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Dissipative Extensions of Fourth Order Differential Operators

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Abstract : In this work, with the help of boundary conditions, we give a description of all maximal dissipative, self adjoint and other extensions of scalar fourth order differential operators on the half line and on the whole line.

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1 Introduction

Extensions of symmetric operators arise in many areas of mathematical physics, like solvable models of quantum mechanics and quantization problems. The extension theory developed originally by J. von Neumann [1]. He gave an affirmative answer to the question under which conditions does a symmetric densely defined operator possess self adjoint extensions and describes all such extensions. As is known a description of various classses of extensions (self adjoint, symmetric etc.) in terms of boundary conditions is an important problem in the spectral theory of differential operators. The problem on the description of all self adjoint exten-

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sions of a symmetric operator in terms of abstract boundary conditions was put forward for the first time in Calkin [2]. Later, Rofe- Beketov [3] described self adjoint extensions of a symmetric operator in terms of abstract boundary conditions with aid of linear relations. Bruk [4] and Kochubei [5] introduced the notion of a space of boundary values. They described all maximal dissipative, acretive, self adjoint extensions of symmetric operators. For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [6].

A description of self adjoint extensions of a second order operator on an infinite interval was obtained by Krein [7] and Fulton [8]. For a scalar fourth order equation and two term differential expressions of arbitrary even order, the same question was investigated by Mirzoev [9], Khol'kin [10]. Gorbachuk [11] obtained a description of self adjoint extensions of Sturm Liouville operators with an operator potential in the absolutely indeterminate case. In the case when the deficiency indices take indeterminate values, a description of self adjoint extensions of differential operators was given in the works of Guseinov and Pashaev [12], Maksudov and Allahverdiev [13], Allahverdiev [14], Mogilevsky [15], Malamud and Mogilevsky [16].

In this paper, a space of boundary value is constructed for scalar fourth order differential operators. We describe all maximal dissipative, acretive, self adjoint and other extensions in terms of boundary conditions.

2 Extensions of Fourth Order Differential Operators on the Half Line

We will consider the differential expression

$$l\left(y\right)=y^{(4)}+q\left(x\right)y,\quad 0\leq x<+\infty \tag{2.1}$$

where $q\left(x\right)$ is a real continuous function in $[0,\infty)$.

Let L_0 denote the closure of the minimal operator generated by (2.1) and by D_0 its domain. Besides, we denote by the set of all functions $y\left(x\right)$ from $L_2\left(0,\infty\right)$ whose first three derivatives are locally absolutely continuous in $[0,\infty)$ and $l\left(y\right)\in L_2\left(0,\infty\right)$; D is the domain of the maximal operator L. Furthermore, $L=L_0^{\ast}$ [17].

Suppose that $q\left(x\right)$ be such that the operator L_0 has defect index $(4,4)$. Let $v_1\left(x\right),v_2\left(x\right),v_3\left(x\right),v_4\left(x\right)$ be four linearly independent solutions of the equation $l\left(y\right)=0$ satisfying the conditions at $x=0$:

$$\begin{aligned}
&v_1\left(0\right)=1,\ v_1'\left(0\right)=0,\ v_1''\left(0\right)=0,\ v_1'''\left(0\right)=0,\\
&v_2\left(0\right)=0,\ v_2'\left(0\right)=1,\ v_2''\left(0\right)=0,\ v_2'''\left(0\right)=0,\\
&v_3\left(0\right)=0,\ v_3'\left(0\right)=0,\ v_3''\left(0\right)=1,\ v_3'''\left(0\right)=0,\\
&v_4\left(0\right)=0,\ v_4'\left(0\right)=0,\ v_4''\left(0\right)=0,\ v_4'''\left(0\right)=1,
\end{aligned}$$

and their Wronskian equals one. Since L_0 has defect index $(4,4)$, $v_1,v_2,v_3,v_4\in L_2\left(0,\infty\right)$.

Let's define by Γ_1, Γ_2 the linear maps from D to C^4 by the formula

$$\Gamma_1 f = \begin{pmatrix} f(0) \\ f'(0) \\ [f, v_2]_\infty \\ [f, v_1]_\infty \end{pmatrix}, \ \Gamma_2 f = \begin{pmatrix} f'''(0) \\ f''(0) \\ [f, v_4]_\infty \\ [f, v_3]_\infty \end{pmatrix} \tag{2.2}$$

where $[y, z]_x = [y'''(x) z(x) - y(x) z'''(x)] - [y''(x) z'(x) - y'(x) z''(x)]$ $(0 \le x < \infty)$.

