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The Modified Viscosity Iteration with m-Generalized Hybrid Mappings and (a, b)-Monotone Mappings for Equilibrium Problems

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Abstract : In this paper, we show the existence of a common element of the set of solutions of an equilibrium problem, the set of fixed points of m-generalized hybrid

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and (a, b)-monotone mappings in Hilbert spaces by using a modified viscosity iteration. First, we prove some strong convergence theorems of our proposed algorithm to converge a common element of the set of solutions of an equilibrium problem, the sets of fixed points of m-generalized hybrid and (a, b)-monotone mappings. Finally, we give numerical examples to illustrate the our results.

Keywords : common fixed point; equilibrium problem; strong convergence; m-generalized hybrid mapping; (a, b)-monotone mapping; modified viscosity iteration
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1 Introduction

From the existence of solutions of the equilibrium problem has used for solving a wide class of real problems for instance economics, finance, optimization and networks (see [3, 5, 6, 7]). In fact, the equilibrium problem contain many problems such as optimization problems, the fixed point problems, the Nash equilibrium problems, variational inequalities, complementary problems, saddle point problems and some others as special cases.

In 1953, Mann [8] introduced an iteration method, which is called the *Mann iteration*, for finding a fixed point of a nonexpansive mapping in a Hilbert space as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \end{cases}$$

for each $n \geq 0$, where $\alpha_n \in [0, 1]$. In 1974, Ishikawa [9] introduced a new iteration procedure, which is called the *Ishikawa iteration*, for approximating a fixed point of a nonexpansive mapping in a Hilbert space as follows:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases}$$

for each $n \geq 0$, where $\alpha_n, \beta_n \in [0, 1]$ with some conditions. Note that, if $\beta_n = 0$ in the Ishikawa iteration, then we have the Mann iteration.

Actually, the Mann and Ishikawa iterations converge weakly to a fixed point of a nonexpansive mapping in a Hilbert space. To overcome this problem, some authors have introduced some iterations to converge strongly to fixed points of the proposed nonlinear mappings and solutions of nonlinear problems.

Especially, Moudafi [4] introduced the *viscosity iteration* $\{x_n\}$ in a Hilbert space H defined as follows: choose an arbitrary initial $x_0 \in H$,

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T x_n$$

for each $n \geq 0$, where $T : H \to H$ is a nonexpansive mapping and $f : H \to H$ is a contraction with a coefficient $\alpha \in [0, 1)$, the sequence $\{\varepsilon_n\}$ is in $(0, 1)$ such that

(a) $\lim_{n\to\infty} \varepsilon_n = 0$;
(b) $\sum_{n-0}^{\infty} \varepsilon_n = \infty$;
(c) $\lim_{n\to\infty} \left(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}\right) = 0$,

and proved that the sequence $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0$$

for all $x \in Fix(T)$, where $Fix(T)$ denotes the set of fixed points of the mapping T.

Since the Moudafi viscosity iteration, some authors have introduced some generalizations of the viscosity iteration in several ways (see [10] and many others). Especially, in 2007, Takahashi [11] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

On the other hand, In 2008, Kohsaka and Takahashi [14] introduced a nonlinear mapping called a *nonspreading mapping* in a smooth strictly convex and reflexive Banach space X as follows:

Let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2$ for all $x, y \in X$ and J is the normalized duality mapping on C. Observe that, if X is a real Hilbert space, then J is the identity mapping and $\phi(x, y) = \|x - y\|^2$ for all $x, y \in X$. So, a nonspreading mapping T in a real Hilbert space $X = H$ is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Since then, some fixed point theorems of such a mapping has been studied by many researchers (see, for example, [9, 10, 11]).

In 2010, Takahashi [12] introduced a new nonlinear mapping, which is called the *hybrid mapping*. Motivated by this mapping, Kocourek et al. [13] introduced the *generalized hybrid mapping* in Hilbert spaces. In 2012, Lin and Wang [14] introduced the (a, b)*-monotone mapping* and proved some weak and strong convergence theorems for this mapping in Hilbert spaces and show that the (a, b)-monotone mapping is not necessary to be a quasi-nonexpansive mapping.

Especially, in 2015, Alizadeh and Moradlou [15] introduced the *m-generalized hybrid mapping* in Hilbert spaces and proved some weak and strong convergence theorems for this mapping. In 2016, they [16] introduced the *modified Ishikawa iteration* for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of the generalized hybrid mapping in Hilbert spaces. Also, in 2016, they [17] used the modified Ishikawa iteration for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a 2-generalized hybrid mapping.

Recently, in 2017, Sadeewong et al. Cho [18] used the modify Ishikawa iteration for finding the set of fixed points of the (a, b)-monotone mapping, the m-generalized hybrid mapping and the set of solutions of the equilibrium problem in Hilbert spaces.

Motivated by works mentioned above, in this paper, using the modify viscosity iteration, we prove some strong convergence theorems for finding an common point of the set of fixed points of the (a, b)-monotone mappings, the m-generalized hybrid mappings and the set of solution of the equilibrium problem in Hilbert spaces. Finally, we give some examples to illustrate the main result in this paper.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ and C be a nonempty closed convex subset of H.

The *equilibrium problem* for a bifunction $F : C \times C \to \mathbb{R}$ is to find a point $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{2.1}$$

The set of solutions of the problem (2.1) is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

A mapping $f : C \to C$ is called a *contraction* if there exists a constant $k \in (0, 1]$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \forall x, y \in C.$$

A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C.$$

The set of fixed points of a mapping $S : C \to C$ is denoted by $Fix(S)$, i.e.,

$$Fix(S) = \{x \in C : Sx = x\}.$$

A mapping $S : C \to C$ with $Fix(S) \neq \emptyset$ is said to be *quasi-nonexpansive* if

$$\|Sx - y\| \leq \|x - y\|, \forall x, y \in C \text{ and } y \in Fix(S).$$

Let $S : C \to H$ be a mapping and a bifunction $F : C \times C \to \mathbb{R}$ be defined by

$$F(x, y) = \langle Sx, y - x \rangle, \forall x, y \in C.$$

Then, it follows that $z \in EP(F) \Leftrightarrow \langle Sz, y-z\rangle \geq 0$, $\forall y \in C$, i.e., z is a solution of the variational inequality

$$\langle Sx, y-x\rangle \geq 0,\ \forall y \in C.$$

So, the formulation (2.1) includes variational inequalities as special cases.

