



## Some Notes on Cone Metric Spaces

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**Abstract :** Recently, several articles have been written on the cone metric spaces. Despite the fact that any cone metric space is equivalent to a usual metric space, we aim in this paper to deal with some of the published articles on cone metric spaces by repairing some gaps, providing new proofs and extending their results to topological vector spaces. Several authors have worked with a class of special cones which known as strongly minihedral cones where the strongly minihedrality condition (that is, each nonempty bounded above subset has a least upper bound) is very restrictive. Another goal of this article is to eliminate or mitigate this condition. Furthermore, we present some examples in order to show that the imagination of many authors that the behavior of the ordering induced by a strongly minihedral cone is just as the behavior of the usual ordering on the real line, that has caused an error in their proofs, is not correct. We establish a relationship between strong minihedrality and total orderness. Finally, a fixed point theorem for a contractive mapping, which generalizes the corresponding result given in [1], is investigated. One can consider the results of this paper as a generalization and correction of some recent papers that have been written in this area.

**Keywords :** cone metric space; first countable; strongly minihedral cone; totally ordered; sequentially compact; contractive mapping.

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## 1 Introduction

Investigation of K-metric space (also known as cone metric space) was introduced by several Russian authors in the middle of 20th century [2]. Ordered normed spaces and cones have applications in applied mathematics, for instance in using Newton's approximation method [3]. They differ from usual metric spaces in the fact that the values of distance functions are not positive real numbers, but elements of a cone in some normed space [4] or topological vector space [5–9]. L.G. Huang and X. Zhang [4], re-introduced cone metric spaces and also went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. They proved some fixed point theorems for this class of spaces.

Recently, several articles have been written on the topological properties of cone metric spaces. In some of them there are some gaps which one of the aims of this paper is to deal with them by providing new proofs and extending the results to general case. Several authors were interested to work with strongly minihedral cones where this condition (that is, each nonempty bounded above subset has the least upper bound) is very restrictive. Another goal of this note is to eliminate or mitigate this condition. Furthermore, some examples are presented in this article to show that the behavior of the ordering induced by a strongly minihedral cone is so different from the behavior of the usual ordering on the real line. Some authors used in their proofs while working on strongly minihedral that the order is total without imposing such an assumption. We shall repair this and discuss the relationship between being strongly minihedral and total orderedness. Finally, a fixed point theorem for a contractive mapping is presented which is an improvement of the corresponding result given in [1].

In the rest of this section we recall some definitions which are needed in the sequel.

Let  $E$  be a vector space with its zero vector  $\theta$ . By a cone  $P \neq \{\theta\}$  we understand a subset of  $E$  such that  $\lambda P \subseteq P$  for all  $\lambda \geq 0$  and  $P \cap -P = \{\theta\}$ . Given a cone  $P \subseteq E$ , we define a *partial ordering*  $\preceq$  with respect to  $P$  by  $x \preceq y$  if and only if  $y - x \in P$ . We shall write  $x \prec y$  (or  $y \succ x$ ) to indicate that  $x \preceq y$  but  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int}P$  if  $P$  has nonempty interior. From now onward, we always suppose that  $E$  is a real topological vector space (t.v.s., for short) unless otherwise explicitly stated, with its zero vector  $\theta$ ,  $P$  is a closed cone with  $\text{int}P \neq \emptyset$ ,  $e \in \text{int}P$  and  $\preceq$  a partial ordering induced by  $P$ . The cone  $P$  is called *strongly minihedral* if for any nonempty subset of  $E$  which is bounded above with respect to the ordering induced by  $P$  has a least upper bound. Also  $P$  is called *regular* if for any increasing sequence which is bounded from above is convergent, that is, if  $\{x_n\}$  is a sequence such that  $x_1 \preceq x_2 \preceq \dots \preceq y$  for some  $y \in E$  then there is  $x \in E$  such that  $\{x_n\}$  converges to  $x$ . A cone  $P$  of a t.v.s  $E$  is said to be *normal* if  $X$  has a local base of zero consisting of  $P$ -full sets, where a subset  $A$  is said to be  $P$ -full if  $\{z \in E : x \preceq z \preceq y\} \subseteq A$  for all  $x, y \in A$  (see, for more details, [10, 11]). One can see that when  $(E, \|\cdot\|)$  is a normed space and  $P$  a cone of  $E$  then being normal is equivalent to finding a constant  $k \geq 1$  such that for all  $x, y \in E$ ,  $\theta \preceq x \preceq y$  implies  $\|x\| \leq k\|y\|$ .

