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# Berinde-Borcut Tripled Best Proximity Points with Generalized Contraction Pairs

**Premyuda Dechboon**[†], **Parinya Sa Ngiamsunthorn**[†], **Poom Kumam**[†,‡,1] **and Parin Chaipunya**[†]

[†]Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Bangkok 10140, Thailand
e-mail : premyuda.s@mail.kmutt.ac.th (P. Dechboon)
parinya.san@kmutt.ac.th (P.S. Ngiamsunthorn)
parin.cha@mail.kmutt.ac.th (P. Chaipunya)

[‡]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Bangkok 10140, Thailand
e-mail : poom.kum@kmutt.ac.th (P. Kumam)

**Abstract :** Given a pair of mappings $F : A^3 \to B$ and $G : B^3 \to A$, where $A$ and $B$ are nonempty subsets of a metric space $X$. We propose an existence theorem for a best proximity point for this pair of mappings by assuming a generalized contractivity condition. We also show that their best proximity points carry a cyclic interrelationship in the following sense: the mapping $(u, v, w) \in A^3 \mapsto (F(u, v, w), F(v, u, v), F(w, v, u)) \in B^3$ maps a best proximity point of $F$ to a best proximity point of $G$, and vice versa.

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[1]Corresponding author.

# 1 Introduction

Fixed point theory concerning contractions and generalized contractions has been an active area of research recently for at least for two decades, and it dates back to the dissertation of Banach in 1922 [1]. Several improvements and extensions were investigated since then in several directions. Let us recall that for a given mapping $T$ defined on a (nonempty) set $X$ into itself, a point $\bar{x} \in X$ is said to be a fixed point of $T$ if $\bar{x} = T\bar{x}$.

The 2-dimensional extension of a fixed point, so-called the *coupled fixed point*, was first introduced by Guo and Lakshmikantham in 1987 [2] and was re-introduced and brought into popularity again in 2006 [3] by Bhaskar and Lakshmikantham. It was given for a mapping $F : X \times X \to X$ (where $X$ is a nonempty set), in the following: a pair $(\bar{x}, \bar{y}) \in X \times X$ is said to be a *coupled fixed point* of $F$ if $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$.

The $n$-dimensional extension ($n \in \mathbb{N}$) of a fixed point was the problem for several mathematicians. Let us recall the setting of $n$-dimensional fixed point as follows: let $X \neq \emptyset$ and $F : X^n \to X$, then $(\bar{x}^1, \ldots, \bar{x}^n) \in X^n$ is called an *$n$-dimensional fixed point* of $F$ if $\bar{x}^i = F(\bar{x}^i, \bar{x}^{i-1}, \ldots, \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{n-i+1})$ for every $i \in \{1, \ldots, n\}$.

In this case, let $\mathbf{X} := X^n$ and define $\mathbf{F} : \mathbf{X} \to \mathbf{X}$ for each $(x^1, \ldots, x^n) \in \mathbf{X}$ by

$$\begin{aligned}\mathbf{F}&(x^1, \ldots, x^n) \\ &:= (F(x^1, \ldots, x^n), \ldots, F(x^i, x^{i-1}, \ldots, x^1, x^2, \ldots, x^{n-i+1}), \ldots, F(x^n, \ldots, x^1)).\end{aligned} \tag{1.1}$$

It is easy to see the equivalence between the $n$-dimensional fixed point of $F$ and the classical fixed point of $\mathbf{F}$. When $n$ is *even*, we usually have the equivalence between the contractivity condition of $\mathbf{F}$ and the corresponding $n$-dimensional version of the contractivity condition for $F$. This equivalence is unlikely for *odd* dimension $n \geq 3$, since $F$ skips some variables in its even coordinate at the solution. For example, if $n = 3$, then $\bar{x}^2 = F(\bar{x}^2, \bar{x}^1, \bar{x}^2)$ is skipping $\bar{x}^3$. Note that this effect grows stronger as the dimension grows larger. In particular, if $n = 7$, then $\bar{x}^4 = F(\bar{x}^4, \bar{x}^3, \bar{x}^2, \bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)$ is skipping $\bar{x}^5, \bar{x}^6, \bar{x}^7$.

On the other hand, the best proximity point was introduced in 1969 [4] as a replacement of fixed point for nonself mappings in metric spaces. For two given (nonempty) subsets $A, B$ of a metric space $(X, d)$ and a mapping $T : A \to B$, a point $\bar{x} \in A$ is called a *best proximity point* of $T$ if $d(\bar{x}, T\bar{x}) = d(A, B)$, where $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$.

Combining both the idea of $n$-dimensional and nonself generalizations of a fixed point, we consider the $n$-dimensional best proximity point. In particular, a point $(\bar{x}^1, \ldots, \bar{x}^n) \in A^n$ is an $n$-dimensional best proximity point of $F : A^n \to B$ if

$$d(x^i, F(\bar{x}^i, \bar{x}^{i-1}, \ldots, \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{n-i+1})) = d(A, B), \quad \forall i \in \{1, \ldots, n\}. \tag{1.2}$$

Likewise, if we write $\mathbf{A} := A^n$ and $\mathbf{B} := B^n$, we can define $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ using the formula (1.1). We can adopt the same equivalence between the $n$-dimensional best proximity point (1.2) and the best proximity point of $\mathbf{F}$. Similar relationship between the classical proximal contractiity condition and its corresponding $n$-dimensional version is also effective.

In this paper, we consider the $n$-dimensional best proximity points (will be call the *tripled best proximity point* in the sequel), for two mappings satisfying the generalized contractivity condition called the *generalized contraction pair*. However, we shall state and prove our main results only for the case $n = 3$, since the proof line is already quite lengthy. The $n$-dimensional extensions can also be carried out by similar proof lines and ideas. We would also like to underline that the two mappings involved in the generalized contraction pair exchange their solutions in the sense that we can always manipulate a solution for one mapping to evaluate a solution for another mapping.

## 2 Preliminaries

In this section, some elementary definitions related to the major results are recognized. Moreover, we denote the set of all positive integers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively, throughout this article. For any nonempty sets $A$ and $B$ in a metric space $(X, d)$, given

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

stands for the length between $A$ and $B$,

$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

Moreover, for $n \in \mathbb{N}$, $A^n$ means Cartesian product of the set $A$ for $n$ times, i.e.,

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}.$$

**Definition 2.1.** Let $T : A \to B$ be a mapping. A point $x \in A$ is said to be a *best proximity point* of $T$ if it satisfies that $d(x, Tx) = d(A, B)$.

It can be observed that a best proximity point absolutely reduces to a fixed point if the mapping becomes a self-mapping.