A mapping $S : C \to C$ is called to be:

(1) *firmly nonexpansive* if

$$\|Sx - Sy\|^2 \leq \langle x-y, Sx-Sy\rangle,\ \forall x, y \in C;$$

(2) *nonspreading* ([19]) if

$$2\|Sx - Sy\|^2 \leq \|Sx-y\|^2 + \|Sy-x\|^2,\ \forall x, y \in C;$$

(3) *hybrid* ([12]) if

$$3\|Sx - Sy\|^2 \leq \|x-y\|^2 + \|Sx-y\|^2 + \|Sy-x\|^2,\ \forall x, y \in C;$$

(4) α*-hybrid* ([1]) if there exists $\alpha \in \mathbb{R}$ such that

$$(1+\alpha)\|Sx - Sy\|^2 - \alpha\|x-Sy\|^2 \leq (1-\alpha)\|x-y\|^2 + \alpha\|y-Sx\|^2,\ \forall x, y \in C;$$

(5) α*-nonexpansive* ([2]) if there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that

$$\|Sx - Sy\|^2 \leq \alpha\|Sx-y\|^2 + \alpha\|x-Sy\|^2 + (1-2\alpha)\|x-y\|^2,\ \forall x, y \in C;$$

(6) *generalized hybrid* or (α, β)*-generalized hybrid* ([16]) if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1-\alpha)\|Sy-x\|^2 \leq \beta\|Sx-y\|^2 + (1-\beta)\|x-y\|^2,\ \forall x, y \in C;$$

(7) m*-generalized hybrid mapping* ([21]) if there exist $\gamma_k, \lambda_k \in \mathbb{R}$ such that

$$\begin{aligned}&\sum_{k=1}^{m}\gamma_k\|S^{m+1-k}x - Sy\|^2 + \Big(1-\sum_{k=1}^{m}\gamma_k\Big)\|x-Sy\|^2\\ &\leq \sum_{k=1}^{m}\lambda_k\|S^{m+1-k}x - y\|^2 + \Big(1-\sum_{k=1}^{m}\lambda_k\Big)\|x-y\|^2,\ \forall x, y \in C;\end{aligned}$$

(8) (a, b)*-monotone* ([14]) if there exist $a \in (\frac{1}{2}, \infty)$ and $b \in (-\infty, a)$ such that

$$\begin{aligned}&\langle x-y, Sx-Sy\rangle\\ &\geq a\|Sx-Sy\|^2 + (1-a)\|x-y\|^2 - b\|x-Sx\|^2 - b\|y-Sy\|^2,\ \forall x, y \in C.\end{aligned}$$

Remark 2.1. From the definitions of nonlinear mappings given above, we have the following:

(1) Every firmly nonexpansive mapping is nonexpansive;

(2) Every firmly nonexpansive mapping is α-nonexpansive for all $\alpha \in [0, \frac{1}{2}]$;

(3) The identity mapping I_X is α-nonexpansive for all $\alpha < 1$;

(4) A mapping $S : C \to X$ is 0-nonexpansive if and only if S is nonexpansive;

(5) The $\frac{1}{2}$-nonexpansive mapping is nonspreading;

(6) The $\frac{1}{3}$-nonexpansive mapping is hybrid;

(7) For all $0 \leq \alpha \leq \frac{2}{3}$, every constant mapping $S : C \to C$ is α-nonexpansive;

(8) The $(1, 0)$-generalized hybrid mapping is nonexpansive;

(9) The $(2, 1)$-generalized hybrid mapping is nonspreading;

(10) The $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid;

(11) Every m-generalized hybrid mapping is quasi-nonexpansive if $Fix(S)$ is nonempty.

Remark 2.2. From the definitions of the (a, b)-monotone, we have the following:

(1) The $(1, \frac{1}{2})$-monotone mapping is nonspreading;

(2) The $(\frac{3}{2}, \frac{1}{2})$-monotone mapping is hybrid;

(3) The $(\frac{\gamma}{\gamma+\lambda-1}, \frac{1}{2})$-monotone mapping is (γ, λ)-generalized hybrid;

(4) If $S : C \to C$ is (a, b)-monotone, then we have

$$\|x - p\|^2 \geq \|Sx - p\|^2 + \frac{1 - 2b}{2a - 1}\|x - Sx\|^2$$

for all $x \in C$ and $p \in Fix(S)$ ([14]).

Throughout this paper, the weak convergence and the strong convergence of $\{x_n\}$ to $x \in H$ denote $x_n \rightharpoonup x$ and $x_n \to x$, respectively, and $\omega_\omega(x_n)$ denotes the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_\omega(x_n) := \{x \in H : \exists\{x_{n_k}\} \subset \{x_n\}; x_{n_k} \rightharpoonup x\}.$$

Remark 2.3. Now, we recall some basic properties of a Hilbert space H as follows:

(1) For all $x, y \in H$, it follows from [22] that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \ \forall \alpha \in \mathbb{R};$$

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(3) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$.

Let C be a closed convex subset of H and P_C be the metric (or nearest point) projection from H onto C, i.e., for all $x \in H$, $P_C x$ is the only point in C such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

It is well known that, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, y - z\rangle \leq 0$$

for all $y \in C$. For more details, we refer to [22, 23].

We need the following lemmas in the proof of our main results in next section.