By a *cone metric space* we mean an ordered pair  $(X, d)$  where,  $X$  is any

nonempty set and  $d : X \times X \rightarrow E$  is a mapping, called *cone metric*, satisfying the following conditions:

- (i)  $\theta \preceq d(x, y)$ , for all  $x, y \in X$ ,  $d(x, y) = \theta$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (iii)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

It is easy to see that if  $(X, d)$  is a cone metric space, then the family  $\{B(x, c)\}_{(x,c) \in X \times \text{int}P}$  is a basis for a topology on  $X$ , where

$$B(x, c) = \{y \in X : d(x, y) \ll c\} \quad \text{and} \quad c \in \text{int}P.$$

**Definition 1.1.** A sequence  $(x_n)$  in a cone metric space  $(X, d)$  is said to *converge* to an element  $x \in X$  if for any  $c \in \text{int}P$  there exists a natural number  $n_0$  such that

$$d(x_n, x) \ll c, \quad \forall n > n_0.$$

**Definition 1.2.** A sequence  $(x_n)$  in a cone metric space  $(X, d)$  is said to be *Cauchy* if for any  $c \in \text{int}P$  there exists a natural number  $n_0$  such that

$$d(x_n, x_m) \ll c, \quad \forall n, m > n_0.$$

Cone metric spaces in which every Cauchy sequence is convergent are called *complete cone metric spaces*.

## 2 Main Results

In this section we give the main results of the paper. We review some results obtained in the papers [1, 12–15] and then we provide some examples which show that there are some gaps in them. Finally, we repair, by giving new proofs, and extend them to the general topological vector space (t.v.s.) case.

**Theorem 2.1.** (see [16, Theorem 2.1] or [3]) *If the underlying cone of an ordered t.v.s is solid and normal, then such the t.v.s must be an ordered normed space.*

We begin with the following proposition which is a t.v.s version of [13, Lemma 14] which was proved in the category of locally convex space by means of seminorms.

**Proposition 2.2.** *Let  $(X, d)$  be a cone metric space over a t.v.s.  $E$  and  $P$  a normal cone of  $E$ . Then the following assertions are true.*

- (i)  $x_n \rightarrow x$  in  $(X, d)$  if and only if  $d(x_n, x) \rightarrow \theta$  in  $E$ .
- (ii)  $\{x_n\}$  is a cauchy sequence in  $(X, d)$  if and only if  $d(x_n, x_m) \rightarrow \theta$  in  $E$ .

*Proof.* To see (i), let  $x_n \rightarrow x$  (in  $X$ ),  $c \in \text{int } P$ , and  $U \in \mathfrak{S}$  (local base of zero of  $P$ -full sets). Then there exist a positive number  $\alpha$  and a natural number  $n_0$  such that  $\alpha c \in U$  and

$$d(x_n, x) \preceq \alpha c, \quad \forall n \geq n_0. \quad (2.1)$$

Since  $U$  is a  $P$ -full set and  $\alpha c \in U$  then the set  $\{z : \theta \preceq z \preceq \alpha c\} \subseteq U$  and so, by (2.1), we get  $d(x_n, x) \in U$  for all  $n \geq n_0$  and hence  $x_n \rightarrow x$  in  $E$ .

Conversely, assume  $d(x_n, x) \rightarrow \theta$  and  $c \in \text{int } P$ . It is clear that  $U = c - \text{int } P$  is a neighborhood of zero and so there exists  $n_0$  such that  $d(x_n, x) \in U = c - \text{int } P$  for all  $n \geq n_0$  and hence  $x_n \rightarrow x$  (in  $X$ ). The proof of (ii) is similar to (i).  $\square$

The following proposition is a t.v.s. version of [13, Lemma 15].

**Proposition 2.3.** *Let  $(X, d)$  be a cone metric space over a normal cone of a topological vector space  $E$ . Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  with  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Then  $d(x_n, y_n) \rightarrow d(x, y)$  (in  $E$ ).*

*Proof.* It follows from Proposition 2.2 that  $d(x_n, x) \rightarrow \theta$  and  $d(y_n, y) \rightarrow \theta$ . Let  $V$  an arbitrary neighborhood of zero. Then there exist an open neighborhood full set (note  $P$  is normal) and a balanced open neighborhood  $W$  such that  $U + W \subseteq V$ . Let  $c \in \text{int } P$ . Then there exist a positive number  $\alpha$  and a positive integer  $n_0$  such that  $-2\alpha c \in W$ ,  $4\alpha c \in U$  and

$$d(x_n, x) \ll \alpha c, \quad d(y_n, y) \ll \alpha c, \quad \forall n > n_0.$$

Hence, by triangle inequality, we have

$$d(x_n, y_n) \preceq d(x_n, x) + d(x, y) + d(y, y_n) \preceq d(x, y) + 2\alpha c, \quad \forall n > n_0 \quad (2.2)$$

and

$$d(x, y) \preceq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \preceq d(x_n, y_n) + 2\alpha c, \quad \forall n > n_0. \quad (2.3)$$

Hence it follows from (2.2) and (2.3) that

$$\theta \preceq d(x, y) - d(x_n, y_n) + 2\alpha c \preceq 4\alpha c.$$

Since  $U$  is a  $P$ -full set with  $4\alpha c \in U$  and  $-2\alpha c \in W$ , then

$$d(x, y) - d(x_n, y_n) = d(x, y) - d(x_n, y_n) + 2\alpha c - 2\alpha c \in U + W \subseteq V, \quad \forall n > n_0.$$

Since  $V$  is an arbitrary open set then the proof is completed.  $\square$

**Remark 2.4.** The proof of sufficiency in Proposition 2.2 which have been done under normality and solidity of the cone can be achieved by Theorem 2.1 and in [4, Lemma 1, Lemma 4] or [17, Lemma 1.5]. Similarly, the proof of Proposition 2.3 can be achieved alternatively by Theorem 2.1 and in [4, Lemma 5] or [17, Lemma 1.7].