**Definition 2.2.** [5] Let $T : A^3 \to B$ be a mapping. A point $(x, y, z) \in A^3$ is said to be a *tripled best proximity point* of $T$ if it satisfies that

$$d(T(x,y,z),x) = d(T(y,x,y),y) = d(T(z,y,x),z) = d(A,B).$$

**Definition 2.3.** [6] The ordered pair $(A, B)$ satisfies the *property UC* if the following holds: if $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$ then $d(x_n, z_n) \to 0$.

**Definition 2.4.** [7] The ordered pair $(A, B)$ satisfies the *property strongly* $UC$ (or *property* $UC^*$) if the pair $(A, B)$ has property $UC$ and the following conditions hold:
if $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that
(1) $\lim_{n\to\infty} d(z_n, y_n) = d(A, B)$
(2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m, y_n) \leq d(A, B) + \varepsilon \text{ for all } m > n \geq N$$

then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

# 3 Main Results

This section is separated to be 3 subsections which consist of introduction of generalized contraction pair, proofs of existence and uniqueness of a tripled best proximity, and some examples supported the main theorems, as following respectively.

## 3.1 Generalized Contraction Pairs

Now, we give the definition of the generalized contraction pair in a metric space and show that a tripled best proximity point of one mapping can be used to determine a tripled best proximity point of another mapping. We view this cyclic exchanging behavior of their solutions is the fundamental property for this class of generalized contraction pairs.

**Definition 3.1.** Suppose that $(X, d)$ is a metric space, $A, B$ are nonempty and closed, and $F : A^3 \to B$ and $T : B^3 \to A$ are given. We say that $(F, G)$ is a *generalized contraction pair over* $(A, B)$ if there exist constants $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$ satisfying, for each $x, y, z \in A$ and $u, v, w \in B$, the following inequality:

$$\begin{aligned} &d(F(x, y, z), G(u, v, w)) \\ &\quad \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, w) + (1 - (\alpha + \beta + \gamma))\, d(A, B). \end{aligned}$$

**Example 3.2.** In metric space $(X, d)$ where $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $A = [1, 5]$ and $B = [-5, -1]$ be closed subsets of $X$. Define $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F(x, y, z) = \frac{-8x - 6y - 4z - 54}{72} \text{ and } G(u, v, w) = \frac{-8u - 6v - 4w + 54}{72}$$

for all $x, y, z \in A$ and $u, v, w \in B$. Since, we have $d(A, B) = 2$ and choose $\alpha = \frac{1}{9}$,

$\beta = \frac{1}{12}$ and $\gamma = \frac{1}{18}$ which $\alpha+\beta+\gamma = \frac{1}{9}+\frac{1}{12}+\frac{1}{18} = \frac{1}{4} < 1$. Then, it follows that

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left|\frac{-8x-6y-4z-54}{72} - \frac{-8u-6v-4w+54}{72}\right| \\ &\le \frac{8|x-u|+6|y-v|+4|z-w|}{72} + \frac{3}{2} \\ &= \frac{|x-u|}{9} + \frac{|y-v|}{12} + \frac{|z-w|}{18} \\ &\quad + \left(1-\left(\frac{1}{9}+\frac{1}{12}+\frac{1}{18}\right)\right)2 \\ &= \alpha d(x,u) + \beta d(y,v) + \gamma d(z,w) \\ &\quad + (1-(\alpha+\beta+\gamma))\, d(A,B), \end{aligned}$$

which implies that $(F,G)$ is a generalized cyclic contraction pair over $(A,B)$.

In the next proposition, we express the fact that a generalized contraction pair defined above has a cyclic relationship at their solutions.

**Proposition 3.3.** *Suppose that $(X,d)$ is a metric space $A,B \subset X$ are nonempty, $F : A^3 \to B$ and $G : B^3 \to A$ are mappings such that $(F,G)$ is a generalized contraction pair over $(A,B)$. Then, we have:*

1. *If $(\bar{x},\bar{y},\bar{z}) \in A^3$ is a tripled best proximity point of $F$, then the point $(F(\bar{x},\bar{y},\bar{z}), F(\bar{y},\bar{x},\bar{y}), F(\bar{z},\bar{y},\bar{x})) \in B^3$ is a tripled best proximity point of $G$.*
2. *If $(\bar{u},\bar{v},\bar{w}) \in B^3$ is a tripled best proximity point of $G$, then the point $(G(\bar{u},\bar{v},\bar{w}), G(\bar{v},\bar{u},\bar{v}), G(\bar{w},\bar{v},\bar{u})) \in A^3$ is a tripled best proximity point of $F$.*

*In particular, $F$ has a tripled proximity point if an only if $G$ does.*

*Proof.* We will only show 1, since 2 can be done similarly. Suppose that $(\bar{x},\bar{y},\bar{z}) \in A^3$ is a tripled best proximity point of $F$. Then, we get

$$\begin{aligned} & d(F(\bar{x},\bar{y},\bar{z}), G(F(\bar{x},\bar{y},\bar{z}), F(\bar{y},\bar{x},\bar{y}), F(\bar{z},\bar{y},\bar{x}))) \\ &= \alpha d(\bar{x}, F(\bar{x},\bar{y},\bar{z})) + \beta d(\bar{y}, F(\bar{y},\bar{x},\bar{y})) + \gamma d(\bar{z}, F(\bar{z},\bar{y},\bar{x})) \\ &\quad +(1-(\alpha+\beta+\gamma)) d(A,B) \\ &= d(A,B). \end{aligned}$$

Hence, this proof is completed. □

## 3.2 Existence and Uniqueness of a Tripled Best Proximity Point

In this subsection, we prove the existence of tripled best proximity points of generalized coupled cyclic contraction mappings, and also the uniqueness is shown with defined sequences as below:

Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$, $F : A^3 \to B$ and $G : B^3 \to A$. A pair $(x, y, z) \in A^3$ is defined as the sequences $\{x_n\}_{n=0}^{\infty}$ $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$, $z_0 = z$ and

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}, z_{2n}), & x_{2n+2} &= G(x_{2n+1}, y_{2n+1}, z_{2n+1}) \\ y_{2n+1} &= F(y_{2n}, x_{2n}, y_{2n}), & y_{2n+2} &= G(y_{2n+1}, x_{2n+1}, y_{2n+1}) \\ z_{2n+1} &= F(z_{2n}, y_{2n}, x_{2n}), & z_{2n+2} &= G(z_{2n+1}, y_{2n+1}, x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Moreover, we set

$$M_n\{x, y, z\} := \max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1})\} \tag{3.1}$$

during the whole of the main results.