Lemma 2.1. ([24]) *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for each $n \in \mathbb{N}$ and $x \in C$, then $\{P_C(x_n)\}$ converges strongly to a point $z \in C$, where P_C stands for the metric projection on H onto C.*

To solve the equilibrium problem, we assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0, \ \forall x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \ \forall x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.2. ([25]) *Let C be a nonempty closed convex subset of H, $F : C\times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4) *and let $r > 0$, $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.3. ([26]) *Let C be a nonempty closed convex subset of H, $F : C\times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4)*. For any $r > 0$ and $x \in H$, define a mapping $W_r : H \to C$ as follows:*

$$W_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then the following statements hold:

(1) *W_r is single-valued;*
(2) *W_r is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|W_r x - W_r y\|^2 \leq \langle W_r x - W_r y, x - y\rangle;$$

(3) *$Fix(W_r) = EP(F)$;*
(4) *EP(F) is closed and convex.*

3 Main Results

In this section, we prove some strong convergence theorems of the modify viscosity iteration for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of the m-generalized hybrid mapping and the (a, b)-monotone mapping in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4), *$S : C \to C$ be an m-generalized hybrid mapping and $f : C \to C$ is a contraction such that $\Theta := Fix(S) \cap EP(F) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\alpha, \beta \in \mathbb{R}$ are such that $0 < \alpha < \alpha_n < \beta < 1$, $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is a sequence in $[d, 1]$ for some $d \in (0, 1)$ such that*

$$\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

If $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and, for all $y \in C$,

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ y_n = (1 - \beta_n)f(x_n) + \beta_n S u_n, \\ x_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n S y_n \end{cases}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $v \in \Theta$, where $v = \lim_{n\to\infty} P_\Theta(x_n)$.

Proof. By Lemma 2.2, the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ are well defined. Since S is an m-generalized hybrid mapping such that $Fix(S) \neq \emptyset$, S is quasi-nonexpansive. So $Fix(S)$ is closed and convex. Also, by the hypothesis, $\Theta \neq \emptyset$. Let $q \in \Theta$. From $u_n = W_{r_n} x_n$, we have

$$\|u_n - q\| = \|W_{r_n} x_n - W_{r_n} q\| \leq \|x_n - q\|. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} \|y_n - q\|^2 &= (1 - \beta_n)\|f(x_n) - q\|^2 + \beta_n\|Su_n - q\|^2 - \beta_n(1 - \beta_n)\|f(x_n) - Su_n\|^2 \\ &= (1 - \beta_n)\|(f(x_n) - f(q)) + (f(q) - q)\|^2 + \beta_n\|Su_n - q\|^2 \\ &\quad - \beta_n(1 - \beta_n)\|f(x_n) - Su_n\|^2 \\ &\leq (1 - \beta_n)(\|f(x_n) - f(q)\|^2 + \|f(q) - q\|^2 \\ &\quad + 2\|f(x_n) - f(q)\|\|f(q) - q\|) + \beta_n\|Su_n - q\|^2 \\ &\quad - \beta_n(1 - \beta_n)\|f(x_n) - Su_n\|^2 \\ &\leq (1 - \beta_n)k^2\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1 - \beta_n)\|f(x_n) - Su_n\|^2 \\ &\leq \|x_n - q\|^2 - \beta_n(1 - \beta_n)\|f(x_n) - Su_n\|^2 \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)Sx_n+\alpha Sy_n-q\|^2\\ &= (1-\alpha_n)\|Sx_n-q\|^2+\alpha_n\|Sy_n-q\|^2-\alpha_n(1-\alpha_n)\|Sx_n-Sy_n\|^2\\ &\le (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|y_n-q\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|Sx_n-Sy_n\|^2\\ &\le (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-Sy_n\|^2\\ &\le \|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\le \|x_n-q\|^2.\end{aligned} \tag{3.3}$$

So, we can conclude that $\lim_{n\to\infty}\|x_n-q\|$ exists. This yields that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. It follows from (3.3) that

$$\|x_{n+1}-q\|^2 \le \|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2.$$

By using the condition $0<\alpha<\alpha_n<\beta<1$, we have

$$\|x_{n+1}-q\|^2 \le \|x_n-q\|^2-\alpha\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2.$$

Also, we have

$$0\le \alpha\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2 \to 0$$

as $n\to\infty$ since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$. Therefore, we have

$$\|f(x_n)-Su_n\|\to 0. \tag{3.4}$$

This yields that

$$\|y_n-f(x_n)\|=\beta_n\|f(x_n)-Su_n\|\to 0 \tag{3.5}$$

as $n\to\infty$. Using (3) of Remark 2.3 and Lemma 2.3, we have

$$\begin{aligned}\|u_n-q\|^2 &= \|W_{r_n}x_n-W_{r_n}q\|^2\\ &\le \langle W_{r_n}x_n-W_{r_n}q, x_n-q\rangle\\ &= \langle u_n-q, x_n-q\rangle\\ &= \frac{1}{2}(\|u_n-q\|^2+\|x_n-q\|^2-\|x_n-u_n\|^2),\end{aligned}$$

and hence

$$\|u_n-q\|^2\le \|x_n-q\|^2-\|x_n-u_n\|^2.$$

Then, by the convexcity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|y_n-q\|^2 &= \|(1-\beta_n)(f(x_n)-q)+\beta_n(Su_n-q)\|^2\\ &\le (1-\beta_n)\|f(x_n)-q\|^2+\beta_n\|Su_n-q\|^2\\ &\le (1-\beta_n)\|f(x_n)-q\|^2+\beta_n\|u_n-q\|^2\\ &\le (1-\beta_n)\|f(x_n)-q\|^2+\beta_n(\|f(x_n)-q\|^2-\|f(x_n)-u_n\|^2)\\ &= \|f(x_n)-q\|^2-\beta_n\|f(x_n)-u_n\|^2.\end{aligned}$$

Therefore, we have

$$\beta_n\|f(x_n) - u_n\|^2 \le \|f(x_n) - q\|^2 - \|y_n - q\|^2. \tag{3.6}$$

Since $\{\beta_n\} \subset [d, 1]$, it follows from (3.6) that

$$\begin{aligned} d\|f(x_n) - u_n\|^2 &\le \beta_n\|f(x_n) - u_n\|^2 \\ &\le \|f(x_n) - q\|^2 - \|y_n - q\|^2 \\ &= (\|f(x_n) - q\| - \|y_n - q\|)(\|f(x_n) - q\| + \|y_n - q\|) \\ &\le \|y_n - f(x_n)\|(\|x_n - q\| + \|y_n - q\|). \end{aligned}$$

By using the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows from (3.5) and the above inequality that

$$\lim_{n\to\infty} \|f(x_n) - u_n\| = 0. \tag{3.7}$$

From

$$\|y_n - q\|^2 \le \|f(x_n) - q\|^2 - \beta_n\|f(x_n) - u_n\|^2,$$

it follows that

$$\|y_n - q\|^2 \le \|x_n - q\|^2 - \beta_n\|x_n - u_n\|^2.$$

Therefore, we have

$$\beta_n\|x_n - u_n\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2.$$

Also, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty} \left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.8}$$

Since $\beta_n S u_n = y_n - (1 - \beta_n)x_n$, we have

$$\begin{aligned} d\|u_n - Su_n\| &\le \beta_n\|Su_n - u_n\| \\ &= \|y_n - (1 - \beta_n)f(x_n) - \beta_n u_n\| \\ &\le \|y_n - u_n\| + (1 - \beta_n)\|f(x_n) - u_n\| \\ &\le \|y_n - f(x_n)\| + \|f(x_n) - u_n\| + \|f(x_n) - u_n\| \\ &= \|y_n - f(x_n)\| + 2\|f(x_n) - u_n\|. \end{aligned}$$

From (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup u$ as $i \to \infty$ and so $\{u_{n_i}\} \rightharpoonup u$ as $i \to \infty$. Since C is closed and convex and $\{u_{n_i}\} \subset C$, we have $u \in C$.