To prove Hausdorffness of the cone metric space, the authors in [1] assumed that if  $(X, d)$  is a cone metric space and  $x \neq y$  are two points in  $X$  then  $d(x, y) = c \succ \theta$  is a member of interior of  $P$ , that is  $c \in \text{int}P$  (see, [1, page 491]) which is not true in general. Next, we state and present a proof for the fact that every cone metric space is Hausdorff.

**Theorem 2.5.** *Every cone metric space  $(X, d)$  is Hausdorff.*

*Proof.* Let  $x \neq y$  be two arbitrary points in  $X$ . If  $B(x, c) \cap B(y, c) \neq \emptyset$ , for all  $c \in \text{int}P$ , then there is  $z_c \in B(x, c) \cap B(y, c)$  and so by triangle inequality we have  $d(x, y) \preceq d(x, z_c) + d(z_c, y) \ll c + c = 2c$ , for all  $c \in \text{int}P \neq \emptyset$ . Hence, for all  $n \in \mathbb{N}$  and a chosen  $e \in \text{int}P \neq \emptyset$  we have

$$\frac{e}{n} - d(x, y) \in \text{int}P.$$

Letting  $n$  tends to  $\infty$  and using that the cone is closed, we conclude that  $-d(x, y) \in P$ . Therefore,  $d(x, y) = \theta$  and hence  $x = y$  which is a contradiction. This completes the proof.  $\square$

**Remark 2.6.** The proof of Theorem 2.5, alternatively can be done by using that being Hausdorff is a topological property and that every *t.v.s* cone metric space is isomorphic to a usual metric space [5].

**Proposition 2.7.** [1] *Every cone metric space is first countable.*

*Proof.* Let  $e \in \text{int}P$  be an arbitrary element. For each  $x \in X$ , the family  $\{B(x, \frac{1}{n}e)\}_{n \in \mathbb{N}}$  is a countable set of neighborhoods. If  $B(x, c)$  is a neighborhood, then there is a natural number  $n$  such that  $\frac{1}{n}e \ll c$  and so  $B(x, \frac{1}{n}e) \subset B(x, c)$  and so the proof is finished.  $\square$

The proof of the locally convex version of Proposition 2.7 can be found in [13].

**Corollary 2.8.** *Let  $(X, d)$  be a cone metric space and  $A$  a subset of  $X$ .  $A$  is closed if and only if  $A$  is sequentially closed.*

In [1, Proposition 2] and [13, Proposition 25], respectively, the authors used the notation  $\overline{B}(x, c)$  for the set  $\{x \in y : d(y, x) \leq c\}$ , which was just a notation and different from the closure of  $B(x, c) = \{y \in X : d(y, x) \ll c\}$ . Below, we give an example showing that the closure of  $B(x, c)$  is a proper subset of  $\overline{B}(x, c)$ .

Let  $X = E = \mathfrak{R}$  and  $P = [0, \infty)$  and define  $d : X \times X \rightarrow \mathfrak{R}$  as follows:

$$\begin{cases} d(x, y) = 1, & \text{if } x \neq y, \\ d(x, y) = 0, & \text{if } x = y. \end{cases}$$

Then  $\overline{B}(x, 1) = \{x\} \neq \{y \in X : d(x, y) \leq 1\} = X = \mathfrak{R}$ .

In fact, in [13], it was proved that the set  $\overline{B}(x, c)$  is sequentially closed. From the proof, one can conclude that the (sequential) closure of  $B(x, c)$  is a subset of

$\overline{B}(x, c)$ . The extension to arbitrary *t.v.s* and not necessary locally convex is easy. Indeed, if  $\{y_n\}$  is a sequence with the properties  $d(y_n, x) \leq c$  and  $y_n \rightarrow y$ , then for positive number  $\epsilon$  there exists  $n_\epsilon$  such that  $d(y_{n_\epsilon}, y) \ll \epsilon c$ . So

$$d(x, y) \leq d(y_{n_\epsilon}, y) + d(y_{n_\epsilon}, x) \leq \epsilon c + c = (1 + \epsilon)c.$$

Hence  $(1 + \epsilon)c - d(x, y) \in P$ . By letting  $\epsilon$  to zero and using the closedness of  $P$  we obtain  $c - d(x, y) \in P$ .

The following Lemma given in [12] and applied, for example in [15, Lemma 1.9] and [14, page 3].

**Lemma 2.9.** (i) *Every strongly minihedral (not necessarily closed) normal cone is regular.*

(ii) *Every strongly minihedral closed cone is normal (is also regular by (i)).*

The assumption that the ordering is totally ordered was missed. The following example shows that total orderness can not be dropped.

