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Some Convergence Theorems of Three-Step Iteration for G-Nonexpansive Mappings on Banach Spaces with Graph

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Abstract : In this paper, the authors prove weak and strong convergence of a sequence $\{x_n\}$ generated by modified Noor iteration, to some common fixed points of three G-nonexpansive mappings defined on a Banach space endowed with a graph.

Keywords : common fixed point; G-nonexpansive mappings; modified Noor iteration; Banach space; directed graph.
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1 Introduction

Let C be a closed convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed graph where $V(G)$, the set of its vertices, coincides with C, and $E(G)$, the set of its edges, contains all loops. A mapping $T : C \to C$ is G-*nonexpansive* if T preserves edges of G, that is $(T(x), T(y)) \in E(G)$ whenever $(x, y) \in E(G)$, and $\|Tx - Ty\| \leq \|x - y\|$ for any $(x, y) \in E(G)$. Noor iteration scheme was introduced by Noor [1] for studying general variational inequalities. In the study, he also gave the convergence criterior of the scheme. Afterward the Noor iteration has been generalized in various ways (see, for example, [2–4]).

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The study in fixed point theory on a metric space endowed with a graph structure was originated by Jachymski [5]. In the study, he gave a generalization of a contraction on a metric space and named it a G-contraction where G is a graph. Since then, some iterative scheme results for G-contraction and G-nonexpansive maps on a Banach spaces endowed with graphs have been studied extensively by many authors. Aleomraninejad, Rezapour and Shahzad [6] showed some results on iterative scheme for G-contractive and G-nonexpansive mappings on graphs. Alfuraidan and Khamsi [7] gave the concept of G-monotone nonexpansive multivalued mappings defined on a metric space with a graph. Alfuraidan [8] gave a new definition of the G-contraction for multivalued mappings on a metric space with a graph and obtained sufficient conditions for the existence of fixed points. In [9], he also gave the existence of a fixed point of monotone nonexpansive mappings defined in Banach space endowed with a graph. Tiammee, Kaewkhao and Suantai [10] proved Browder's convergence theorem for G-nonexpansive mappings in a Banach space with a directed graph. They also proved strong convergence of the Halpern iteration for G-nonexpansive mappings. Tripak [11] proved weak and strong convergence by using the Ishikawa iterations. Recently, Suparatulatorn, Cholamjiak and Suantai [12] proved weak and strong convergence of a sequence generated by a modified S-iteration process.

Our purpose for this paper is to establish weak and strong convergence results for a sequence generated by $x_0 \in X$ and the modified Noor iteration for G-nonexpansive mappings:

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

2 Preliminaries

In this section, we recall some of standard notations and terminologies, and some needed results.

Consider a directed graph G with the set of vertices $V(G)$ and the set of edges $E(G)$. We assume that the graph has no parallel edges. Then we can write each edge as an ordered pair of vertices. We define a transitive graph as follows.

Definition 2.1. A directed graph $G = (V(G), E(G))$ is said to be *transitive* if $(x, z) \in E(G)$ whenever (x, y) and (y, z) are in $E(G)$.

Definition 2.2. Let C be a nonempty convex subset of a Banach space X, $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Then a mapping $T : C \to C$ is *G-nonexpansive* (see, [7, Definition 2.3]) if it satisfies the following conditions:

(i) T is edge-preserving; that is, $(Tx, Ty) \in E(G)$ for all $(x, y) \in E(G)$,

(ii) $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in E(G)$.

Definition 2.3. [13] Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. We say the mappings T_i $(i = 1, 2, 3)$ on C satisfy *Condition B* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$,

$$\max\{\|x - T_1x\|, \|x - T_2x\|, \|x - T_3x\|\} \geq f(d(x, F))$$

where $F := F(T_1) \cap F(T_2) \cap F(T_3)$ and $F(T_i)$ $(i = 1, 2, 3)$ are the sets of fixed points of T_i.

Definition 2.4. [13] Let C be a subset of a metric space (X, d). A mapping T is said to be *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

Definition 2.5. A Banach space X is said to satisfy *Opial's property* if the following inequality holds for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Definition 2.6. Let X be a Banach space. A mapping T with domain D and range R in X is *demiclosed at zero* if, for any sequence $\{x_n\}$ in D such that $\{x_n\}$ converges weakly to $x \in D$ and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$.

Lemma 2.7. [14] *Let X be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose sequences $\{x_n\}$ and $\{y_n\}$ are in X such that $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\limsup_{n\to\infty} \|\alpha x_n + (1 - \alpha_n)y_n\| = c$ hold for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.8. [15] *Let X be a Banach space and $R > 1$ be fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_R(0) := \{x \in X | \|x\| \leq R\}$, and $\lambda \in [0, 1]$.