**Lemma 3.4.** *Let $A$, $B$ be nonempty subsets of a metric space $(X, d)$, $F : A^3 \to B$, $G : B^3 \to A$ and the ordered pair $(F, G)$ be a generalized cyclic contraction mapping. If $(x_0, y_0, z_0) \in A^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are the sequences then*

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\to d(A, B), & d(x_{2n+1}, x_{2n+2}) &\to d(A, B), \\ d(y_{2n}, y_{2n+1}) &\to d(A, B), & d(y_{2n+1}, y_{2n+2}) &\to d(A, B), \\ d(z_{2n}, z_{2n+1}) &\to d(A, B), & d(z_{2n+1}, z_{2n+2}) &\to d(A, B) \end{aligned}$$

*as $n \to \infty$.*

*Proof.* For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(x_{2n}, y_{2n}, z_{2n})) \\ &\leq \alpha d(x_{2n-1}, x_{2n}) + \beta d(y_{2n-1}, y_{2n}) + \gamma d(z_{2n-1}, z_{2n-1}) \\ &\quad +(1-(\alpha+\beta+\gamma))d(A, B) \\ &= \alpha d(F(x_{2n-2}, y_{2n-2}, z_{2n-2}), G(x_{2n-1}, y_{2n-1}, z_{2n-1})) \\ &\quad +\beta d(F(y_{2n-2}, x_{2n-2}, y_{2n-2}), G(y_{2n-1}, x_{2n-1}, y_{2n-1})) \\ &\quad +\gamma d(F(z_{2n-2}, y_{2n-2}, x_{2n-2}), G(z_{2n-1}, y_{2n-1}, x_{2n-1})) \\ &\quad +(1-(\alpha+\beta+\gamma))d(A, B) \\ &\leq \alpha\left[\alpha d(x_{2n-2}, x_{2n-1}) + \beta d(y_{2n-2}, y_{2n-1}) + \gamma d(z_{2n-2}, z_{2n-1})\right. \\ &\quad \left.+(1-(\alpha+\beta+\gamma))d(A, B)\right] + \beta\left[\alpha d(y_{2n-2}, y_{2n-1})\right. \\ &\quad +\beta d(x_{2n-2}, x_{2n-1}) + \gamma d(y_{2n-2}, y_{2n-1}) \\ &\quad \left.+(1-(\alpha+\beta+\gamma))d(A, B)\right] + \gamma\left[\alpha d(z_{2n-2}, z_{2n-1})\right. \\ &\quad +\beta d(y_{2n-2}, y_{2n-1}) + \gamma d(x_{2n-2}, x_{2n-1}) \\ &\quad \left.+(1-(\alpha+\beta+\gamma))d(A, B)\right] + (1-(\alpha+\beta+\gamma))d(A, B) \\ &= \left(\alpha^2+\beta^2+\gamma^2\right) d(x_{2n-2}, x_{2n-1}) + (2\alpha\beta + 2\beta\gamma) d(y_{2n-2}, y_{2n-1}) \\ &\quad +(2\alpha\gamma) d(z_{2n-2}, z_{2n-1}) + \left(1-(\alpha+\beta+\gamma)^2\right) d(A, B) \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\le \left(\alpha^2+\beta^2+\gamma^2\right)\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\quad +(2\alpha\beta+2\beta\gamma)\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\quad +(2\alpha\gamma)\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\quad +\left(1-(\alpha+\beta+\gamma)^2\right)d(A,B)\\
&=(\alpha+\beta+\gamma)^2\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\quad +\left(1-(\alpha+\beta+\gamma)^2\right)d(A,B).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&d(x_{2n},x_{2n+1})\\
&\quad\le (\alpha+\beta+\gamma)^2\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\qquad +\left(1-(\alpha+\beta+\gamma)^2\right)d(A,B).
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
&d(y_{2n},y_{2n+1})\\
&\quad\le (\alpha+\beta+\gamma)^2\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\qquad +\left(1-(\alpha+\beta+\gamma)^2\right)d(A,B)
\end{aligned}
$$

and

$$
\begin{aligned}
&d(z_{2n},z_{2n+1})\\
&\quad\le (\alpha+\beta+\gamma)^2\max\{d(x_{2n-2},x_{2n-1}),d(y_{2n-2},y_{2n-1}),d(z_{2n-2},z_{2n-1})\}\\
&\qquad +\left(1-(\alpha+\beta+\gamma)^2\right)d(A,B).
\end{aligned}
$$

By (3.1) and we continuously use (3.2), inequalities are following

$$
\begin{aligned}
d(x_{2n},x_{2n+1}) &\le (\alpha+\beta+\gamma)^2M_{2n-2}\{x,y,z\}+(1-(\alpha+\beta+\gamma)^2)d(A,B)\\
&\le (\alpha+\beta+\gamma)^2\max\big\{(\alpha+\beta+\gamma)^2M_{2n-4}\{x,y,z\}\\
&\quad +(1-(\alpha+\beta+\gamma)^2)d(A,B),(\alpha+\beta+\gamma)^2M_{2n-4}\{x,y,z\}\\
&\quad +(1-(\alpha+\beta+\gamma)^2)d(A,B),(\alpha+\beta+\gamma)^2M_{2n-4}\{x,y,z\}\\
&\quad +(1-(\alpha+\beta+\gamma)^2)d(A,B)\big\}+(1-(\alpha+\beta+\gamma)^2)d(A,B)\\
&= (\alpha+\beta+\gamma)^4\max\{M_{2n-4}\{x,y,z\},M_{2n-4}\{x,y,z\},\\
&\quad M_{2n-4}\{x,y,z\}\}+\left(1+(\alpha+\beta+\gamma)^2\right)(1-(\alpha+\beta\\
&\quad +\gamma)^2)\,d(A,B)\\
&= (\alpha+\beta+\gamma)^4M_{2n-4}\{x,y,z\}+\left(1-(\alpha+\beta+\gamma)^4\right)d(A,B)\\
&\le (\alpha+\beta+\gamma)^{2n}M_0\{x,y,z\}+\left(1-(\alpha+\beta+\gamma)^{2n}\right)d(A,B).
\end{aligned}
$$

Thus, we get that

$$
d(x_{2n},x_{2n+1})\le(\alpha+\beta+\gamma)^{2n}M_0\{x,y,z\}+\left(1-(\alpha+\beta+\gamma)^{2n}\right)d(A,B).
$$

In the same way, it can be shown that

$$d(x_{2n+1}, x_{2n+2}) \leq (\alpha+\beta+\gamma)^{2n} M_1\{x, y, z\} + \left(1 - (\alpha+\beta+\gamma)^{2n}\right) d(A, B).$$

By using the squeeze theorem, we obtain that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B) \text{ and } \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = d(A, B).$$

Similarly, we get $d(y_{2n}, y_{2n+1}) \to d(A, B)$, $d(y_{2n+1}, y_{2n+2}) \to d(A, B)$, $d(z_{2n}, z_{2n+1}) \to d(A, B)$ and $d(z_{2n+1}, z_{2n+2}) \to d(A, B)$ as $n \to \infty$. □