Now, we show that $u \in \Theta$. Since S is an m-generalized hybrid mapping, we have

$$\sum_{k=1}^{m} \alpha_k \|S^{m+1-k}x - Sy\|^2 + \Big(1 - \sum_{k=1}^{m} \alpha_k\Big)\|x - Sy\|^2 \\ \leq \sum_{k=1}^{m} \beta_k \|S^{m+1-k}x - y\|^2 + \Big(1 - \sum_{k=1}^{m} \beta_k\Big)\|x - y\|^2$$

and hence

$$0 \leq \sum_{k=1}^{m} \beta_k \|S^{m+1-k}x - y\|^2 + \Big(1 - \sum_{k=1}^{m} \beta_k\Big)\|x - y\|^2 \\ - \sum_{k=1}^{m} \alpha_k \|S^{m+1-k}x - Sy\|^2 - \Big(1 - \sum_{k=1}^{m} \alpha_k\Big)\|x - Sy\|^2.$$

If we replace x and y by u_n and u in the above inequality, respectively, then we have

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \beta_k \|S^{m+1-k}u_n - u\|^2 + \Big(1 - \sum_{k=1}^{m} \beta_k\Big)\|u_n - u\|^2 \\ &\quad - \sum_{k=1}^{m} \alpha_k \|S^{m+1-k}u_n - Su\|^2 - \Big(1 - \sum_{k=1}^{m} \alpha_k\Big)\|u_n - Su\|^2 \\ &= \sum_{k=1}^{m} \beta_k (\|S^{m+1-k}u_n\|^2 - 2\langle S^{m+1-k}u_n, u\rangle + \|u\|^2) \\ &\quad + \Big(1 - \sum_{k=1}^{m} \beta_k\Big)(\|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2) \\ &\quad - \sum_{k=1}^{m} \alpha_k (\|S^{m+1-k}u_n\|^2 - 2\langle S^{m+1-k}u_n, Su\rangle + \|Su\|^2) \\ &\quad - \Big(1 - \sum_{k=1}^{m} \alpha_k\Big)(\|u_n\|^2 - 2\langle u_n, Su\rangle + \|Su\|^2) \\ &= \|u\|^2 - \|Su\|^2 + 2\langle u_n, Su - u\rangle \\ &\quad + \sum_{k=1}^{m} \beta_k (\|S^{m+1-k}u_n\|^2 - 2\langle S^{m+1-k}u_n, u\rangle - \|u_n\|^2 + 2\langle u_n, u\rangle) \\ &\quad + \sum_{k=1}^{m} \alpha_k (\|u_n\|^2 - 2\langle u_n, Su\rangle - \|S^{m+1-k}u_n\|^2 \\ &\quad + 2\langle S^{m+1-k}u_n, Su\rangle) \end{aligned}$$

$$
\begin{aligned}
&= \|u\|^2 - \|Su\|^2 + 2\langle u_n, Su - u\rangle \\
&\quad + \sum_{k=1}^{m}(\beta_k - \alpha_k)(\|S^{m+1-k}u_n\|^2 - \|u_n\|^2) \\
&\quad - 2\sum_{k=1}^{m}\beta_k\langle S^{m+1-k}u_n - u_n, u\rangle + 2\sum_{k=1}^{m}\alpha_k\langle S^{m+1-k}u_n - u_n, Su\rangle \\
&\le \|u\|^2 - \|Su\|^2 + 2\langle u_n, Su - u\rangle \\
&\quad + \sum_{k=1}^{m}(\beta_k - \alpha_k)\Big[(\|S^{m+1-k}u_n - u_n\|)(\|S^{m+1-k}u_n\| + \|u_n\|)\Big] \\
&\quad - 2\sum_{k=1}^{m}\beta_k\langle S^{m+1-k}u_n - u_n, u\rangle + 2\sum_{k=1}^{m}\alpha_k\langle S^{m+1-k}u_n - u_n, Su\rangle.
\end{aligned}
$$

Now, substituting n by n_i in the above inequality, we have

$$
\begin{aligned}
0 &\le \|u\|^2 - \|Su\|^2 + 2\langle u_{n_i}, Su - u\rangle \\
&\quad + \sum_{k=1}^{m}(\beta_k - \alpha_k)\Big[(\|S^{m+1-k}u_{n_i} - u_{n_i}\|)(\|S^{m+1-k}u_{n_i}\| + \|u_{n_i}\|)\Big] \\
&\quad - 2\sum_{k=1}^{m}\beta_k\langle S^{m+1-k}u_{n_i} - u_{n_i}, u\rangle + 2\sum_{k=1}^{m}\alpha_k\langle S^{m+1-k}u_{n_i} - u_{n_i}, Su\rangle, \ \forall i \in \mathbb{N}.
\end{aligned} \tag{3.10}
$$

Since $u_{n_i} \rightharpoonup u$ as $i \to \infty$, it follows from (3.9) and (3.10) that

$$
\begin{aligned}
0 &\le \|u\|^2 - \|Su\|^2 + 2\langle u_n, Su - u\rangle \\
&= -\|u\|^2 + 2\langle u, Su\rangle - \|Su\|^2 \\
&= -(\|u - Su\|^2).
\end{aligned}
$$

So, we have $Su = u$, i.e., $u \in Fix(S)$.