Lemma 2.9. [16] *Let X be a Banach space which satisfies Opial's property and let $\{x_n\}$ be a sequence in X. Let x, y in X be such that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exist. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to x and y, respectively, then $x = y$.*

3 Main Results

Throughout the section, we let C be a nonempty closed convex subset of a uniformly convex Banach space X endowed with a directed graph G such that $V(G) = C$ and $E(G)$ is convex. We also suppose that the graph G is transitive. Let T_1, T_2 and T_3 be G-nonexpansive mappings from C to C with $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let x_0 be an arbitrary point in C and let $\{x_n\}$ be a sequence generated by x_0 and the following iterations:

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

Lemma 3.1. *Suppose that $p \in F$.*

(a) *If $(x_0, p) \in E(G)$, then $(x_n, p), (y_n, p), (z_n, p) \in E(G)$ for $n = 0, 1, 2, \ldots$.*

(b) *If $(p, x_0) \in E(G)$, then $(p, x_n), (p, y_n), (p, z_n) \in E(G)$ for $n = 0, 1, 2, \ldots$.*

Proof. Suppose that $(x_0, p) \in E(G)$. We prove part (a) by using the mathematical induction. Since T_3 is edge-preserving, $(T_3x_0, p) \in E(G)$. Write $(z_0, p) = (1 - \gamma_0)(x_0, p) + \gamma_0(T_3x_0, p)$. Since $E(G)$ is convex, $(z_0, p) \in E(G)$. Since T_2 is edge-preserving, $(T_2z_0, p) \in E(G)$. Since $E(G)$ is convex, $(y_0, p) = (1 - \beta_0)(x_0, p) + \beta_0(T_2z_0, p)$ is in $E(G)$.

Now suppose that $(x_n, p), (y_n, p), (z_n, p) \in E(G)$. Since T_1 is edge-preserving, $(T_1y_n, p) \in E(G)$. Since $E(G)$ is convex, $(x_{n+1}, p) = (1-\alpha_n)(x_n, p)+\alpha_n(T_1y_n, p)$ is in $E(G)$. Similarly, we have (z_{n+1}, p) and (y_{n+1}, p) are in $E(G)$. Hence we have finished the proof of part (a). Part (b) can be proved in a similar fashion. □

Lemma 3.2. *If $p \in F$ and $(x_0, p) \in E(G)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists.*

Proof. Using the definition of z_n, we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T_3 x_n - p\| \\ &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T_3x_n - p)\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|T_3x_n - T_3p\|. \end{aligned}$$

By Lemma 3.1, we know that (x_n, p) is in $E(G)$. Since T_3 is G-nonexpansive, we have

$$\|z_n - p\| \le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| = \|x_n - p\|.$$

Similarly, we have

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T_2 z_n - p\| \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(T_2 z_n - p)\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|T_2 z_n - T_2 p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| = \|x_n - p\|\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n T_1 y_n - p\| \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T_1 y_n - p)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|T_1 y_n - T_1 p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| = \|x_n - p\|.\end{aligned}$$

Since the sequence $\{\|x_n - p\|\}$ is nonincreasing and bounded below, $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Lemma 3.3. *If $p \in F$, $(x_0, p), (p, x_0) \in E(G)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, 3$.*

Proof. By Lemma 3.2, $\lim_{n\to\infty}\|x_n - p\|$ exists. As a result, the sequence $\{x_n - p\}$ is bounded. By Lemma 3.1, (y_n, p) is in $E(G)$. Since T_1 is G-nonexpansive and $\|y_n - p\| \le \|x_n - p\|$, we have

$$\|T_1 y_n - p\| = \|T_1 y_n - T_1 p\| \le \|y_n - p\| \le \|x_n - p\|$$

and therefore the sequence $\{T_1 y_n - p\}$ is also bounded. By the definition of x_{n+1},

$$\|x_{n+1} - p\| = \|(1-\alpha_n)(x_n - p) + \alpha_n(T_1 y_n - p)\|.$$

By Lemma 2.8, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T_1 y_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|T_1 y_n - x_n\|).$$