**Example 2.10.** Let  $E = \ell^\infty = \{x = (x_1, x_2, \dots, x_n, \dots) : \sup |x_n| < \infty\}$  and

$$\|x\| = \sup |x_n|.$$

Clearly, the set  $P = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \geq 0, \forall n = 1, 2, \dots\}$  is a cone of  $E$ . Also  $\theta \preceq x = (x_1, x_2, \dots, x_n, \dots) \preceq y = (y_1, y_2, \dots, y_n, \dots)$  implies that  $\|x = (x_1, x_2, \dots, x_n, \dots)\| \leq \|y = (y_1, y_2, \dots, y_n, \dots)\|$  and so  $P$  is a normal cone. If  $A$  is a bounded above subset of  $E$  then it is easy to see that  $\sup A = (a_1, a_2, \dots, a_n, \dots)$  where

$$a_n = \sup\{x_n : x = (x_1, x_2, \dots, x_n, \dots), x \in A\}, \quad \forall n = 1, 2, 3, \dots$$

Hence  $P$  is strongly minihedral. Now we show that  $P$  is not regular. To see this, consider the sequence

$$a^{(1)} = (1, 0, 0, 0, \dots), \quad a^{(2)} = (1, 1, 0, 0, \dots), \quad a^{(n)} = (1, 1, \dots, \underbrace{1}_{n^{th} \text{ place}}, 0, 0, \dots), \dots$$

Obviously,  $a^{(n)} \in E$  and  $a^{(n)} \preceq a^{(n+1)}$ , for all  $n = 1, 2, \dots$  and so  $\{a^{(n)}\}$  is an increasing sequence which  $a = (1, 1, 1, \dots)$  is an upper bound for it. But the sequence is not a cauchy sequence because  $\|a^{(n+1)} - a^{(n)}\| = 1$ . Consequently  $\{a^{(n)}\}$  is not a convergent sequence and so  $P$  is not a regular cone. Clearly, the investigated ordering via the cone  $P$  is not totally ordered.

**Remark 2.11.** If one review the proofs given for [1, Lemma 2.6] and [1, Lemma 5] will realize that the total orderness assumption is missed there as well. In fact, total orderness guarantees the existence of an element  $a_M$  of an ordering space such that

$$a - c \ll a_M \ll a \tag{2.4}$$

where  $a$  is the supremum of an increasing sequence  $\{a_n\}$  and  $c$  is an element of the interior of the cone  $P$ . Even the relation (2.4) does not hold when the sequence  $\{a_n\}$  is constant. For example if we take  $E = \mathfrak{R}$  (the set of real numbers) and  $P = [0, \infty)$  then nonempty subset  $A$  of  $\mathfrak{R}$  has a supremum in  $\mathfrak{R}$ , say  $\sup A = \alpha$ , if and only if the following conditions are satisfied:

- (C1)  $\alpha$  is an upper bounded (that is,  $x \leq \alpha$ , for all  $x \in A$ )
- (C2) for each  $c \in \mathfrak{R}$  with  $c > 0$  ( that is  $c \in \text{int}P$ ) there is  $x \in A$  such that  $\alpha - c < x$ .

In general, if the ordering induced by a cone  $P$  is totally ordered, then  $x \prec y$  implies that  $y - x \in \text{int}P$  and  $\alpha = \sup A$  if and only if  $\alpha$  fulfils (C1) and (C2) it is worth mentioning that the above behavior may even fail for arbitrary strongly minihedral normal cones as can be seen in Example 2.10, or simply let  $E = \mathfrak{R}^2$  and  $P = \{(x, y) : 0 \leq x \leq y\}$  with Euclidean norm. Then, the cone  $P$  is normal and strongly minihedral for which the ordering induced by  $P$  on  $E$  is neither totally order (for instance,  $a = (1, 3)$  and  $b = (2, 2)$  are not comparable) nor  $x < y$  implies  $y - x \in \text{int}P$  (for example  $(0, 0) \prec (1, 1)$  and  $(1, 1) \notin \text{int}P$ ).

Now the question that under which conditions a minihedral cone induces a totally ordering is posed. One can see that if we add condition (C2) to a minihedral cone then the ordering induces is a totally ordering and this is an answer to the question. The simple example  $Q$  (the rational numbers) with usual ordering shows that totally ordering solely cannot imply the existence least upper bound for any nonempty bounded above subset (for example,  $A = \{x \in Q : 0 \leq x \leq \sqrt{2}\}$ ). This means that the totally ordering cannot always ensure the strongly minihedralness.

In summary, it is obvious that the totally ordering (induced by  $P$ ) and minihedral property imply the relation (C2).

The next example shows that the normal condition cannot lonely imply the regular condition (that is a normal cone is not necessary regular).

**Example 2.12.** Let  $E = C[0, 1]$  (the class of all real continuous function on  $[0, 1]$ ) with the supremum norm and  $P = \{f \in E : f(x) \geq 0, \forall x \in [0, 1]\}$ . Then  $P$  is a cone with normal constant  $k = 1$  which is not regular. This is clear, since the sequence  $\{f_n(x) = x^n\}$  is monotonically decreasing, but not uniformly convergent to zero function  $\theta$  and  $\inf\{f_n(x) = x^n : x \in [0, 1], n = 1, 2, \dots\} \notin E$ . This means  $P$  is not strongly minihedral.

