frac{t}{3}\Big)\geqslant\alpha$$

and

$$\nu\Big(F(rx,ry)-F(sx,sy),\frac{t}{3}\Big)\leqslant 1-\alpha.$$

Consider rational number r such that $0<|r-s|<\delta$ and $|r^2-s^2|<1-\alpha$, then we will have

$$\begin{aligned}
&\mu(F(sx,sy)-s^2(x,y),t)\geqslant\\
&\quad\mu\Big(F(sx,sy)-F(rx,ry),\frac{t}{3}\Big)*\mu\Big(F(rx,ry)-r^2F(x,y),\frac{t}{3}\Big)\\
&\quad*\mu\Big(r^2F(x,y)-s^2F(x,y),\frac{t}{3}\Big)\geqslant\alpha*1*\mu\Big(F(x,y),\frac{t}{3(1-\alpha)}\Big)
\end{aligned}$$

and

$$\nu(F(sx,sy)-s^2(x,y),t)\leqslant(1-\alpha)\circ 0\circ\nu\Big(F(x,y),\frac{t}{3(1-\alpha)}\Big).$$

When $\alpha\to 1$ and using the definition of IFNS, we get

$$\mu(F(sx,sy)-s^2F(x,y),t)=1\quad\text{and}\quad\nu(F(sx,sy)-s^2F(x,y),t)=0.$$

So we conclude that

$$F(sx,sy)=s^2F(x,y).\qquad\square$$

In the following we prove a result similar to Theorem 4.2 for case $p>2$.

Theorem 4.3. *Let X be a normed space and (Y,μ,ν) be an intuitionistic fuzzy Banach space. Let (Z,μ',ν') be an IFNS and let $p>2$ and $z_0\in Z$. Let f :*

$X \times X \to Y$ be a mapping such that satisfies in (4.1)*. Then there exists a unique mapping $F : X \times X \to Y$ satisfies* (1.1) *such that*

$$\begin{cases} \mu(F(x,y) - f(x,y), t) \geqslant *^{\infty}\mu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(2^p-4)}{8}t\Big) \\ \qquad *^{\infty}\mu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad *^{\infty}\mu\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \nu(F(x,y) - f(x,y), t) \leqslant \circ^{\infty}\nu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(2^p-4)}{8}t\Big) \\ \qquad \circ^{\infty}\nu'\Big(2(\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \\ \qquad \circ^{\infty}\nu'\Big((\|x\|^p + \|y\|^p)z_0, \frac{(4-2^p)}{8}t\Big) \end{cases} \tag{4.4}$$

for all $x, y \in X$ and all $t > 0$. Furthermore, if for some $x, y \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(r) := 4^n f(\frac{rx}{2^n}, \frac{ry}{2^n})$ is intuitionistic fuzzy continuous for some $x, y \in X$ and all $n \in \mathbb{N}$, then the mapping $r \to F(rx, ry)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous, in this case, $F(rx, ry) = r^2F(x, y)$ for all $r \in \mathbb{R}$.

Proof. Define a mapping $\varphi : X \times X \times X \times X \to Z$ by $\varphi(x, y, z, w) = (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0$ for all $x, y, z, w \in X$. Then

$$\mu'\Big(\varphi\Big(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\Big), t\Big) = \mu'\Big(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p)z_0, t\Big)$$

for all $x, y \in X$ and all $t > 0$. From $p > 2$, then $2^p > 4$. By Theorem 3.3, there exists a unique mapping F which satisfies (1.1) and (4.4). The rest of the proof is similar as in Theorem 4.2. □

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