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On the Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeros

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Abstract : In this paper, we prove an L_p-inequality concerning the polar derivative of a polynomial with restricted zeros.

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1 Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n and $P^{'}(z)$ be its derivative, then according to a famous result known as Berstein's inequality (see [1, 2])

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

In (1) equality holds only for $P(z) = \alpha z^n$, $|\alpha| \neq 0$, that is, if and only if $P(z)$ has all zeros at the origin. Inequality (1) was extended to L_p-norm, $p \geq 1$ by Zygmund

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[3], who proved that, if $P(z)$ is a polynomial of degree n, then

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}|P^{'}(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq n\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}. \tag{2}$$

In (2) equality holds only for $P(z) = \alpha z^n,\ |\alpha| \neq 0$. If we let $p \to \infty$ in (2), we get inequality (1).

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ denote the polar derivative of $P(z)$ with respect to a point α.

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and generalizes the ordinary derivative $P^{'}(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P^{'}(z). \tag{3}$$

As an extension of (1) to the polar derivative, Aziz and Shah ([4], Theorem 4 with $k = 1$) have shown that, if $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| > 1$ and for $|z| = 1$,

$$|D_\alpha P(z)| \leq n|\alpha|\max_{|z|=1}|P(z)|. \tag{4}$$

Inequality (4) becomes equality for $P(z) = az^n, a \neq 0$. If we divide the two sides of (4) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (1).

As a generalization of (2) to the polar derivative. Aziz et all [5] proved the following result:

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$*

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \leq\ n(|\alpha|+1)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{5}$$

For the class of polynomials having no zeros in $|z| < 1$, Inequality (2) can be improved. In fact, it was shown by De-Bruijn [6] that, if $P(z) \neq 0$ in $|z| < 1$, then for $p \geq 1$

$$\left\{\int_0^{2\pi}\left|P^{'}(e^{i\theta})\right|^p d\theta\right\}^{1/p} \leq\ nC_p\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \tag{6}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\phi}\right|^p d\phi\right\}^{-1/p}. \tag{7}$$

Inequality (6) is best possible with equality for $P(z) = az^n + b, |a| = |b|$. By letting $p \to \infty$, in (6), it follows that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1}|P^{'}(z)| \leq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{8}$$

Inequality (8) was conjuctured by $Erd\ddot{o}s$ and later verified by Lax [7]. Aziz [8] extended (8) to the polar derivative of a polynomial and proved that, if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |P(z)|. \tag{9}$$

The estimate (9) is best possible with equality for $P(z) = z^n + 1$. If we divide both sides of (9) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (8). As an extension to the polar derivative, the following generalizations of (6) and (9) has been proved:

Theorem 1.2. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$*

$$\left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \leq n(|\alpha| + 1)C_p \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \tag{10}$$

where C_p is defined by (7).

In this paper, we prove a result which generalize the above theorem and there by obtain compact generalizations of many polynomial inequalities as well. In fact, we prove:

Theorem 1.3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and $p \geq 1$*

$$\left\{\int_0^{2\pi} \left|e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - k)}{k+1}\beta P(e^{i\theta})\right|^p d\theta\right\}^{1/p}$$

$$\leq n(1+|\alpha|+2\frac{(|\alpha| - k)}{k+1}|\beta|)C_p \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \tag{11}$$

where C_p is defined by (7).

Or equivalently,

$$||e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - k)}{k+1}\beta P(e^{i\theta})||_p \leq n(1 + |\alpha| + 2\frac{(|\alpha| - k)}{k+1}|\beta|)\frac{||P(e^{i\theta})||_p}{||1 + e^{i\phi}||_p}.$$

Remark 1.4. *In Theorem* 1.3*, if we choose $\beta = 0$ and $k = 1$, we get immediately Theorem* 1.2*.*

If we choose $k = 1$ in Theorem 1.3, then we obtain the following corollary:

Corollary 1.5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $p \geq 0$*

$$\left\{\int_0^{2\pi} \left|e^{i\theta} D_\alpha P(e^{i\theta}) + n(|\alpha| - 1)\frac{\beta}{2}P(e^{i\theta})\right|^p d\theta\right\}^{1/p}$$

$$\leq n(1 + |\alpha| + (|\alpha| - 1)|\beta|)C_p \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \tag{12}$$

where C_p is defined by (7).