Next we present *t.v.s* version of Lemma 2.9.

**Lemma 2.13.** *Let  $E$  be a t.v.s. space and  $P$  a cone inducing a total orderness on  $E$ . Then The following statements are true.*

- (i) *If  $P$  is strongly minihedral (not necessarily closed) normal cone then it is regular.*
- (ii) *If  $P$  is strongly minihedral closed then it is normal (is also regular by (i)).*

*Proof.* (i) Let  $\{x_n\}$  be an increasing sequence with least upper bound  $x$ , that is  $x = \sup\{x_n : n = 1, 2, \dots\}$  (note that  $P$  is a strongly minihedral cone). We claim that  $\lim_{n \rightarrow \infty} x_n = x$ . Indeed let  $U$  be a balanced open full-set of zero (note  $P$  is a normal cone) and  $e \in \text{int}P$ . Then there is a positive number  $\alpha$  such that  $\alpha c \in U$ . Since  $x - \alpha c \preceq x$  and  $x = \sup\{x_n : n = 1, 2, \dots\}$ , then it follows from the definition of the supremum that there exists  $x_{n_0}$  such that  $x_{n_0} \not\preceq x - \alpha c$  and so  $x - \alpha c \preceq x_{n_0}$  (note that  $\preceq$  is totally ordered). Then

$$x - \alpha c \preceq x_{n_0} \preceq x_n, \quad \forall n > n_0$$

and hence

$$\theta \preceq x - x_n \preceq \alpha c, \quad \forall n > n_0.$$

Then since  $U$  is a full-set (note  $\theta, \alpha c \in U$ ) we get

$$x - x_n \in U \quad \forall n > n_0,$$

and so

$$x_n \in x - U (= x + U), \quad \forall n > n_0,$$

(note  $U$  is a balanced set) and hence that  $x_n \rightarrow x$ .

The proof of (ii) is similar to the proof of (ii) in [12, Lemma 1]. □

**Remark 2.14.** Alternatively, by means of Theorem 2.1, the proof of Lemma 2.13 can be proved following the normed space proof version of Lemma 2.9 presented in [12] after taking into account the total orderness assumption.

Let  $A$  be a subset of a cone metric space  $X$ . We say that  $A$  is *bounded* if there exists  $c \in \text{int}P$  such that  $d(x, y) \preceq c$ , for all  $x, y \in A$ . This definition is equivalent to the following statement:

$$\exists c \in \text{int}P, a \in E; d(a, x) \preceq c, \quad \forall x \in A.$$

It worth noting that in the aforementioned definition we do not need the existence of the least upper bound for the set  $\{d(x, y) : x, y \in A\}$  whence it was necessary in [1, Definition 6]. Moreover, the definition is an extension of the boundedness for metric spaces.

The following proposition is a t.v.s. version of in [1, Proposition 3] and in [13, Proposition 29] by eliminating the strongly minihedral assumption.

**Proposition 2.15.** *Let  $(X, d)$  be a cone metric space over a t.v.s.  $E$  and  $P$  a normal cone of  $E$ . Then  $A \subset X$  is bounded if and only if the set  $\{d(x, y) : x, y \in A\}$  is a bounded subset of  $E$ .*

*Proof.* Let  $A$  be bounded and  $U$  a balanced open (full-set) neighborhood of zero (note  $P$  is normal). Since  $A$  is bounded then there exists  $c \in \text{int}P$  such that

$$d(x, y) \preceq c, \quad \forall x, y \in A. \tag{2.5}$$



There is a positive number  $\alpha$  such that  $\alpha c \in U$ . From (2.5), we get  $\alpha d(x, y) \preceq \alpha c$ . Since  $U$  is a full set and  $\theta, \alpha c \in U$ , we deduce

$$\alpha d(x, y) \in U, \quad \forall x, y \in A.$$

This means that  $U$  absorbs the set  $\{d(x, y) : x, y \in A\}$  and so  $\{d(x, y) : x, y \in A\}$  is a bounded subset of  $E$ . Conversely, suppose  $\{d(x, y) : x, y \in A\}$  is a bounded subset of  $E$  and on the contrary that  $A$  is not bounded. Then for all  $e \in \text{int}P$  and natural number  $m$  there exists  $(x_{n_m}, y_{n_m}) \in A \times A$  such that  $d(x_{n_m}, y_{n_m}) \not\preceq me$  and so

$$me - d(x_{n_m}, y_{n_m}) \in E \setminus P \subset E \setminus \text{int}P.$$

Hence,

$$e - \frac{1}{m}d(x_{n_m}, y_{n_m}) \in E \setminus P \subset E \setminus \text{int}P,$$

(note  $E \setminus P$  stands for the complement of  $P$  with respect to  $E$  and  $E \setminus P, E \setminus \text{int}P$  are closed under positive scalar multiplication),  $E \setminus \text{int}P$  is closed, and

$$e - \frac{1}{m}d(x_{n_m}, y_{n_m}) \rightarrow e$$

that  $e \in E \setminus \text{int}P$  which is contradicted by  $e \in \text{int}P$  and so  $A$  is a bounded subset of  $E$ . This completes the proof.  $\square$

**Definition 2.16.** [1] Let  $(X, d)$  be a cone metric space and  $c \in \text{int}P$ . A finite subset  $M = \{e_1, e_2, \dots, e_n\}$  of  $X$  is called a  $c$ -net for  $A \subseteq X$  if for each  $x \in A$  there exists  $e_{i_0} \in M$  such that  $d(x, e_{i_0}) \ll c$ .