**Lemma 3.5.** *Let $A$, $B$ be nonempty subsets of a metric space $(X, d)$. Let $F : A^3 \to B$, $G : B^3 \to A$ be mappings and the ordered pair $(F, G)$ be a generalized cyclic contraction mapping. If $(x_0, y_0, z_0) \in A^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are the sequences then*

$$\begin{aligned} &d(x_{2m}, x_{2n+1}) \\ &\quad \leq d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) + (1 - (\alpha+\beta+\gamma)^2) d(A, B) \\ &\quad +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m}, x_{2n+1}), d(y_{2m}, y_{2n+1}), d(z_{2m}, z_{2n+1})\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &d(y_{2m}, y_{2n+1}) \\ &\quad \leq d(y_{2m}, y_{2m+2}) + d(y_{2n+3}, y_{2n+1}) + (1 - (\alpha+\beta+\gamma)^2) d(A, B) \\ &\quad +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m}, x_{2n+1}), d(y_{2m}, y_{2n+1}), d(z_{2m}, z_{2n+1})\}, \end{aligned} \tag{3.4}$$

*and*

$$\begin{aligned} &d(z_{2m}, z_{2n+1}) \\ &\quad \leq d(z_{2m}, z_{2m+2}) + d(z_{2n+3}, z_{2n+1}) + (1 - (\alpha+\beta+\gamma)^2) d(A, B) \\ &\quad +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m}, x_{2n+1}), d(y_{2m}, y_{2n+1}), d(z_{2m}, z_{2n+1})\} \end{aligned} \tag{3.5}$$

*for all $m, n \in \mathbb{N}$.*

*Proof.* Let $m, n \in \mathbb{N}$, it follows that

$$\begin{aligned} d(x_{2m}, x_{2n+1}) &\leq d(x_{2m}, x_{2m+2}) + d(x_{2m+2}, x_{2n+3}) + d(x_{2n+3}, x_{2n+1}) \\ &= d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) \\ &\quad +d(G(x_{2m+1}, y_{2m+1}, z_{2m+1}), F(x_{2n+2}, y_{2n+2}, z_{2n+2})) \\ &\leq d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) + \alpha d(x_{2m+1}, x_{2n+2}) \\ &\quad +\beta d(y_{2m+1}, y_{2n+2}) + \gamma d(z_{2m+1}, z_{2n+2}) \\ &\quad +(1 - (\alpha+\beta+\gamma)) d(A, B) \\ &= d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) \\ &\quad +\alpha d(F(x_{2m}, y_{2m}, z_{2m}), G(x_{2n+1}, y_{2n+1}, z_{2n+1})) \\ &\quad +\beta d(F(y_{2m}, x_{2m}, y_{2m}), G(y_{2n+1}, x_{2n+1}, y_{2n+1})) \\ &\quad +\gamma d(F(z_{2m}, y_{2m}, x_{2m}), G(z_{2n+1}, y_{2n+1}, x_{2n+1})) \\ &\quad +(1 - (\alpha+\beta+\gamma)) d(A, B) \end{aligned}$$

$$
\begin{aligned}
\leq\ & d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) \\
& +\alpha(\alpha d(x_{2m}, x_{2n+1}) + \beta d(y_{2m}, y_{2n+1}) + \gamma d(z_{2m}, z_{2n+1}) \\
& +(1-(\alpha+\beta+\gamma))d(A,B)) \\
& +\beta(\alpha d(y_{2m}, y_{2n+1}) + \beta d(x_{2m}, x_{2n+1}) + \gamma d(y_{2m}, y_{2n+1}) \\
& +(1-(\alpha+\beta+\gamma))d(A,B)) \\
& +\gamma(\alpha d(z_{2m}, z_{2n+1}) + \beta d(y_{2m}, y_{2n+1}) + \gamma d(x_{2m}, x_{2n+1}) \\
& +(1-(\alpha+\beta+\gamma))d(A,B)) \\
& +(1-(\alpha+\beta+\gamma))d(A,B) \\
=\ & d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) \\
& +(\alpha^2+\beta^2+\gamma^2)d(x_{2m}, x_{2n+1}) \\
& +(2\alpha\beta+2\beta\gamma)d(y_{2m}, y_{2n+1}) + (2\alpha\gamma)d(z_{2m}, z_{2n+1}) \\
& +(1-(\alpha+\beta+\gamma)^2)d(A,B) \\
\leq\ & d(x_{2m}, x_{2m+2}) + d(x_{2n+3}, x_{2n+1}) + (1-(\alpha+\beta+\gamma)^2)d(A,B) \\
& +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m}, x_{2n+1}), d(y_{2m}, y_{2n+1}), d(z_{2m}, z_{2n+1})\},
\end{aligned}
$$

which imply that (3.3) holds. For the other inequalities, (3.4) and (3.5), can be shown in the same way. □

**Lemma 3.6.** *Let $A$, $B$ be nonempty subsets of a metric space $(X, d)$ such that the pair $(A, B)$ and $(B, A)$ satisfy the property strongly $UC$. Let $F : A^3 \to B$, $G : B^3 \to A$ be mappings and the ordered pair $(F, G)$ be a generalized cyclic contraction mapping. If $(x_0, y_0, z_0) \in A^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are the sequences then the sequences $\{x_{2n}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$, $\{z_{2n}\}_{n=0}^{\infty}$, $\{x_{2n+1}\}_{n=0}^{\infty}$, $\{y_{2n+1}\}_{n=0}^{\infty}$ and $\{z_{2n+1}\}_{n=0}^{\infty}$ are Cauchy sequences.*

*Proof.* By Lemma 3.4, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$ as $n \to \infty$. By using the property strongly $UC$ of the pairs $(A, B)$ and $(B, A)$ then $d(x_{2n}, x_{2n+2}) \to 0$ and $d(x_{2n+1}, x_{2n+3}) \to 0$ as $n \to \infty$. Next, we will prove that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon$ for all $m > n \geq N$. Suppose on the contrary that there exists $\epsilon_0 > 0$ and $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0. \tag{3.6}$$

We choose the smallest $m_k$ with $m_k > n_k \geq k$, for all $k \in \mathbb{N}$, and (3.6) holds which imply that $d(x_{2m_k-2}, x_{2n_k+1}) \leq d(A, B) + \epsilon_0$. Then, it follows that

$$
\begin{aligned}
d(A, B) + \epsilon_0 &< d(x_{2m_k}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k-2}) + d(A, B) + \epsilon_0.
\end{aligned}
$$