Lemma 2.1. *For arbitrary* $y, z \in D$,

$$(Ly, z)_{L^2} - (y, Lz)_{L^2} = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4}.$$

Proof. We know that every $f, g \in D$,

$$[f, g](x) = \begin{vmatrix} [v_2, g]_x & [g, v_4]_x \\ [v_2, f]_x & [f, v_4]_x \end{vmatrix} + \begin{vmatrix} [v_1, g]_x & [g, v_3]_x \\ [v_1, f]_x & [f, v_3]_x \end{vmatrix} \tag{2.3}$$

(see [18]). For every $y, z \in D$, we have Green's formula

$$(Ly, z)_{L^2} - (y, Lz)_{L^2} = [y, \overline{z}]_\infty - [y, \overline{z}]_0.$$

Then

$$\begin{aligned} (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4} \quad = \quad & y(0)\, \overline{z}'''(0) - \overline{z}(0)\, y'''(0) \\ & + y''(0)\, \overline{z}'(0) - \overline{z}''(0)\, y'(0) \\ & + [y, v_2]_\infty [\overline{z}, v_4]_\infty - [\overline{z}, v_2]_\infty [y, v_4]_\infty \\ & + [y, v_1]_\infty [\overline{z}, v_3]_\infty - [\overline{z}, v_1]_\infty [y, v_3]_\infty. \end{aligned}$$

From the conditions (2.3), we have

$$(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4} = [y, \overline{z}]_\infty - [y, \overline{z}]_0.$$

Hence

$$(Ly, z)_{L^2} - (y, Lz)_{L^2} = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4}.$$

This completes the proof. ∎

Lemma 2.2. *For any complex numbers* $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ *and* β_3, *there is a function* $y \in D$ *satisfying*

$$\begin{aligned} y(0) = \alpha_0,\ y'(0) = \alpha_1,\ y''(0) = \alpha_2,\ y'''(0) = \alpha_3 \\ [y, v_1]_\infty = \beta_0,\ [y, v_2]_\infty = \beta_1,\ [y, v_3]_\infty = \beta_2,\ [y, v_4]_\infty = \beta_3. \end{aligned} \tag{2.4}$$

Proof. Let f be an arbitrary element of $L_2(0, \infty)$ satisfying

$$\begin{aligned} (f, v_1)_{L^2} = \beta_0 - \alpha_3,\ (f, v_3)_{L^2} = \beta_2 - \alpha_1, \\ (f, v_2)_{L^2} = \beta_1 + \alpha_2,\ (f, v_4)_{L^2} = \beta_3 + \alpha_0. \end{aligned} \tag{2.5}$$

There is such an f, even among the linear combinations of v_1, v_2, v_3 and v_4. If we set $f = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$ then conditions (2.5) are a system of equations in the constants c_1, c_2, c_3, c_4 whose determinant is the Gram determinant of the linearly independent functions v_1, v_2, v_3 v_4 and is therefore nonzero. Let $y(x)$ denote the soulution of $l(y) = f$ satisfying the initial conditions $y(0) = \alpha_0,\ y'(0) = \alpha_1,\ y''(0) = \alpha_2,\ y'''(0) = \alpha_3$. We suppose that $y(x)$ is the desired element. Applying Green's formula to $y(x)$ and v_j we get

$$(f, v_j)_{L^2} = (l(y), v_j)_{L^2} = [y, v_j]_\infty - [y, v_j]_0,\ j = 1, 2, 3, 4.$$

But $l(v_j) = 0,\ j = 1, 2, 3, 4$. Since $y(0) = \alpha_0,\ y'(0) = \alpha_1,\ y''(0) = \alpha_2,\ y'''(0) = \alpha_3$, we have

$$[y, v_j]_0 = \left\{ \begin{array}{cc} \alpha_3, & \text{for } j = 1 \\ -\alpha_2, & \text{for } j = 2 \\ \alpha_1, & \text{for } j = 3 \\ -\alpha_0, & \text{for } j = 4 \end{array} \right\}.$$

Hence

$$\begin{aligned} (f, v_1)_{L^2} &= [y, v_1]_\infty - \alpha_3,\ (f, v_2)_{L^2} = [y, v_2]_\infty + \alpha_2, \\ (f, v_3)_{L^2} &= [y, v_3]_\infty - \alpha_1,\ (f, v_4)_{L^2} = [y, v_4]_\infty + \alpha_0. \end{aligned}$$