The following definition is slightly different from [1, Definition 8] by relaxing the strongly minihedral condition on the cone.

**Definition 2.17.** Let  $(X, d)$  be a cone metric space. A subset  $A$  of  $X$  is called *totally bounded* if for each  $c \in \text{int}P$  there exist bounded subsets  $M_1, \dots, M_k$  of  $X$  such that

$$A \subseteq \cup_{i=1}^k M_i \quad \text{and} \quad d(x, y) \ll c, \quad \forall x, y \in M_i, \quad \forall i = 1, 2, \dots, k.$$

It is clear that a finite set is totally bounded and a totally bounded set is bounded while the converse is not true as the following example.

**Example 2.18.** Take  $X = E = \mathfrak{R}$  and  $P = [0, \infty)$  with discrete metric (that is  $d(x, y) = 1$ , if  $x \neq y$  and  $d(x, y) = 0$ , if  $x = y$ ). It is clear that  $X$  is a bounded subset (take  $c=2$ ) but it is not totally bounded because if we take  $c = \frac{1}{2}$  then  $M_i$ , in the definition of totally boundedness, should be singleton and it is impossible to cover an infinite set with a finite family of singletons.

It follows from the proof of Proposition 4 in [1] that the strongly minihedral assumption is necessary but it has been omitted from the hypothesis of the proposition (see [1, page 492]). In the following, we will prove Proposition 4 in [1] based upon Definitions 2.12 and 2.13 without using the notion of the strongly minihedral and its proof is slightly different from that given in [1]. However, for the sake of completeness we present the proof here again.

**Proposition 2.19.** *Let  $(X, d)$  be a cone metric space and  $A$  a subset of it. Then  $A$  is totally bounded if and only if for each  $c \in \text{int}P$ ,  $A$  has a  $c$ -net.*

*Proof.* Let  $A$  be totally bounded and  $c \in \text{int}P$ . Then there exist bounded subsets  $M_1, \dots, M_k$  such that  $A \subset \cup_{i=1}^k M_i$  and  $d(x, y) \ll c$ , for all  $x, y \in \cup_{i=1}^k M_i$  and so  $M = \cup_{i=1}^k M_i$  is a  $c$ -net for  $A$ . To see the converse, let  $c \in \text{int}P$ . Hence there exists  $M = \{e_1, \dots, e_n\}$  such that for each  $x \in A$  there exists  $e_i \in M$  ( $1 \leq i \leq n$ ) such that  $d(x, e_i) \ll c$ . Now we take  $M_i = B(e_i, c)$  for  $i = 1, 2, \dots, n$ . It is clear  $d(x, y) \ll c$  for all  $x, y \in M_i$ , for all  $i = 1, 2, \dots, n$  and  $A \subset \cup_{i=1}^n M_i$ . This completes the proof.  $\square$

The following definition is slightly different from [1, Definition 9].

**Definition 2.20.** Let  $(X, d)$  be a cone metric space. An element  $c \in \text{int}P$  is called *Lebesgue* of an open cover  $\Lambda = \{G_i\}$  for a subset  $A$  of  $(X, d)$  if for each subset  $B$  of  $A$  with  $d(x, y) \prec c$ , for all  $x, y \in B$ , there exists  $G_{i_0} \in \Lambda$  such that  $B \subseteq G_{i_0}$ .

A subset  $B$  of a cone metric space  $(X, d)$  is called *compact* if every open cover for  $B$  can be reduced to finite subcover. Also  $B$  is said to be *sequentially compact* if for any sequence  $\{x_n\}$  in  $B$  there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  which is convergent to a point in  $B$ .

**Proposition 2.21.** [1] *Let  $(X, d)$  be a cone metric space and  $A \subset X$ . If  $A$  is sequentially compact, then it is totally bounded.*

The following proposition is converse of Proposition 2.21.