Next, we show that $u \in EP(F)$. Since $u_n = W_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0$$

for all $y \in C$. From the condition (A2), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n)$$

for all $y \in C$ and so

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(y, u_{n_i}) \tag{3.11}$$

for all $y \in C$. It follows from (3.8), (3.11) and the condition (A4) that

$$0 \ge F(y, u)$$

for all $y \in C$. Suppose that $t \in (0,1]$, $y \in C$ and $y_t = ty + (1-t)u$. Then $y_t \in C$ and so $F(y_t, u) \leq 0$. Hence we ahve

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, u) \leq tF(y_t, y)$$

and, dividing by t, we have $F(y_t, y) \geq 0$ for all $y \in C$. So, by taking the limit as $t \downarrow 0$ and using the condition (A3), we have $u \in EP(F)$.

Finally, we prove that $x_n \to v$ as $n \to \infty$. Since

$$\begin{aligned}
\|y_n - v\|^2 &= \|(1-\beta_n)(f(x_n) - v) + \beta_n(Su_n - v)\|^2 \\
&\leq (1-\beta_n)\|f(x_n) - v\|^2 + \beta_n\|Su_n - v\|^2 \\
&\leq (1-\beta_n)\|f(x_n) - v\|^2 + \beta_n\|u_n - v\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq (1-\alpha_n)\|Sx_n - v\|^2 + \alpha_n\|Sy_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|y_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(1-\beta_n)\|f(x_n) - v\|^2 + \alpha_n\beta_n\|u_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(1-\beta_n)k\|(x_n) - v\|^2 + \alpha_n\beta_n\|x_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(1-\beta_n)\|(x_n) - v\|^2 + \alpha_n\beta_n\|x_n - v\|^2 \\
&\leq \|x_n - v\|^2.
\end{aligned}$$

Therefore, the sequence $\{x_n\}$ converges strongly to v, where v $= \lim_{n\to\infty} P_\Theta(x_n)$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4) *and $S : C \to C$ be an m-generalized hybrid self mapping such that $Fix(S) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\alpha, \beta \in \mathbb{R}$ are such that $0 < \alpha < \alpha_n < \beta < 1$, $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is a sequence in $[d,1]$ for some $d \in (0,1)$ such that*

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

If $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and, for all $y \in C$,

$$\begin{cases} u_n \in C \text{ such that } \langle y - u_n, u_n - x_n\rangle \geq 0, \\ y_n = (1-\beta_n)f(x_n) + \beta_n Su_n, \\ x_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Sy_n. \end{cases}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $v \in Fix(S)$, where

$$v = \lim_{n\to\infty} P_{Fix(S)}(x_n).$$

Proof. Let $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.1, we have the result. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying the conditions (A1)–(A4) and $S: C\to C$ be an m-generalized hybrid, $T: C\to C$ be an (a,b)-monotone mapping and $f: C\to C$ is a contraction such that $\Omega := Fix(S)\cap Fix(T)\cap EP(F)\neq\emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\alpha,\beta\in\mathbb{R}$ are such that*

$$\liminf_{n\to\infty}\alpha_n\Big(\frac{1-2b}{2a-1}\Big)>0,\quad 0<\alpha<\alpha_n<\beta<1,$$

$\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\to\infty} r_n>0$ and $\{\beta_n\}$ is a sequence in $[d,1]$ for some $d\in(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\beta_n)>0.$$

If $\{x_n\}$ is a sequence generated by $x_1=x\in C$ and, for all $y\in C$,

$$\begin{cases} u_n\in C \text{ such that } F(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\\ y_n=(1-\beta_n)f(x_n)+\beta_n Su_n,\\ x_{n+1}=(1-\alpha_n)Sx_n+\alpha_n Ty_n,\end{cases}$$

for all $n\in\mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $v\in\Omega$, where $v=\lim_{n\to\infty}P_\Omega(x_n)$.

Proof. By Lemma 2.2, the sequences $\{u_n\},\{y_n\}$ and $\{x_n\}$ are well defined. Since $S: C\to C$ is an m-generalized hybrid mapping and $T: C\to C$ is an (a,b)-monotone mapping such that $Fix(S)$ and $Fix(T)\neq\emptyset$, S is quasi-nonexpansive. So, $Fix(S)$ and $Fix(T)$ are closed and convex. Also, by the hypothesis $\Omega\neq\emptyset$, let $q\in\Omega$. From $u_n=W_{r_n}x_n$, we have

$$\|u_n-q\|=\|W_{r_n}x_n-W_{r_n}q\|\leq\|x_n-q\|. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\|y_n-q\|^2 &= (1-\beta_n)\|f(x_n)-q\|^2+\beta_n\|Su_n-q\|^2-\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &= (1-\beta_n)\|(f(x_n)-f(q))+(f(q)-q)\|^2+\beta_n\|Su_n-q\|^2\\ &\quad -\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\leq (1-\beta_n)(\|f(x_n)-f(q)\|^2+\|f(q)-q\|^2\\ &\quad +2\|f(x_n)-f(q)\|\|f(q)-q\|)+\beta_n\|Su_n-q\|^2\\ &\quad -\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\leq (1-\beta_n)k^2\|x_n-q\|^2+\beta_n\|x_n-q\|^2-\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\leq \|x_n-q\|^2-\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2.\end{aligned} \tag{3.13}$$

Since T ia an (a,b)-monotone mapping, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)Sx_n+\alpha Ty_n-q\|^2\\ &= (1-\alpha_n)\|Sx_n-q\|^2+\alpha_n\|Ty_n-q\|^2-\alpha_n(1-\alpha_n)\|Sx_n-Ty_n\|^2\\ &\leq (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|y_n-q\|^2-\alpha_n\Big(\frac{1-2b}{2a-1}\Big)\|y_n-Ty_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-Ty_n\|^2\\ &\leq (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\quad -\alpha_n\Big(\frac{1-2b}{2a-1}\Big)\|y_n-Ty_n\|^2\\ &\leq \|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|f(x_n)-Su_n\|^2\\ &\leq \|x_n-q\|^2.\end{aligned} \tag{3.14}$$

So, we can conclude that $\lim_{n\to\infty}\|x_n-q\|$ exists. This yields that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Thus, similar to Theorem 3.1, we have $u\in C$.

Now, we show that $u\in\Omega$. Since S is an m-generalized hybrid mapping, in Theorem 3.1, we have $u\in Fix(S)\cap EP(F)$.