2 Lemmas

For the proof of our main theorem, we need the following lemmas. The first is due to Zireh [9].

Lemma 2.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| < k, k \leq 1$ and $P(z)$ a polynomial of degree atmost n. If $|P(z)| \leq |Q(z)|$ for $|z| = k \leq 1$, then for every α, $\beta \in C$ with $|\alpha| \geq k$, $|\beta| \leq 1$*

$$\left|zD_\alpha P(z) + n\frac{(|\alpha| - k)}{k+1}\beta P(z)\right| \;\leq\; \left|zD_\alpha Q(z) + n\frac{(|\alpha| - k)}{k+1}\beta Q(z)\right|.$$

The next lemma is due to Aziz and Rather [5] (see also [10]).

Lemma 2.2. *If $P(z)$ is a polynomial of degree n such that $P(0) \neq 0$ and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then for every $p \geq 0$ and ϕ real*

$$\int_0^{2\pi}\int_0^{2\pi}\left|Q^{'}(e^{i\theta}) + e^{i\phi}P^{'}(e^{i\theta})\right|^p d\theta d\phi \;\leq\; n^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta.$$

3 Proof of the Theorem

Proof of Theorem 1.3. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < k \leq 1$. By Lemma 2.1, for complex numbers α, β with $|\alpha| \geq k$, $|\beta| \leq 1$, we have

$$\left|zD_\alpha P(z) + n\frac{(|\alpha| - k)}{k+1}\beta P(z)\right| \;\leq\; \left|zD_\alpha Q(z) + n\frac{(|\alpha| - k)}{k+1}\beta Q(z)\right|. \tag{13}$$

On the other hand, for every real ϕ and $\zeta \geq 1$, we have

$$|\zeta + e^{i\phi}| \geq |1 + e^{i\phi}|.$$

This implies for any $p \geq 0$,

$$\left\{\int_0^{2\pi}\left|\zeta + e^{i\phi}\right|^p d\phi\right\}^{1/p} \geq \left\{\int_0^{2\pi}\left|1 + e^{i\phi}\right|^p d\phi\right\}^{1/p}. \tag{14}$$

If $e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta}) \neq 0$, we can take

$$\zeta = \frac{e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta})}{e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})},$$

where according to (13), $|\zeta| \geq 1$. Now

$$\int_0^{2\pi}\left|e^{i\theta}D_\alpha Q(e^{i\theta})+n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta})+e^{i\phi}[e^{i\theta}D_\alpha P(e^{i\theta})+n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})]\right|^p d\phi$$

$$= \left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right|^p \int_0^{2\pi} \left|\zeta + e^{i\phi}\right|^p d\phi.$$

$$\geq \left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right|^p \int_0^{2\pi} \left|1 + e^{i\phi}\right|^p d\phi.$$

If $e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta}) = 0$, then the later inequality is trivally true. Integrating both sides of the above inequality with respect to θ in $[0, 2\pi)$, we obtain

$$\int_0^{2\pi}\int_0^{2\pi} \left|e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta}) + e^{i\phi}[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})]\right|^p d\theta d\phi$$
$$\geq \int_0^{2\pi} \left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right|^p d\theta \int_0^{2\pi} \left|1 + e^{i\phi}\right|^p d\phi. \qquad (15)$$

Now for $0 \leq \theta < 2\pi$,

$$\left|e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta}) + e^{i\phi}\left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right]\right|$$

$$= \left|\left[e^{i\theta}\left\{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q^{'}(e^{i\theta})\right\} + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta})\right.\right.$$
$$\left.\left. + e^{i\phi}\left[e^{i\theta}\left\{nP(e^{i\theta}) + (\alpha - e^{i\theta})P^{'}(e^{i\theta})\right\} + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right]\right|\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$= \left|\left[e^{i\theta}\left\{nQ(e^{i\theta}) - e^{i\theta}Q^{'}(e^{i\theta})\right\} + \alpha e^{i\theta}Q^{'}(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta})\right]\right.$$
$$\left. + e^{i\phi}\left[e^{i\theta}\left\{nP(e^{i\theta}) - e^{i\theta}P^{'}(e^{i\theta})\right\} + \alpha e^{i\theta}P^{'}(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right]\right.\ldots\ldots\ldots(17)$$