According to (2.5), we have

$$[y, v_1]_\infty = \beta_0,\ [y, v_2]_\infty = \beta_1,\ [y, v_3]_\infty = \beta_2,\ [y, v_4]_\infty = \beta_3.$$

This completes the proof. ∎

We recall that a triple (H, Γ_1, Γ_2) is called a space of boundary values of a closed symmetric operator A on a Hilbert space H if Γ_1, Γ_2 are linear maps from $D(A^*)$ to H with equal deficiency numbers and such that:

i) Green's formula is valid

$$(A^* f, g)_H - (f, A^* g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathbb{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathbb{H}},\quad f, g \in D(A^*).$$

ii) For any $F_1, F_2 \in H$, there is a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1,\ \Gamma_2 f = F_2$ (see [5], [19]).

Theorem 2.3. *The triple* $(\mathbb{C}^4, \Gamma_1, \Gamma_2)$ *defined by* (2.2) *is a boundary spaces of the operator* L_0.

Proof. First condition of the definition of a space of boundary value follows from Lemma 2.1 and second condition follows from Lemma 2.2. ∎

Corollary 2.4. *For any contraction* K *in* C^4 *the restriction of the operator* L *to the set of functions* $y \in D$ *satisfying either*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0 \tag{2.6}$$

or

$$(K - I)\,\Gamma_1 y - i\,(K + I)\,\Gamma_2 y = 0 \tag{2.7}$$

is respectively the maximal dissipative and accretive extension of the operator L_0*. Conversely, every maximal dissipative (accretive) extension of the operator* L_0 *is the restriction of* L *to the set of functions* $y \in D$ *satisfying (2.6) ((2.7)), and the extension uniquely determines the contraction* K*. Conditions (2.6) ((2.7)) in which* K *is an isometry describe the maximal symmetric extensions of* L_0 *in* $L_2\,(0, \infty)$*. If* K *is unitary, then these conditions define selfadjoint extensions.*

In particular, the boundary conditions

$$y'''\,(0) + h_1 y\,(0) = 0,$$

$$y'\,(0) + h_2\; y''\,(0) = 0,$$

$$[y, v_2]_\infty - h_3 [y, v_4]_\infty = 0,$$

$$[y, v_1]_\infty - h_4 [y, v_3]_\infty = 0$$

with $\operatorname{Im} h_1 \geq 0$ or $h_1 = \infty,\ \operatorname{Im} h_2 \geq 0$ or $h_2 = \infty,\ \operatorname{Im} h_3 \geq 0$ or $h_3 = \infty$ and $\operatorname{Im} h_4 \geq 0$ or $h_4 = \infty$ ($\operatorname{Im} h_1 = 0$ or $h_1 = \infty,\ \operatorname{Im} h_2 = 0$ or $h_2 = \infty,\ \operatorname{Im} h_3 = 0$ or $h_3 = \infty$ and $\operatorname{Im} h_4 = 0$ or $h_4 = \infty$) describe the maximal dissipative (selfadjoint) extensions of L_0 with seperated boundary conditions.

3 Extensions of Fourth Order Differential Operators on the Whole Line

Let us think the differential expression

$$l\,(y) = y^{(4)} + q\,(x)\,y, \quad -\infty < x < +\infty \tag{3.1}$$

where $q\,(x)$ is a real continuous function in $(-\infty, \infty)$.

We denote by L_0 the closure of the minimal operator [17] generated by (3.1) and by D_0 its domain. Further, we denote by the set of all functions $y\,(x)$ from $L_2\,(-\infty, \infty)$ whose first three derivatives are locally absolutely continuous in $(-\infty, \infty)$ and $l\,(y) \in L_2\,(-\infty, \infty)$; D is the domain of the maximal operator L and $L = L_0^*$ [17].

Suppose that $q\,(x)$ be such that the operator L_0 has defect index $(2, 2)$. We denote by L_0^- and L_0^+ the closures of the minimal operator generated by expression $l\,(y)$ on the intervals $(-\infty, 0]$ and $[0, \infty)$ respectively. The deficiency number $def L_0$ of the operator L_0 is determined by the formula [16]

$$def L_0 = def L_0^+ + def L_0^- - 4.$$

Then

$$def L_0^+ + def L_0^- = 6.$$

If we put

$$m_+ = def L_0^+,\ m_- = def L_0^-,$$

then we get $m_+ + m_- = 6$.