Next, we show that $u\in Fix(T)$. Since T is an (a,b)-monotone mapping, we have

$$\langle x-y,Tx-Ty\rangle\geq a\|Tx-Ty\|^2+(1-a)\|x-y\|^2-b\|x-Tx\|^2-b\|y-Ty\|^2$$

and hence

$$b\|x-Tx\|^2+b\|y-Ty\|^2\geq a\|Tx-Ty\|^2+(1-a)\|x-y\|^2-\langle x-y,Tx-Ty\rangle$$

If we replace x and y by u_n and u in the above inequality, respectively, then we have

$$\begin{aligned}& b\|u_n-Tu_n\|^2+b\|u-Tu\|^2\\ &\geq a\|Tu_n-Tu\|^2+(1-a)\|u_n-u\|^2-\langle u_n-u,Tu_n-Tu\rangle\\ &= a\langle Tu_n-Tu,Tu_n-Tu\rangle+(1-a)\langle u_n-u,u_n-u\rangle-\langle u_n-u,Tu_n-Tu\rangle\\ &\quad +a\langle u_n-u,Tu_n-Tu\rangle-a\langle u_n-u,Tu_n-Tu\rangle\\ &= a\langle Tu_n-Tu,Tu_n-Tu-u_n+u\rangle+(1-a)\langle u_n-u,u_n-u-Tu_n+Tu\rangle\\ &= a\langle Tu_n-Tu,Tu_n-u_n\rangle+a\langle T_n-Tu,u-Tu\rangle\\ &\quad +(1-a)\langle u_n-u,u_n-Tu_n\rangle+(1-a)\langle u_n-u,Tu-u\rangle\\ &= a\langle Tu_n-Tu,Tu_n-u_n\rangle+a\langle Tu_n-u_n,u-Tu\rangle\\ &\quad +a\langle u_n-u,u-Tu\rangle+a\langle u-Tu,u-Tu\rangle\quad +(1-a)\langle u_n-u,u_n-Tu_n\rangle\\ &+(1-a)\langle u_n-u,Tu-u\rangle\end{aligned}$$

Now, substituting n by n_i, we have

$$\begin{aligned} &b\|u_{n_i} - Tu_{n_i}\|^2 + b\|u - Tu\|^2 \\ &\geq a\langle Tu_{n_i} - Tu, Tu_{n_i} - u_{n_i}\rangle + a\langle Tu_{n_i} - u_{n_i}, u - Tu\rangle \\ &\quad + a\langle u_{n_i} - u, u - Tu\rangle + (1-a)\langle u_{n_i} - u, u_{n_i} - Tu_{n_i}\rangle \\ &\quad + a\langle u - Tu, u - Tu\rangle + (1-a)\langle u_{n_i} - u, Tu - u\rangle \end{aligned} \tag{3.15}$$

for all $i \in \mathbb{N}$. Since $u_{n_i} \rightharpoonup u$ as $i \to \infty$, it follows from (3.15) that

$$b\|u - Tu\|^2 \geq a\|u - Tu\|^2$$

Since $b < a$, we have $\|u - Tu\| = 0$, that is, $u = Tu$, which implies $u \in Fix(T)$.

Finally, we prove that the sequence $\{x_n\}$ converges strongly to a point v, where $v = \lim_{n\to\infty} P_\Omega(x_n)$. Since

$$\|y_n - v\|^2 \leq (1-\beta_n)\|x_n - v\|^2 + \beta_n\|u_n - v\|^2,$$

we have

$$\begin{aligned} \|x_{n+1} - v\|^2 &\leq (1-\alpha_n)\|Sx_n - v\|^2 + \alpha_n\|Ty_n - v\|^2 \\ &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|y_n - v\|^2 \\ &- \alpha_n\Big(\frac{1-2b}{2a-1}\Big)\|y_n - Ty_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(1-\beta_n)\|x_n - v\|^2 + \alpha_n\beta_n\|u_n - v\|^2 \\ &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(1-\beta_n)\|x_n - v\|^2 + \alpha_n\beta_n\|x_n - v\|^2 \\ &\leq (1-\alpha_n)\|x_n - v\|^2. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ converges strongly to a point v, where $v = \lim_{n\to\infty} P_\Omega(x_n)$. This completes the proof. □

Corollary 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4), *$S : C \to C$ be an m-generalized hybrid mapping, $T : C \to C$ be an (a,b)-monotone mapping and $f : C \to C$ is a contraction with $Fix(S) \cap Fix(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\alpha, \beta \in \mathbb{R}$ are such that*

$$\liminf_{n\to\infty} \alpha_n\Big(\frac{1-2b}{2a-1}\Big) > 0, \quad 0 < \alpha < \alpha_n < \beta < 1,$$

$\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is a sequence in $[d,1]$ for some $d \in (0,1)$ such that

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

If $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and for all $y \in C$,

$$\begin{cases} u_n \in C \ \textit{such that} \ \langle y - u_n, u_n - x_n\rangle \geq 0, \\ y_n = (1-\beta_n)f(x_n) + \beta_n Su_n, \\ x_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n. \end{cases}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $v \in Fix(S) \cap Fix(T)$, where $v = \lim_{n\to\infty} P_{Fix(S)\cap Fix(T)}(x_n)$.

Proof. Let $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.3. Then we have the result. □

4 Numerical Examples

In this section, we give some examples to illustrate Theorem 3.1 and 3.3.