Since $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, we have $P(z) = z^n\overline{Q\left(\frac{1}{\bar{z}}\right)}$ and it can be easily verified that for $0 \leq \theta < 2\pi$

$$nP(e^{i\theta}) - e^{i\theta}P^{'}(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q^{'}(e^{i\theta})}$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q^{'}(e^{i\theta}) = e^{i(n-1)\theta}\overline{P^{'}(e^{i\theta})}.$$

From (17), we have

$$\left|e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta Q(e^{i\theta}) + e^{i\phi}\left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right]\right|$$

$$= \left| \left[e^{i\theta} \left\{ e^{i(n-1)\theta} \overline{P'(e^{i\theta})} \right\} \right] + \alpha e^{i\theta} \left[Q^{'}(e^{i\theta}) + e^{i\phi} P^{'}(e^{i\theta}) \right] \right.$$

$$\left. + n \frac{(|\alpha| - k)}{k+1} \beta \left[Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta}) \right] + e^{i\phi} e^{i\theta} e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} \right| \ldots\ldots\ldots\ldots\ldots (18)$$

Therefore (15) in conjuction with (18) gives,

$$\left\{ \int_0^{2\pi} \int_0^{2\pi} \left| \left[e^{i\theta} e^{i(n-1)\theta} \left\{ \overline{P'(e^{i\theta})} + e^{i\phi} \overline{Q'(e^{i\theta})} \right\} \right] + \alpha e^{i\theta} \left[Q^{'}(e^{i\theta}) + e^{i\phi} P^{'}(e^{i\theta}) \right] \right. \right.$$

$$\left. \left. + n \frac{(|\alpha| - k)}{k+1} \beta \quad \left[Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta}) \right] \right|^p d\theta d\phi \right\}^{1/p}$$

$$\geq \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - k)}{k+1} \beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} \left| 1 + e^{i\phi} \right|^p d\phi \right\}^{1/p}.$$

By Minkowski inequality, we have

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - k)}{k+1} \beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} \left| 1 + e^{i\phi} \right|^p d\phi \right\}^{1/p}$$

$$\leq \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| Q^{'}(e^{i\theta}) + e^{i\phi} P^{'}(e^{i\theta}) \right|^p d\theta d\phi \right\}^{1/p}$$

$$+ \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| \alpha \left\{ Q^{'}(e^{i\theta}) + e^{i\phi} P^{'}(e^{i\theta}) \right\} \right|^p d\theta d\phi \right\}^{1/p}$$

$$+ \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| n \frac{(|\alpha| - k)}{k+1} \beta \left\{ Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta}) \right\} \right|^p d\theta d\phi \right\}^{1/p}$$

$$= \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| Q^{'}(e^{i\theta}) + e^{i\phi} P^{'}(e^{i\theta}) \right|^p d\theta d\phi \right\}^{1/p} \left\{ 1 + |\alpha| \right\} + \left| n \frac{(|\alpha| - k)}{k+1} \beta \right|$$

$$\left\{ \int_0^{2\pi} \int_0^{2\pi} \left| Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta}) \right|^p d\theta d\phi \right\}^{1/p}.$$

By Lemma 2, we have

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - k)}{k+1} \beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} \left| 1 + e^{i\phi} \right|^p d\phi \right\}^{1/p}$$

$$\leq \left\{ 2n^p \pi \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{1/p} \left\{ 1 + |\alpha| \right\} + 2n/(k+1) \left| (|\alpha| - k)\beta \right|$$

$$\left\{ 2\pi \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{1/p}$$

$$= \left[n(1 + |\alpha|) + 2n \frac{(|\alpha| - k)}{k+1} |\beta| \right] \left\{ 2\pi \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{1/p}.$$

This implies,

$$\left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta})+n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})\right|^p d\theta\right\}^{1/p}\leq n(1+|\alpha|+2\frac{(|\alpha|-k)}{k+1}|\beta|)C_p \left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p},$$

where C_p is defined by (7). Or equivalently,

$$||e^{i\theta}D_\alpha P(e^{i\theta})+n\frac{(|\alpha|-k)}{k+1}\beta P(e^{i\theta})||_p\leq n(1+|\alpha|+2\frac{(|\alpha|-k)}{k+1}|\beta|)\frac{||P(e^{i\theta})||_p}{||1+e^{i\phi}||_p}.$$

This completes the proof.

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