Firstly, we take $m_- = 2$. Then $m_+ = 4$, i.e., the operator L_0^+ have deficiency index $(4,4)$. We shall assume that there is at least one point $a \in (-\infty, \infty)$ of regular type for the operator L_0^+. Thus the equation $l(y) = ay,\ (0 \leq x \leq \infty)$ has 4 linearly independent solutions owning to $L_2(0,\infty)$.

Let $v_1(x), v_2(x), v_3(x), v_4(x)$ denote the solutions of $l(y) = 0$ satisfying the initial conditions:

$$\begin{aligned}
&v_1(0) = 1,\ v_1'(0) = 0,\ v_1''(0) = 0,\ v_1'''(0) = 0,\\
&v_2(0) = 0,\ v_2'(0) = 1,\ v_2''(0) = 0,\ v_2'''(0) = 0,\\
&v_3(0) = 0,\ v_3'(0) = 0,\ v_3''(0) = 1,\ v_3'''(0) = 0,\\
&v_4(0) = 0,\ v_4'(0) = 0,\ v_4''(0) = 0,\ v_4'''(0) = 1.
\end{aligned}$$

We let

$$y(x) = v_1(x)c_1 + v_2(x)c_2 + v_3(x)c_3 + v_4(x)c_4, \tag{3.2}$$

$c_1, c_2, c_3, c_4 \in C,\ y(x) \in L_2(0,\infty)$. D_+ denotes the domain of a corresponding maximal operator $L^+ = \left(L_0^+\right)^*$. Then $y(x) \in D_+$.

Following Section 2, the mappings $\Gamma_1, \Gamma_2 : D \to C^2$ are defined as

$$\Gamma_1 f = \begin{pmatrix} [f, v_2]_\infty \\ [f, v_1]_\infty \end{pmatrix},\ \Gamma_2 f = \begin{pmatrix} [f, v_4]_\infty \\ [f, v_3]_\infty \end{pmatrix} \tag{3.3}$$

where $[y,z]_x = [y'''(x)\, z(x) - y(x) z'''(x)] - [y''(x) z'(x) - y'(x)\, z''(x)] \quad (0 \leq x < \infty)$.

Lemma 3.1. *For arbitrary $y, z \in D$,*

$$(Ly, z)_{L^2} - (y, Lz)_{L^2} = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^2} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^2}.$$

Proof. Since the operator L_0^- have deficiency index $(2,2)$, Green's formula takes the following form:

$$(Ly, z)_{L^2} - (y, Lz)_{L^2} = [y, z]_\infty \tag{3.4}$$

for all $y, z \in D$. From Lemma 2.1, we have

$$\begin{aligned}
(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^2} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^2} &= [y, v_2]_\infty [\overline{z}, v_4]_\infty - [\overline{z}, v_2]_\infty [y, v_4]_\infty \\
&\quad + [y, v_1]_\infty [\overline{z}, v_3]_\infty - [\overline{z}, v_1]_\infty [y, v_3]_\infty \\
&= [y, z]_\infty .
\end{aligned}$$

This completes the proof. ∎

Lemma 3.2. *Given any complex number $\beta_0, \beta_1, \beta_2$ and β_3, there is a function $y \in D$ satisfying the conditions:*

$$[y,v_1]_\infty = \beta_0,\ [y,v_2]_\infty = \beta_1,\ [y,v_3]_\infty = \beta_2,\ [y,v_4]_\infty = \beta_3. \tag{3.5}$$

Proof. From Lemma 2.2, there is a function $y_+(x) \in D_+$ satisfying the conditions:

$$\begin{aligned} y_+(0) &= \alpha_0, \quad y_+'(0) = \alpha_1, \\ y_+''(0) &= \alpha_2, \quad y_+'''(0) = \alpha_3, \qquad \alpha_0,\alpha_1,\alpha_2,\alpha_3 \in \mathbb{C}, \\ [y_+,v_1]_\infty &= \beta_0,\ [y_+,v_2]_\infty = \beta_1, \\ [y_+,v_3]_\infty &= \beta_2,\ [y_+,v_4]_\infty = \beta_3. \end{aligned} \tag{3.6}$$

Let us consider a regular operator L_{01}^- which is the closure of a minimal operator generated by $l(y)$ on the interval $[-1,0]$. Since L_{01}^- have deficiency index $(4,4)$, there must be a function $y_-(x) \in D\left(\left(L_{01}^-\right)^*\right)$ such that

$$\begin{aligned} y_-(0) = \alpha_0,\ y_-'(0) = \alpha_1,\ y_-''(0) = \alpha_2,\ y_-'''(0) = \alpha_3, \\ y_-(-1) = y_-'(-1) = y_-''(-1) = y_-'''(-1) = 0. \end{aligned} \tag{3.7}$$