Letting $k \to \infty$, we then get $d(x_{2m_k}, x_{2n_k+1}) \to d(A,B) + \epsilon_0$. By using (3.3), (3.4) and (3.5), we then get

$$\begin{aligned}
& \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\} \\
\leq\ & \max\{d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) + (1-(\alpha+\beta+\gamma)^2)d(A,B) \\
& +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\}, \\
& d(y_{2m_k}, y_{2m_k+2}) + d(y_{2n_k+3}, y_{2n_k+1}) + (1-(\alpha+\beta+\gamma)^2)d(A,B) \\
& +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\}, \\
& d(z_{2m_k}, z_{2m_k+2}) + d(z_{2n_k+3}, z_{2n_k+1}) + (1-(\alpha+\beta+\gamma)^2)d(A,B) \\
& +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\}\} \\
=\ & \max\{d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}), d(y_{2m_k}, y_{2m_k+2}) \\
& +d(y_{2n_k+3}, y_{2n_k+1}), d(z_{2m_k}, z_{2m_k+2}) + d(z_{2n_k+3}, z_{2n_k+1})\} \\
& +(\alpha+\beta+\gamma)^2 \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\} \\
& +(1-(\alpha+\beta+\gamma)^2)d(A,B),
\end{aligned}$$

that is,

$$\begin{aligned}
& \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\} \\
& \leq \frac{1}{1-(\alpha+\beta+\gamma)^2} \max\{d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}), d(y_{2m_k}, y_{2m_k+2}) \\
& \quad + d(y_{2n_k+3}, y_{2n_k+1}), d(z_{2m_k}, z_{2m_k+2}) + d(z_{2n_k+3}, z_{2n_k+1})\} + d(A,B).
\end{aligned}$$

Since $d(x_{2m_k}, x_{2n_k+1}) \leq \max\{d(x_{2m_k}, x_{2n_k+1}), d(y_{2m_k}, y_{2n_k+1}), d(z_{2m_k}, z_{2n_k+1})\}$ then we obtain that

$$\begin{aligned}
d(x_{2m_k}, x_{2n_k+1}) \quad \leq \quad & \frac{1}{1-(\alpha+\beta+\gamma)^2} \max\{d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}), \\
& d(y_{2m_k}, y_{2m_k+2}) + d(y_{2n_k+3}, y_{2n_k+1}), d(z_{2m_k}, z_{2m_k+2}) \\
& +d(z_{2n_k+3}, z_{2n_k+1})\} + d(A,B).
\end{aligned}$$

Letting $k \to \infty$, we now have $d(A,B) + \epsilon_0 \leq d(A,B)$, a contradiction. Thus, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_{2m}, x_{2n+1}) \leq d(A,B) + \epsilon$ for all $m > n \geq N$ as desired. Since we have $d(x_{2n}, x_{2n+1}) \to d(A,B)$ from Lemma 3.4. So, using Definition 2.4 then we get that $\lim_{n\to\infty} d(x_{2m}, x_{2n}) = 0$, i.e., the sequences $\{x_{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence. Similarly, it is easily able to prove that $\{y_{2n}\}_{n=0}^{\infty}$, $\{z_{2n}\}_{n=0}^{\infty}$, $\{x_{2n+1}\}_{n=0}^{\infty}$, $\{y_{2n+1}\}_{n=0}^{\infty}$ and $\{z_{2n+1}\}_{n=0}^{\infty}$ are also Cauchy sequences. □

**Lemma 3.7.** *Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ such that the pair $(A,B)$ and $(B,A)$ satisfy the property strongly $UC$. Let $F : A^3 \to B$, $G : B^3 \to A$ be mappings. If $(u,v,w)$ and $(u^*,v^*,w^*)$ are tripled best proximity* *points of $F$ and $G$, respectively, then*

$$\begin{aligned} u &= G(F(u,v,w),F(v,u,v),F(w,v,u)),\\ v &= G(F(v,u,v),F(u,v,w),F(v,u,v)),\\ w &= G(F(w,v,u),F(v,u,v),F(u,v,w)),\\ u^* &= F(G(u^*,v^*,w^*),G(v^*,u^*,v^*),G(w^*,v^*,u^*)),\\ v^* &= F(G(v^*,u^*,v^*),G(u^*,v^*,w^*),G(v^*,u^*,v^*)),\\ w^* &= F(G(w^*,v^*,u^*),G(v^*,u^*,v^*),G(u^*,v^*,w^*)) \end{aligned}$$

*for all $n \in \mathbb{N} \cup \{0\}$.*

*Proof.* Since $d(u,F(u,v,w)) = d(v,F(v,u,v)) = d(w,F(w,v,u)) = d(A,B)$ and $d(u^*,G(u^*,v^*,w^*)) = d(v^*,G(v^*,u^*,v^*)) = d(w^*,G(w^*,v^*,u^*)) = d(A,B)$, then

$$\begin{aligned} & d(F(u,v,w),G(F(u,v,w),F(v,u,v),F(w,v,u)))\\ \leq\ & \alpha d(u,F(u,v,w)) + \beta d(v,F(v,u,v)) + \gamma d(w,F(w,v,u))\\ & +(1-(\alpha+\beta+\gamma))d(A,B) = d(A,B), \end{aligned}$$

$$\begin{aligned} & d(F(v,u,v),G(F(v,u,v),F(u,v,w),F(v,u,v)))\\ \leq\ & \alpha d(v,F(v,u,v)) + \beta d(u,F(u,v,w)) + \gamma d(v,F(v,u,v))\\ & +(1-(\alpha+\beta+\gamma))d(A,B) = d(A,B), \end{aligned}$$

and
$$\begin{aligned} & d(F(w,v,u),G(F(w,v,u),F(v,u,v),F(u,v,w)))\\ \leq\ & \alpha d(w,F(w,v,u)) + \beta d(v,F(v,u,v)) + \gamma d(u,F(u,v,w))\\ & +(1-(\alpha+\beta+\gamma))d(A,B) = d(A,B). \end{aligned}$$

After that, we use the property strongly $UC$ then this proof is complete for some above equations. However, the others can be done in the same way. □

**Theorem 3.8.** *Let $A$, $B$ be nonempty closed subsets of a complete metric space $(X,d)$ such that the pair $(A,B)$ and $(B,A)$ satisfy the property strongly $UC$. Let $F : A^3 \to B$, $G : B^3 \to A$ be mappings and the ordered pair $(F,G)$ be a generalized cyclic contraction mapping. If $(x_0,y_0,z_0) \in A^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are the sequences then $F$ has a unique tripled best proximity point $(u,v,w) \in A^3$ and $G$ also has a unique tripled best proximity point $(u^*,v^*,w^*) \in B^3$. Moreover, we have $x_{2n} \to u$, $y_{2n} \to v$, $z_{2n} \to w$, $x_{2n+1} \to u^*$, $y_{2n+1} \to v^*$ and $z_{2n+1} \to w^*$. Furthermore, we obtain that*

$$d(u,u^*) + d(v,v^*) + d(w,w^*) = 3d(A,B).$$

*Proof.* To show that there exists $(u,v,w) \in A^3$ such that $d(u,F(u,v,w)) = d(v,F(v,u,v)) = d(w,F(w,v,u)) = d(A,B)$. By Lemma 3.4, we have $d(x_{2n},x_{2n+1})$