**Proposition 2.22.** *Let  $(X, d)$  be a complete cone metric space and  $A \subset X$ . If  $A$  is totally bounded then it is sequentially compact.*

*Proof.* Let  $A$  be totally bounded and  $\{x_n\}$  a sequence in  $A$ . On the contrary assume that  $\{x_n\}$  does not have any Cauchy subsequence ( $(X, d)$  is complete). Hence for each subsequence  $\{x_{n_k}\}$  there exist  $c \in \text{int}P$  and natural numbers  $k_1, k_2$  such that  $c - d(x_{n_{k_1}}, x_{n_{k_2}}) \notin \text{int}P$ . Since  $A$  is totally bounded then there is  $M = \{e_1, e_2, \dots, e_n\}$  so that, for each  $x \in A$ , there exists  $e_i \in M$  with  $\frac{c}{2} - d(x, e_i) \in \text{int}P$  and then  $c - d(x_{n_{k_1}}, x_{n_{k_2}}) \in \text{int}P$  (note that  $\{n_k\}$  is infinite and  $M$  is finite) which is a contradiction.  $\square$

**Proposition 2.23.** [1] *Let  $(X, d)$  be a cone metric space and  $A$  is sequentially compact subset of  $X$ . Then every open covering for  $A$  has a Lebesgue element.*

In the following, we show that a sequential compact set of a cone metric space is compact set. The proof is slightly different and repair the proof given in [1, Proposition 7]. Moreover, we do not assume that the cone has strongly minihedral property.

**Proposition 2.24.** *Let  $(X, d)$  be a cone metric space. Then every sequentially compact subset  $A \subset X$  is compact.*

*Proof.* Let  $\Lambda = \{G_i\}_{i \in I}$  be an open cover for  $A$ . By Proposition 2.17 there exists  $c \in \text{int}P$  such that for any  $B \subseteq A$  with  $d(x, y) \prec c$ , for all  $x, y \in B$ , there is  $i \in I$  with  $B \subset G_i$ . It follows from to be totally boundedness of  $A$  that there exist  $N_1, \dots, N_k$  so that  $A \subset \cup_{i=1}^k N_i$  with  $d(x, y) \ll c$ , for all  $x, y \in N_i$  for all  $i = 1, 2, \dots, k$ . Now we take  $B_i = A \cap N_i$ , for all  $i = 1, 2, \dots, k$ , then  $B_i \subseteq A$  and  $d(x, y) \ll c$  for all  $x, y \in B_i$  and  $i = 1, 2, \dots, k$ . Hence, for each  $i \in \{1, 2, \dots, k\}$  there exists  $G_i \in \Lambda$  such that  $B_i \subset G_i$  and so  $A = \cup_{i=1}^k B_i \subset \cup_{i=1}^k G_i$  and this completes the proof.  $\square$

**Definition 2.25.** [4] Let  $(X, d)$  be a cone metric space. A mapping  $T : X \rightarrow X$  is called (*Banach*) *contractive* if

$$d(Tx, Ty) \preceq d(x, y), \quad \forall x, y \in X.$$

**Definition 2.26.** [1] A mapping  $T : X \rightarrow X$  on a complete cone metric space is said to be *diametrically contractive* if

$$\delta(TA) = \sup\{d(T(x), T(y)) : x, y \in A\} \prec \delta(A),$$

for all closed bounded subsets  $A \subset X$  such that  $\delta(A) = \sup\{d(x, y) : x, y \in X\}$  exists and  $\delta(A) > 0$ .

Remark that the Definition 2.26 does not indicate the existence of the least upper bound for the set  $\{d(T(x), T(y)) : x \in A\}$  (that is  $\delta(TA)$  exists). Also it follows from the Definition 2.26 that each closed bounded subset  $A$  of  $X$  with  $\delta(A)$  exists and the diameter of its image under  $T$ , i.e.,  $\delta(T(A))$  satisfies  $\delta(T(A)) \prec \delta(A)$ . In fact it is not always true that every nonempty subset of a set which has a least upper bound has also a least upper bound. For example if we take  $X = C[0, 1]$ , the set of all continuous mappings on  $[0, 1]$ ,  $P = \{f \in X : f(x) \leq 0, \forall x \in [0, 1]\}$ . It is clear that the set  $B = \{f \in X : f \geq 1\}$  has a least upper bound which is the constant function one, note for all  $f \in B$ , we have  $1 - f \leq 0$  so  $f \preceq 1$  while the subset  $C = \{1 + x, 1 + x^2, 1 + x^3, \dots\}$  of  $B$  does not have a least upper bound in  $X$ . Indeed, if we take  $g$  as the least upper bound of  $C$  then we get  $1 + x^k \preceq g, \forall k \in N$  which means  $g(x) \leq 1 + x^k, \forall k \in N, \forall x \in [0, 1]$ . For fixed  $x \in [0, 1]$ , we have  $g(x) \leq 1 + x^k, \forall k \in N$ . Hence we have

- $g(x) \leq 1 \quad \forall x \in [0, 1];$
- $g(1) \leq 2,$

which it is possible to choose uncountable functions to verify the aforementioned properties  $g$  and then  $g$  is not unique. This is contradicted by being the least upper bound.

Depending on the above discussion, the following condition is different from and weaker than the condition introduced in Definition 2.26:

$$\delta(T(A)) \prec \delta(A), \quad (2.6)$$

for any bounded closed subset of  $X$  with both  $\delta(A)$  and  $\delta(T(A))$  exist.