Now we let

$$y(x) = \left\{ \begin{array}{cc} 0, & -\infty < x \le -1, \\ y_-(x), & -1 \le x \le 0, \\ y_+(x), & 0 \le x < \infty \end{array} \right\}.$$

The conditions (3.6), (3.7) then assure that the functions $y(x)$ are smooth at the points $x=-1$, $x=0$. Then

$$[y,v_1]_\infty = \beta_0,\ [y,v_2]_\infty = \beta_1,\ [y,v_3]_\infty = \beta_2,\ [y,v_4]_\infty = \beta_3.$$

This completes the proof. ∎

Thus we have proved the following theorem.

Theorem 3.3. *Let the operator L_0^- have deficiency index $(2,2)$ and the operator L_0^+ have deficiency index $(4,4)$. It follows that the triple $\left(\mathbb{C}^2,\Gamma_1,\Gamma_2\right)$ defined by (3.3) is a space of boundary values of an operator L_0.*

Corollary 3.4. *For any contraction K in C^2 the restriction of the operator L to the set of functions $y \in D$ satisfying either*

$$(K-I)\,\Gamma_1 y + i\,(K+I)\,\Gamma_2 y = 0 \tag{3.8}$$

or

$$(K-I)\,\Gamma_1 y - i\,(K+I)\,\Gamma_2 y = 0 \tag{3.9}$$

is respectively the maximal dissipative or accretive extension of the operator L_0. Conversely, every maximal dissipative (accretive) extension of the operator L_0 is the restriction of L to the set of functions $y \in D$ satisfying (3.8) ((3.9)), and the contraction K is uniquely determined by the extension. Conditions (3.8) ((3.9)) in which K is an isometry describe the maximal symmetric extensions of L_0 in $L_2(0,\infty)$. If K is unitary, then these conditions define selfadjoint extensions.

Let L_0^- have deficiency index $(4,4)$ and L_0^+ have deficiency index $(4,4)$. We shall assume that there is at least one point $a_\pm \in (-\infty,\infty)$ of regular type for the operator $L_0^\pm$. The equation $l(y) = a_\pm y$ has 4 linearly independent solutions belonging to $L_2(R_\pm)$, where $R_+ = [0,\infty),\ R_- = (-\infty, 0]$. Let $v_1(x), v_2(x), v_3(x), v_4(x)$ denote the solutions of $l(y) = 0$ satisfying the initial conditions:

$$\begin{aligned}
&v_1(0) = 1,\ v_1'(0) = 0,\ v_1''(0) = 0,\ v_1'''(0) = 0,\\
&v_2(0) = 0,\ v_2'(0) = 1,\ v_2''(0) = 0,\ v_2'''(0) = 0,\\
&v_3(0) = 0,\ v_3'(0) = 0,\ v_3''(0) = 1,\ v_3'''(0) = 0,\\
&v_4(0) = 0,\ v_4'(0) = 0,\ v_4''(0) = 0,\ v_4'''(0) = 1.
\end{aligned}$$

We let

$$y_\pm(x) = v_{1\pm}(x)\, c_1^\pm + v_{2\pm}(x)\, c_2^\pm + v_{3\pm}(x)\, c_3^\pm + v_{4\pm}(x)\, c_4^\pm, y_\pm(x) \in L_2(R_\pm),$$

where $c_1^\pm, c_2^\pm, c_3^\pm, c_4^\pm \in C$. We denote by Γ_1, Γ_2 the linear maps from D to C^4 defined by the formula

$$\Gamma_1 f = \begin{pmatrix} -[f_-, v_2]_\infty \\ [f_+, v_2]_\infty \\ -[f_-, v_1]_\infty \\ [f_+, v_1]_\infty \end{pmatrix}, \quad \Gamma_2 y = \begin{pmatrix} [f_-, v_4]_\infty \\ [f_+, v_4]_\infty \\ [f_-, v_3]_\infty \\ [f_+, v_3]_\infty \end{pmatrix} \tag{3.10}$$

where $[y,z]_x = [y'''(x)\, z(x) - y(x) z'''(x)] - [y''(x)\, z'(x) - y'(x)\, z''(x)] \quad (0 \le x < \infty)$.