$\to d(A,B)$ as $n \to \infty$. Since $\{x_{2n}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ and $\{z_{2n}\}_{n=0}^{\infty}$ are Cauchy sequences by Lemma 3.6 and we also have $A$ is closed. Then, there exist $u, v, w \in A$ such that $x_{2n} \to u$, $y_{2n} \to v$ and $z_{2n} \to w$. Then $d(A,B) \le d(u, x_{2n-1}) \le d(u, x_{2n}) + d(x_{2n}, x_{2n-1})$. It follows that $d(u, x_{2n-1}) \to d(A,B)$ as $n \to \infty$. Similarly, $d(v, y_{2n-1}) \to d(A,B)$ and $d(w, z_{2n-1}) \to d(A,B)$ as $n \to \infty$. First, we will show that $d(u, F(u,v,w)) + d(v, F(v,u,v)) + d(w, F(w,v,u)) \le 3d(A,B)$. Thus,

$$\begin{aligned}
& d(x_{2n}, F(u,v,w)) + d(y_{2n}, F(v,u,v)) + d(z_{2n}, F(w,v,u)) \\
= \ & d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(u,v,w)) + d(G(y_{2n-1}, x_{2n-1}, y_{2n-1}), F(v,u,v)) \\
& + d(G(z_{2n-1}, y_{2n-1}, x_{2n-1}), F(w,v,u)) \\
\le \ & \alpha d(x_{2n-1}, u) + \beta d(y_{2n-1}, v) + \gamma d(z_{2n-1}, w) + (1 - (\alpha+\beta+\gamma))d(A,B) \\
& + \alpha d(y_{2n-1}, v) + \beta d(x_{2n-1}, u) + \gamma d(y_{2n-1}, v) + (1 - (\alpha+\beta+\gamma))d(A,B) \\
& + \alpha d(z_{2n-1}, w) + \beta d(y_{2n-1}, v) + \gamma d(x_{2n-1}, u) + (1 - (\alpha+\beta+\gamma))d(A,B).
\end{aligned}$$

Letting $n \to \infty$ then we obtain that

$$\begin{aligned}
& d(u, F(u,v,w)) + d(v, F(v,u,v)) + d(w, F(w,v,u)) \\
\le \ & 3(\alpha+\beta+\gamma)d(A,B) + 3(1 - (\alpha+\beta+\gamma))d(A,B) = 3d(A,B),
\end{aligned}$$

that is, $d(u, F(u,v,w)) = d(v, F(v,u,v)) = d(w, F(w,v,u)) = d(A,B)$ which imply that $(u,v,w) \in A^3$ is a tripled best proximity point of $F$. By Proposition 3.3, we can also show that $(u^*, v^*, w^*) \in B^3$ is a tripled best proximity point of $G$. Next, we consider the uniqueness of best proximity points of both mappings $F$ and $G$. Suppose that there is another best proximity point of $F$, called $(p,q,r)$, i.e., $d(p, F(p,q,r)) = d(q, F(q,p,q)) = d(r, F(r,q,p)) = d(A,B)$. Then, we use Lemma 3.7 to get these

$$\begin{aligned}
d(u, F(p,q,r)) &= d(G(F(u,v,w), F(v,u,v), F(w,v,u)), F(p,q,r)) \\
&\le \alpha d(F(u,v,w), p) + \beta d(F(v,u,v), q) + \gamma d(F(w,v,u), r) \\
&\quad + (1 - (\alpha+\beta+\gamma))d(A,B) \\
&= \alpha d(F(u,v,w), G(F(p,q,r), F(q,p,q), F(r,q,p))) \\
&\quad + \beta d(F(v,u,v), G(F(q,p,q), F(p,q,r), F(q,p,q))) \\
&\quad + \gamma d(F(w,v,u), G(F(r,q,p), F(q,p,q), F(p,q,r))) \\
&\quad + (1 - (\alpha+\beta+\gamma))d(A,B) \\
&\le \alpha(\alpha d(u, F(p,q,r)) + \beta d(v, F(q,p,q)) + \gamma d(w, F(r,q,p)) \\
&\quad + (1 - (\alpha+\beta+\gamma))d(A,B)) \\
&\quad + \beta(\alpha d(v, F(q,p,q)) + \beta d(u, F(p,q,r)) + \gamma d(v, F(q,p,q)) \\
&\quad + (1 - (\alpha+\beta+\gamma))d(A,B)) \\
&\quad + \gamma(\alpha d(w, F(r,q,p)) + \beta d(v, F(q,p,q)) + \gamma d(u, F(p,q,r)) \\
&\quad + (1 - (\alpha+\beta+\gamma))d(A,B)) + (1 - (\alpha+\beta+\gamma))d(A,B) \\
&\le (\alpha+\beta+\gamma)^2 \max\{d(u, F(p,q,r)), d(v, F(q,p,q)), \\
&\quad d(w, F(r,q,p))\} + (1 - (\alpha+\beta+\gamma)^2)d(A,B),
\end{aligned}$$

similarly, we can also show that

$$\begin{aligned} d(v, F(q,p,q)) &\leq (\alpha+\beta+\gamma)^2 \max\{d(u, F(p,q,r)), d(v, F(q,p,q)), \\ &\quad d(w, F(r,q,p))\} + (1-(\alpha+\beta+\gamma)^2)d(A,B) \\ \text{and} \quad d(w, F(r,q,p)) &\leq (\alpha+\beta+\gamma)^2 \max\{d(u, F(p,q,r)), d(v, F(q,p,q)), \\ &\quad d(w, F(r,q,p))\} + (1-(\alpha+\beta+\gamma)^2)d(A,B). \end{aligned}$$