Example 4.1. *Let $H = \mathbb{R}$ and $C = [-10, 10]$. Define a bifunction $F : C \times C \to \mathbb{R}$ by*

$$F(u, y) := -5u^2 + uy + 4y^2$$

for all $u, y \in C$. We see that F satisfies the conditions (A1)-(A4) as follows:

(A1) $F(u, u) = -5u^2 + u^2 + 4u^2 = 0$ *for all* $u \in [-10, 10]$;

(A2) $F(u, y)+F(y, u) = -y^2+2uy-u^2 = -(y-u)^2 \leq 0$ *for all* $u, y \in [-10, 10]$, *i.e., F is a monotone;*

(A3) *for each* $u, y \in [-10, 10]$,

$$\begin{aligned}
&\lim_{t\downarrow 0} F(tz + (1-t)u, y)\\
&= \lim_{t\downarrow 0}(-5(tz + (1-t)u)^2 + (tz + (1-t)u)y + 4y^2)\\
&= -5u^2 + uy + 4y^2\\
&= F(u, y);
\end{aligned}$$

(A4) *for all* $u \in [-10, 10]$, $y \mapsto (-5u^2 + uy + 4y^2)$ *is convex and lower semi-continuous.*

From Lemma 2.3, W_r is single-valued. Let $u = W_r x$ for any $y \in [-10, 10]$ and $r > 0$. Then we have

$$\begin{aligned}
F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle &= 4ry^2 + (ru + u - x)y - 5ru^2 - u^2 + ux\\
&\geq 0
\end{aligned}$$

Let $G(y) = 4ry^2 + (ru + u - x)y - 5ru^2 - u^2 + ux$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4r, b = ru + u - x$ and $c = -5ru^2 - u^2 + ux$. Hence we have $\triangle = b^2 - 4ac \leq 0$, i.e.,

$$\begin{aligned}
\triangle &= (ru + u - x)^2 - 16r(-5ru^2 - u^2 + ux)\\
&= ((9r + 1)u - x)^2\\
&\leq 0.
\end{aligned}$$

Thus it follows that $u = \frac{x}{9r+1}$ *and so* $W_r x = \frac{x}{9r+1}$. *Therefore, we have*

$$u_n = W_{r_n} x_n = \frac{x_n}{9r+1}.$$

Since $Fix(W_{r_n}) = \{0\}$, *from Lemma 2.3, it follows that* $EP(F) = \{0\}$.

Define a mapping $S : C \to C$ *by* $Sx = \dfrac{x}{n}$, $\forall x \in C$, $\forall n \in \mathbb{N}$ *and* $Fix(S) = \{0\}$. *Therefore,* S *is an* m*-generalized hybrid mapping, where*

$$\gamma_k = \frac{1}{m}, \ \ \lambda_k = \frac{1}{n^2 m}, \ \ \lambda_1 = \frac{n^2-1}{n^2}$$

for each $k = 1, 2, \cdots, m$. *Define a mapping* $f : C \to C$ *by* $f(x) = e^{-e^{-x}}$ *for all* $x \in C$. *Then* f *is a contraction.*

Now, we consider different parameters into 3 Cases:

Case 1: $\alpha_n = \frac{1}{4} + \frac{1}{4n}$ *and* $\beta_n = \frac{1}{2} - \frac{1}{5n}$;
Case 2: $\alpha_n = \frac{1}{20} + \frac{1}{5n}$ *and* $\beta_n = \frac{1}{10} - \frac{1}{6n}$;
Case 3: $\alpha_n = \frac{1}{100} + \frac{1}{6n}$ *and* $\beta_n = \frac{1}{50} - \frac{1}{7n}$.

Assume that $r_n = \frac{1}{36}$. *Then the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{r_n\}$ *satisfy the conditions in Theorem 3.1 and stop criterion to* $\|x_n - x^*\| \leq 5 \times 10^{-3}$, *where*

$$x^* \in \theta := Fix(S) \cap EP(F).$$

Since $u_n = \frac{4}{5} x_n$, *we have*

$$y_n := (1-\beta_n) e^{-e^{-x_n}} + \Big(\frac{1}{n}\Big)\Big(\frac{4}{5}\Big)\Big(\beta_n\Big) x_n.$$

Also, we have

$$x_{n+1} := (1-\alpha_n)\Big(\frac{1}{n}\Big) x_n + \Big(\frac{1}{n}\Big)\alpha_n \Big[(1-\beta_n) e^{-e^{-x_n}} + \Big(\frac{1}{n}\Big)\Big(\frac{4}{5}\Big)\beta_n x_n\Big]. \quad (4.1)$$

Table 1. *The numerical results for* $x_1 = -0.1$.

n	x_n (Case 1)	x_n (Case 2)	x_n (Case 3)
1	-0.1	-0.1	-0.1
2	0.0539	0.0146	-0.0149
3	0.0621	0.0338	0.0110
4	0.0392	0.0241	0.0118
5	0.0234	0.0143	0.0077
6	0.0156	0.0089	0.0047
7	0.0116	0.0061	0.0031
8	0.0092	0.0046	0.0022
9	0.0077	0.0037	0.0017
10	0.0066	0.0031	0.0013
11	0.0058	0.0027	0.0011
12	0.0051	0.0023	0.0009
13	0.0046	0.0021	0.0008

Since $\Theta = \{0\}$, we have $P_\Theta(x_n) = 0$ for all $n \geq 1$. See Figure 1 for the initial value $x_1 = -0.1$. We use package program for compute this example.

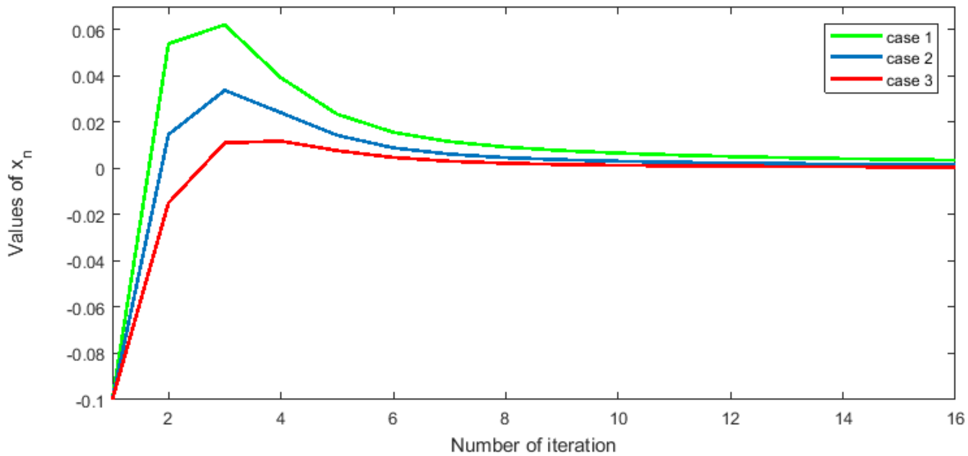

FIGURE 1

From Figure 4.1, We choose the initial value $x_1 = -0.1$. The green line show the speed of convergence for both $\alpha_n = \frac{1}{4} + \frac{1}{4n}$ and $\beta_n = \frac{1}{2} - \frac{1}{5n}$. The blue line show the speed of convergence for both $\alpha_n = \frac{1}{20} + \frac{1}{5n}$ and $\beta_n = \frac{1}{10} - \frac{1}{6n}$. The red line show the speed of convergence for both $\alpha_n = \frac{1}{100} + \frac{1}{6n}$ and $\beta_n = \frac{1}{50} - \frac{1}{7n}$. So, the red line converge fastest.