Lemma 3.5. *For every* $y, z \in D$,

$$(Ly, z) - (y, Lz) = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4}.$$

Proof. For every $y, z \in D$, we have

$$(Ly, z) - (y, Lz) = [y,z]_\infty - [y,z]_{-\infty}.$$

From Lemma 2.1, we get

$$\begin{aligned}
(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4} = \ & [y_+, v_2]_\infty [\overline{z}_+, v_4]_\infty - [\overline{z}_+, v_2]_\infty [y_+, v_4]_\infty \\
& + [y_+, v_1]_\infty [\overline{z}_+, v_3]_\infty - [\overline{z}_+, v_1]_\infty [y_+, v_3]_\infty \\
& + [y_-, v_2]_{-\infty} [\overline{z}_-, v_4]_{-\infty} - [\overline{z}_-, v_2]_{-\infty} [y_-, v_4]_{-\infty} \\
& + [y_-, v_1]_{-\infty} [\overline{z}_-, v_3]_{-\infty} - [\overline{z}_-, v_1]_{-\infty} [y_-, v_3]_{-\infty} \\
= \ & [y_+, z_+]_\infty - [y_-, z_-]_{-\infty},
\end{aligned}$$

$$[y,z]_\infty = [y_+, z_+]_\infty, \quad [y,z]_{-\infty} = [y_-, z_-]_{-\infty}.$$

This completes the proof. ∎

Lemma 3.6. *For any complex numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ and β_3, there is a function $y \in D$ satisfying*

$$\begin{aligned} -[y_-, v_1]_{-\infty} &= \alpha_0, \quad -[y_-, v_2]_{-\infty} = \alpha_1, \\ [y_-, v_3]_{-\infty} &= \alpha_2, \qquad [y_-, v_4]_{-\infty} = \alpha_3, \\ [y_+, v_1]_{\infty} &= \beta_0, \qquad [y_+, v_2]_{\infty} = \beta_1, \\ [y_+, v_3]_{\infty} &= \beta_2, \qquad [y_+, v_4]_{\infty} = \beta_3. \end{aligned} \tag{3.11}$$

Proof. From Lemma 2.2, there is a function $y_+(x) \in D_+,\ y_-(x) \in D$ satisfying the conditions:

$$\begin{aligned} y_-(0) &= y_-'(0) = y_-''(0) = y_-'''(0) = 0, \\ y_+(0) &= y_+'(0) = y_+''(0) = y_+'''(0) = 0, \\ -[y_-, v_1]_{-\infty} &= \alpha_0, \ -[y_-, v_2]_{-\infty} = \alpha_1, \\ [y_-, v_3]_{-\infty} &= \alpha_2, \quad [y_-, v_4]_{-\infty} = \alpha_3, \\ [y_+, v_1]_{\infty} &= \beta_0, \quad [y_+, v_2]_{\infty} = \beta_1, \\ [y_+, v_3]_{\infty} &= \beta_2, \quad [y_+, v_4]_{\infty} = \beta_3. \end{aligned} \tag{3.12}$$

We let

$$y(x) = \left\{ \begin{array}{ll} y_-(x), & -\infty \le x \le 0, \\ y_+(x), & 0 \le x < \infty, \end{array} \right\}.$$

The conditions (3.12) ensure that the function $y(x)$ is smooth at the point $x = 0$. Hence $y(x) \in D$ and the conditions (3.11) are satisfied. ∎

Thus we have proved the following theorem.

Theorem 3.7. *Let the operator L_0^- have deficiency index $(4,4)$ and the operator L_0^+ have deficiency index $(4,4)$. It follows that the triple $\left(\mathbb{C}^4, \Gamma_1, \Gamma_2\right)$ defined by (3.10) is a boundary spaces of the operator L_0.*

Corollary 3.8. *For any contraction K in C^4 the restriction of the operator L to the set of functions $y \in D$ satisfying either*

$$(K - I)\,\Gamma_1 y + i\,(K + I)\,\Gamma_2 y = 0 \tag{3.13}$$

or

$$(K - I)\,\Gamma_1 y - i\,(K + I)\,\Gamma_2 y = 0 \tag{3.14}$$

is respectively the maximal dissipative or accretive extension of the operator L_0. Conversely, every maximal dissipative (accretive) extension of the operator L_0 is the restriction of L to the set of functions $y \in D$ satisfying (3.13) (3.14), and the contraction K is uniquely determined by the extension. Conditions (3.13) (3.14) in which K is an isometry describe the maximal symmetric extensions of L_0 in $L_2(0,\infty)$. If K is unitary, then these conditions define selfadjoint extensions.

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