Thus, it follows that

$$\max\{d(u, F(p,q,r)), d(v, F(q,p,q)), d(w, F(r,q,p))\} \leq d(A,B).$$

Since $d(u, F(p,q,r)) \leq \max\{d(u, F(p,q,r)), d(v, F(q,p,q)), d(w, F(r,q,p))\}$, so, $d(u, F(p,q,r)) \leq d(A,B)$. By the property strongly $UC$, we can conclude that $p$ and $u$ is the same. Similarly, $q = v$, $r = w$ can be easily presented, i.e., the mapping $F$ has the only one best proximity point. For the mapping $G$, it also has a unique best proximity point as $F$ by exhibiting in the same way. Last, we will prove that $d(u,u^*)+d(v,v^*)+d(w,w^*) = 3d(A,B)$. Since, $d(u,u^*)+d(v,v^*)+d(w,w^*) \geq 3d(A,B)$ then it is enough to show that $d(u,u^*)+d(v,v^*)+d(w,w^*) \leq 3d(A,B)$. Since, $\{x_{2n+2}\}_{n=0}^{\infty}$ is a subsequence of $\{x_{2n}\}_{n=0}^{\infty}$ and we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(F(x_{2n}, y_{2n}, z_{2n}), G(x_{2n+1}, y_{2n+1}, z_{2n+1})) \\ &\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(z_{2n}, z_{2n+1}) \\ &\quad +(1-(\alpha+\beta+\gamma))d(A,B), \end{aligned} \tag{3.7}$$

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(F(y_{2n}, x_{2n}, y_{2n}), G(y_{2n+1}, x_{2n+1}, y_{2n+1})) \\ &\leq \alpha d(y_{2n}, y_{2n+1}) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(y_{2n}, y_{2n+1}) \\ &\quad +(1-(\alpha+\beta+\gamma))d(A,B) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \text{and} \quad d(z_{2n+1}, z_{2n+2}) &= d(G(z_{2n}, y_{2n}, x_{2n}), F(z_{2n+1}, y_{2n+1}, x_{2n+1})) \\ &\leq \alpha d(z_{2n}, z_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(x_{2n}, x_{2n+1}) \\ &\quad +(1-(\alpha+\beta+\gamma))d(A,B). \end{aligned} \tag{3.9}$$

Combining (3.7), (3.8), (3.9) and then letting $n \to \infty$ with Lemma 3.4, we have $d(u,u^*)+d(v,v^*)+d(w,w^*) \leq 3d(A,B)$. Therefore, $d(u,u^*)+d(v,v^*)+d(w,w^*) = 3d(A,B)$ which completes the proof. □

Next, we consider compact subsets of a metric spaces which contain a tripled best proximity point.

**Corollary 3.9.** *Let $A$, $B$ be nonempty compact subsets of a metric space $(X,d)$ such that the pair $(A,B)$ and $(B,A)$ satisfy the property strongly $UC$. Let $F : A^3 \to B$, $G : B^3 \to A$ be mappings and the ordered pair $(F,G)$ be a generalized cyclic contraction mapping. If $(x_0, y_0, z_0) \in A^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and*

*$\{z_n\}_{n=0}^{\infty}$ are the sequences then $F$ has a tripled best proximity point $(u, v, w) \in A^3$ and $G$ also has a tripled best proximity point $(u^*, v^*, w^*) \in B^3$. Moreover, we have $x_{2n} \to u$, $y_{2n} \to v$, $z_{2n} \to w$, $x_{2n+1} \to u^*$, $y_{2n+1} \to v^*$ and $z_{2n+1} \to w^*$.*

*Proof.* Since $x_{2n}$, $y_{2n}$, and $z_{2n}$ are in $A$ and $x_{2n+1}$, $y_{2n+1}$, and $z_{2n+1}$ are in $B$ for all $n \in \mathbb{N} \cup \{0\}$. Since $A$ is compact, $x_{2n}$, $y_{2n}$, and $z_{2n}$ have the convergent subsequences $x_{2n_k}$, $y_{2n_k}$, and $z_{2n_k}$, respectively. Then there exists $(u, v, w) \in A^3$ such that $x_{2n_k} \to u$, $y_{2n_k} \to v$, and $z_{2n_k} \to w$. Since $d(A, B) \leq d(u, x_{2n_k-1}) \leq d(u, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$. Then $d(u, x_{2n_k-1}) \to d(A, B)$ whenever $k \to \infty$ by using Lemma 3.4. Similarly, we get $d(v, y_{2n_k-1}) \to d(A, B)$ and $d(w, z_{2n_k-1}) \to d(A, B)$ whenever $k \to \infty$. Since

$$\begin{aligned} d(x_{2n_k}, F(u, v, w)) &= d(G(x_{2n_k-1}, y_{2n_k-1}, z_{2n_k-1}), F(u, v, w)) \\ &\leq \alpha d(u, x_{2n_k-1}) + \beta d(v, y_{2n_k-1}) + \gamma d(w, z_{2n_k-1}) \\ &\quad +(1 - (\alpha + \beta + \gamma))d(A, B), \end{aligned}$$

then letting $k \to \infty$, we get $d(u, f(u, v, w)) \leq d(A, B)$. Hence, $d(u, f(u, v, w)) = d(A, B)$. For the other, $d(v, f(v, u, v)) = d(A, B)$ and $d(w, f(w, v, u)) = d(A, B)$. That is $(u, v, w)$ is a tripled best proximity point of $F$. A tripled best proximity point $(u^*, v^*, w^*)$ of $G$ can be shown in the same way, by Proposition 3.3. Therefore, these complete the proof. □

Note that if $d(A, B) = 0$ then the generalized contraction pair $(F, G)$ over $(A, B)$ will become as the following: for some constants $\alpha, \beta, \gamma > 0$ with $\alpha+\beta+\gamma < 1$,

$$d(F(x, y, z), G(u, v, w)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, w) \tag{3.10}$$

for all $x, y, z, u, v, w \in A \cap B$.

This inequality will be used in the next theorem which guarantees both existence and uniqueness of a common tripled fixed point of mappings $F$ and $G$ in $(A \cap B)^3$.

**Theorem 3.10.** *Let $A$, $B$ be nonempty closed subsets of a complete metric space $(X, d)$ with $A \cap B \neq \emptyset$ such that the pair $(A, B)$ and $(B, A)$ satisfy the property strongly $UC$. Let $F : (A \cup B)^3 \to A \cup B$, $G : (A \cup B)^3 \to A \cup B$ be mappings and the ordered pair $(F, G)$ be a generalized cyclic contraction mapping. If $(x_0, y_0, z_0) \in (A \cup B)^3$ and $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are the sequences then $F$ and $G$ have a unique common tripled fixed point in $(A \cap B)^3$.*

*Proof.* By Theorem 3.8, $F$ has a tripled best proximity point $(u, v, w) \in A^3$ and $G$ also has a tripled best proximity point $(u^*, v^*, w^*) \in B^3$. Then,

$$d(u, F(u, v, w)) = d(v, F(v, u, v)) = d(w, F(w, v, u)) = d(A, B)$$

and also

$$d(u^*, G(u^*, v^*, w^*)) = d(v^*, G(v^*, u^*, v^*)) = d(w^*, G(w^*, v^*, u^*)) = d(A, B).$$