Under condition (2.6), existence of a closed bounded subset whose diameter exists but the diameter of its image under  $T$  does not exist, is possible. However, existence of such subset  $A$  is not possible under the diametrically contractive condition in Definition 2.26.

**Theorem 2.27.** [4] *Let  $(X, d)$  be a sequentially compact cone metric space  $P$  a normal cone with normal constant  $k$ . If the mapping  $T : X \rightarrow X$  satisfies the contractive condition,*

$$d(Tx, Ty) \prec d(x, y), \quad \forall x, y \in X, x \neq y,$$

*then  $T$  has a unique fixed point.*

**Theorem 2.28.** [1] *Let  $(X, d)$  be a sequentially compact cone metric space with strongly minihedral cone and  $T : X \rightarrow X$  be diametrically contractive mapping. Then  $T$  has a fixed point.*

It is clear that the family of all Banach contractive mappings and diametrically contractive mappings is a subset of the family of all contractive mappings. Then it is important that we obtain fixed point theorems for this class of mappings. In the next theorem, we present a fixed point theorem for a contractive mapping without considering neither the normality condition nor strongly minihedral condition on the cone. The proof of the next theorem is slightly different from that one given in [1, Theorem 2.22].

**Theorem 2.29.** *Let  $(X, d)$  be a sequentially compact cone metric space. If the mapping  $T : X \rightarrow X$  is a contractive mapping, that is,*

$$d(Tx, Ty) \prec d(x, y), \quad \forall x, y \in X, x \neq y,$$

*then  $T$  has a unique fixed point.*

*Proof.* Set  $\Gamma = \{A \subseteq X : A \text{ is nonempty, sequentially compact and } T(A) \subseteq A\}$ . First,  $X \in \Gamma$  and hence  $\Gamma$  is nonempty. We consider inclusion ordering on  $\Gamma$ . In fact, for  $A, B \in \Gamma$ ,

$$A \preceq B \Leftrightarrow B \subseteq A.$$

We show that  $(\Gamma, \preceq)$  fulfils all the conditions of Zorn's lemma. To see this, let  $S = \{A_i\}_{i \in I}$  be a chain in  $\Gamma$ . It follows from finite intersection property that  $\bigcap_{i \in I} A_i$  is nonempty and compact. Moreover,

$$T(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} T(A_i) \subseteq \bigcap_{i \in I} A_i,$$

and so  $\bigcap_{i \in I} A_i$  belongs to  $\Gamma$  and it also is an upper bound for the chain  $S$ . Then each chain in  $\Gamma$  has an upper bound in  $\Gamma$ . Hence by Zorn's lemma  $\Gamma$  has a maximal element  $B$  with respect to the relation  $\preceq$  which is minimal element with respect to the relation  $\subseteq$ . Since  $B \in \Gamma$ , then  $T(B) \subset B$  and so it follows from the minimality that  $T(B) = B$  (note the image of a compact set by  $T$  is compact and  $T(T(B)) \subseteq T(B)$ , and then  $T(B) \in \Gamma$ ). So  $B$  is singleton (note  $B \in \Gamma$ , then  $B$  is nonempty and since  $T$  is a contractive mapping and  $T(B) = B$ , then it cannot have more than one element). Hence  $T$  has a fixed point. This fixed point is unique because  $T$  is a contractive mapping. This completes the proof.  $\square$

The following lemma given in [1]. From its proof one can understand that the normal condition for the cone is necessary and shall be added it (because the authors used from [1, Lemma 1]).

**Lemma 2.30.** [1] *Let  $(X, d)$  be a cone metric space,  $P$  be strongly minihedral and  $A \subset X$  be bounded. Then  $\delta(\overline{A}) = \delta(A)$ .*

Now this question will be raised that it is possible one can prove the previous lemma without the normal condition on the cone. The following lemma will give a positive answer.

**Lemma 2.31.** *Let  $(X, d)$  be a cone metric space, and  $A \subset X$  with  $\alpha = \delta(A)$ . Then  $\overline{A}$  is bounded and  $\delta(\overline{A}) = \delta(A)$ .*

*Proof.* Assume  $x, y$  are arbitrary elements of  $\overline{A}$  and  $e \in \text{int}P$  a fixed element. Hence if  $\epsilon$  is a positive number, then there exist  $a, b \in A$  such that  $d(x, a) \ll \epsilon e$  and  $d(y, b) \ll \epsilon e$  and so it follows from the triangle inequality that

$$d(x, y) \preceq d(x, a) + d(a, b) + d(b, y) \preceq 2\epsilon e + \alpha.$$

Since  $\epsilon$  is an arbitrary positive number, we get  $d(x, y) \preceq \alpha$ . Since  $x, y$  were arbitrary members of  $\overline{A}$ , we deduce that  $\alpha$  is an upper bound for  $\overline{A}$ . Now if  $\beta$  is another upper bound of  $\overline{A}$ , then it is also an upper bound for  $A$  (because  $A \subseteq \overline{A}$ ) and so  $\alpha \preceq \beta$ . Hence  $\sup \overline{A} = \alpha$  and the proof is finished.  $\square$

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