Example 4.2. *Let $H = \mathbb{R}$ and $C = [-10, 10]$. Define a bifunction $F : C \times C \to \mathbb{R}$ by*

$$F(u, y) := -5u^2 + uy + 4y^2$$

for all $u, y \in C$. Thus, similar to Example 4.1, F satisfies the conditions (A1)–(A4) *and we have*

$$u_n = W_{r_n} x_n = \frac{x_n}{9r + 1}, \quad Fix(W_{r_n}) = EP(F) = \{0\}.$$

Define a mapping $S : C \to C$ by $Sx = \dfrac{x}{n}$, $\forall x \in C$, $\forall n \in \mathbb{N}$. Then $Fix(S) = \{0\}$ and S is an m-generalized hybrid mapping, where

$$\gamma_k = \frac{1}{m}, \quad \lambda_k = \frac{1}{n^2 m}, \quad \lambda_1 = \frac{n^2 - 1}{n^2}$$

for each $k = 1, 2, \cdots, m$. Define a mapping $T : C \to C$ by $Tx = -\frac{\sqrt{2}}{2}x$, $\forall x \in C$, $\forall n \in \mathbb{N}$. Then $Fix(T) = \{0\}$ and T is an (a, b)-monotone mapping, where $a = 2$ and $b = 1$. Define a mapping $f : C \to C$ by $f(x) = e^{-e^{-x}}$ for all $x \in C$. Then f is a contraction.

Now, we consider different parameters into 3 Cases:
Case 1: $\alpha_n = \frac{1}{2000} + \frac{1}{10n}$ *and* $\beta_n = \frac{1}{5} - \frac{1}{7n}$;
Case 2: $\alpha_n = \frac{1}{300} + \frac{1}{25n}$ *and* $\beta_n = \frac{1}{30} - \frac{1}{8n}$;
Case 3: $\alpha_n = \frac{1}{500} + \frac{1}{50n}$ *and* $\beta_n = \frac{1}{50} - \frac{1}{9n}$.

Assume that $r_n = \frac{1}{36}$. *Then the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{r_n\}$ *satisfy the conditions in Theorem 3.3 and stop criterion to* $\|x_n - x^*\| \le 5 \times 10^{-3}$, *where*

$$x^* \in \Omega := Fix(S) \cap Fix(T) \cap EP(F).$$

Since $u_n = \frac{4}{5}x_n$, *we have*

$$y_n := (1-\beta_n)e^{-e^{-x_n}} + \left(\frac{1}{n}\right)\left(\frac{4}{5}\right)\left(\beta_n\right)x_n.$$

Also, we have

$$x_{n+1} := (1-\alpha_n)\left(\frac{1}{n}\right)x_n + (-\frac{\sqrt{2}}{2})\alpha_n\left[(1-\beta_n)e^{-e^{-x_n}} + \left(\frac{1}{n}\right)\left(\frac{4}{5}\right)\beta_n x_n\right].$$

Table 2. *The numerical results for* $x_1 = -0.1$.

n	x_n (Case 1)	x_n (Case 2)	x_n (Case 3)
1	-0.1	-0.1	-0.1
2	-0.1114	-0.1070	-0.1035
3	-0.0626	-0.0578	-0.0541
4	-0.0270	-0.0231	-0.0200
5	-0.0119	-0.0091	-0.0068
6	-0.0067	-0.0047	-0.0029
7	-0.0047	-0.0033	-0.0019

Since $\Omega = \{0\}$, *we have* $P_\Omega(x_n) = 0$ *for all* $n \ge 1$. *See Figure 2 for the initial values* $x_1 = -0.1$. *We use package program for compute this example.*

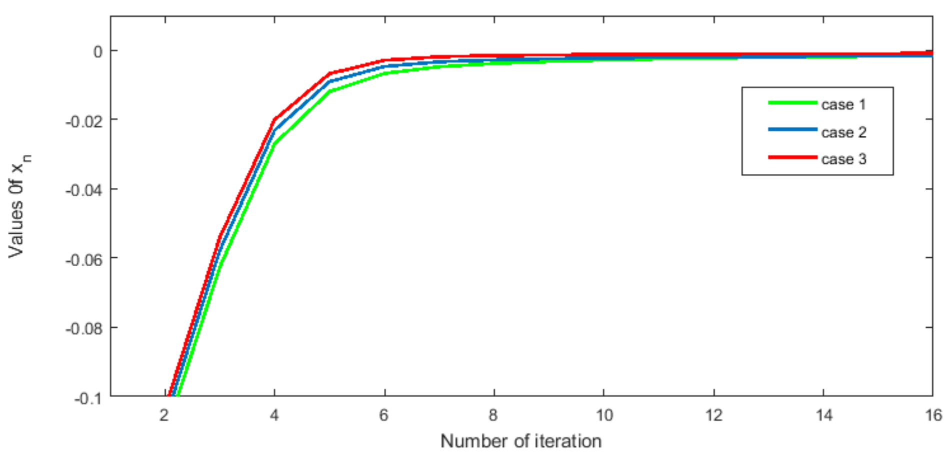

FIGURE 2

From Figure 2, We choose the initial value $x_1 = -0.1$. The green line show the speed of convergence for both $\alpha_n = \frac{1}{2000} + \frac{1}{10n}$ and $\beta_n = \frac{1}{5} - \frac{1}{7n}$. The blue line show the speed of convergence for both $\alpha_n = \frac{1}{300} + \frac{1}{25n}$ and $\beta_n = \frac{1}{30} - \frac{1}{8n}$. The red line show the speed of convergence for both $\alpha_n = \frac{1}{500} + \frac{1}{50n}$ and $\beta_n = \frac{1}{50} - \frac{1}{9n}$. So, the red line converge fastest.

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