Since $A \cap B \neq \emptyset$ then $d(A,B) = 0$, i.e., $u = F(u,v,w)$, $v = F(v,u,v)$, $w = F(w,v,u)$, $u^* = G(u^*,v^*,w^*)$, $v^* = G(v^*,u^*,v^*)$ and $w^* = G(w^*,v^*,u^*)$ which imply that $(u,v,w)$ is a tripled fixed point of $F$ and $(u^*,v^*,w^*)$ is a tripled fixed point of $G$. By Theorem 3.8, we also have $d(u,u^*)+d(v,v^*)+d(w,w^*) = 3d(A,B)$. So, if $d(A,B) = 0$, we get $d(u,u^*) + d(v,v^*) + d(w,w^*) = 0$ which means $u = u^*$, $v = v^*$ and $w = w^*$. Hence, $(u,v,w) \in (A \cap B)^3$ is a common tripled fixed point of $F$ and $G$. For the uniqueness, suppose there exists $(p,q,r)$ which is another common tripled fixed point of $F$ and $G$, i.e.,

$$\begin{aligned} p = &\quad F(p,q,r) &= G(p,q,r) \\ q = &\quad F(q,p,q) &= G(q,p,q) \\ r = &\quad F(r,q,p) &= G(r,q,p). \end{aligned}$$

Since, we also have

$$\begin{aligned} u = &\quad F(u,v,w) &= G(u,v,w) \\ v = &\quad F(v,u,v) &= G(v,u,v) \\ w = &\quad F(w,v,u) &= G(w,v,u). \end{aligned}$$

Thus, we get these

$$\begin{aligned} d(u,p) &= d(F(u,v,w),F(p,q,r)) \\ &\leq \alpha d(u,p) + \beta d(v,q) + \gamma d(w,r) + (1-(\alpha+\beta+\gamma))d(A,B) \\ &\leq (\alpha+\beta+\gamma)\max\{d(u,p),d(v,q),d(w,r)\}. \end{aligned}$$

Similarly, it can be shown that

$$d(v,q) \quad \leq \quad (\alpha+\beta+\gamma)\max\{d(u,p),d(v,q),d(w,r)\}$$

and

$$d(w,r) \quad \leq \quad (\alpha+\beta+\gamma)\max\{d(u,p),d(v,q),d(w,r)\},$$

which imply that

$$\max\{d(u,p),d(v,q),d(w,r)\} \leq 0.$$

Hence, $d(u,p) = 0$, i.e., $u = p$. Moreover, $v = q$, $w = r$ are also proved in the same way. That is, $(u,v,w)$ and $(p,q,r)$ is same. □

## 3.3 Some Examples

Next, we illustrate examples satisfying Theorem 3.8 and Theorem 3.10, respectively.

**Example 3.11.** In metric space $(X,d)$ where $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Let $X$ be a set of real numbers. Consider closed subsets $A = [0.5,2]$ and $B = [-2,-0.5]$ of $X$. It is easy to see that $d(A,B) = 1$. Define $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F(x,y,z) = \frac{-x-4y-3z-8}{24} \quad \text{and} \quad G(u,v,w) = \frac{-u-4v-3w+8}{24}$$

for all $x, y, z \in A$ and $u, v, w \in B$. We now choose $\alpha = \frac{1}{24}$, $\beta = \frac{1}{6}$ and $\gamma = \frac{1}{8}$ which $\alpha + \beta + \gamma = \frac{1}{24} + \frac{1}{6} + \frac{1}{8} = \frac{1}{3} < 1$. Then, we get

$$\begin{aligned} d(F(x,y,z), G(u,v,w)) &= \Big|\frac{-x-4y-3z-8}{24} - \frac{-u-4v-3w+8}{24}\Big| \\ &\leq \frac{|x-u| + 4|y-v| + 3|z-w|}{24} + \frac{2}{3} \\ &= \frac{|x-u|}{24} + \frac{|y-v|}{6} + \frac{|z-w|}{8} + \left(1 - \left(\frac{1}{24} + \frac{1}{6} + \frac{1}{8}\right)\right), \end{aligned}$$

which implies that $(F, G)$ is a generalized cyclic contraction mapping. Since the pairs $(A, B)$ and $(B, A)$ satisfy the property strongly $UC$. By Theorem 3.8, all its assumptions hold then $F$ has a unique tripled best proximity point $(u, v, w) \in A^3$ and $G$ also has a unique tripled best proximity point $(u^*, v^*, w^*) \in B^3$. Moreover, it can be seen that $F(0.5, 0.5, 0.5) = -0.5$, $G(-0.5, -0.5, -0.5) = 0.5$ and $d(F(0.5, 0.5, 0.5), G(-0.5, -0.5, -0.5)) = 1$. That is, $(0.5, 0.5, 0.5)$ is a tripled best proximity point of $F$ and $(-0.5, -0.5, -0.5)$ is a tripled best proximity point of $G$.

**Example 3.12.** Let $X = \mathbb{R}$. Define the discrete metric $d : X \times X \to [0, 1]$ on $X$ by $d(x, y) = |x| + |y|$ for all $x, y \in X$. Let $A = \bigcap_{n\in\mathbb{N}} \mathcal{B}\left(-1, 1 + \frac{1}{n}\right)$ and $B = \bigcap_{n\in\mathbb{N}} \mathcal{B}\left(1, 1 + \frac{1}{n}\right)$ be subsets of $X$. Define $F : (A \cup B)^3 \to A \cup B$ and $G : (A \cup B)^3 \to A \cup B$ by

$$F(x,y,z) = -0.4x - 0.1y - 0.3z \quad \text{and} \quad G(u,v,w) = 0$$

for all $x, y, z, u, v, w \in A \cup B$. Then, we again consider

$$\begin{aligned} d(F(x,y,z), G(u,v,w)) &= |-0.4x - 0.1y - 0.3z + 0| \\ &= |-0.4x - 0.1y - 0.3z| \\ &\leq 0.4(|x| + |u|) + 0.1(|y| + |v|) + 0.3(|z| + |w|) \\ &= \alpha d(x,u) + \beta d(y,v) + \gamma d(z,w), \end{aligned}$$

which implies that $(F, G)$ is a generalized cyclic contraction mapping where $\alpha = 0.4$, $\beta = 0.1$ and $\gamma = 0.3$. Since $d(A, B) = 0$, by Theorem 3.10, all its assumptions hold then $F$ and $G$ has a common tripled fixed points in $(A \cup B)^3$ which is (0,0,0).

# 4 Conclusion

Finally, we get the existence and uniqueness theorem of a tripled best proximity point under the property strongly $UC$ of distinct closed subsets in a complete metric spaces. Furthermore, we can reduce the theorem to be the existence and uniqueness of a common tripled fixed point by adding the condition that the considered sets have nonempty intersection.

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