Since $\|T_1 y_n - p\| \le \|x_n - p\|$ and $\alpha_n \in [\delta, 1-\delta]$,

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \delta^2 g(\|T_1 y_n - x_n\|)$$

or equivalently

$$g(\|T_1 y_n - x_n\|) \le \frac{\|x_n - p\|^2 - \|x_{n+1} - p\|^2}{\delta^2}.$$

It follows that $\lim_{n\to\infty} g(\|T_1 y_n - x_n\|) = 0$ and therefore

$$\lim_{n\to\infty} \|T_1 y_n - x_n\| = 0.$$

Notice that

$$\|x_n - p\| \leq \|x_n - T_1 y_n\| + \|T_1 y_n - p\| \leq \|x_n - T_1 y_n\| + \|y_n - p\|.$$

Taking the limit inferior on both sides of the inequality, we have $c \leq \liminf_{n\to\infty} \|y_n - p\|$, where $c = \lim_{n\to\infty} \|x_n - p\|$. Since $\|y_n - p\| \leq \|x_n - p\|$, we have $\limsup_{n\to\infty} \|y_n - p\| \leq c$. Thus

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

By the definition of y_n, we obtain

$$\lim_{n\to\infty} \|(1-\beta_n)(x_n - p) + \beta_n(T_2 z_n - p)\| = c.$$

By Lemma 3.1 and the G-nonexpansiveness of T_2,

$$\|T_2 z_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|.$$

Thus

$$\limsup_{n\to\infty} \|T_2 z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c.$$

By Lemma 2.7,

$$\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0.$$

By Lemma 3.1 and the G-nonexpansiveness of T_2,

$$\|x_n - p\| \leq \|x_n - T_2 z_n\| + \|T_2 z_n - p\| \leq \|x_n - T_2 z_n\| + \|z_n - p\|.$$

Taking the limit inferior on both sides of the inequality, we have $c \leq \liminf_{n\to\infty} \|z_n - p\|$. Since $\|z_n - p\| \leq \|x_n - p\|$, we also have $\limsup_{n\to\infty} \|z_n - p\| \leq c$. Thus

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

By the definition of z_n, we obtain

$$\lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n(T_3 x_n - p)\| = c.$$

By Lemma 3.1 and the G-nonexpansiveness of T_3,

$$\|T_3 x_n - p\| \leq \|x_n - p\|.$$

Thus

$$\limsup_{n\to\infty} \|T_3 x_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c.$$

By Lemma 2.7, we have

$$\lim_{n\to\infty}\|T_3x_n - x_n\| = 0.$$

By Lemma 3.1, both (x_n, p) and (p, z_n) are in $E(G)$. Since G is transitive, $(x_n, z_n) \in G$. By the G-nonexpansiveness of T_2,

$$\begin{aligned}\|T_2x_n - x_n\| &\le \|T_2x_n - T_2z_n\| + \|T_2z_n - x_n\| \\ &\le \|x_n - z_n\| + \|T_2z_n - x_n\| \\ &= \gamma_n\|T_3x_n - x_n\| + \|T_2z_n - x_n\|\end{aligned}$$

and hence

$$\lim_{n\to\infty}\|T_2x_n - x_n\| = 0.$$

Using Lemma 3.1 and the G-nonexpansiveness of T_1, we also have

$$\begin{aligned}\|T_1x_n - x_n\| &\le \|T_1x_n - T_1y_n\| + \|T_1y_n - x_n\| \\ &\le \|x_n - y_n\| + \|T_1y_n - x_n\| \\ &= \beta_n\|T_2z_n - x_n\| + \|T_1y_n - x_n\|\end{aligned}$$

and therefore

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = 0.$$

Hence the lemma is proved. □

Theorem 3.4. *Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, the mappings T_1, T_2 and T_3 satisfy Condition B, and $(x_0, p), (p, x_0) \in E(G)$ for each $p \in F$. Then $\{x_n\}$ converges strongly to some common fixed point of T_1, T_2 and T_3.*

Proof. Since $\|x_{n+1} - p\| \le \|x_n - p\|$, $d(x_{n+1}, F) \le d(x_n, F)$. Thus $\lim_{n\to\infty} d(x_n, F)$ exists. Since T_1, T_2, T_3 satisfy Condition B and $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$ for $i = 1, 2, 3$, there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\lim_{n\to\infty} f(d(x_n, F)) = 0$ and hence

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Hence there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F$ satisfying

$$\|x_{n_{j+1}} - p_j\| \le \|x_{n_j} - p_j\| \le \frac{1}{2^j}.$$

Hence

$$\|p_{j+1} - p_j\| \le \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \le \frac{3}{2^{j+1}}.$$

Consequently $\{p_j\}$ is a Cauchy sequence whose limit is denoted by q. Since F is closed, the limit q must be in F. Since $\|x_{n_j} - q\| \le \|x_{n_j} - p_j\| + \|p_j - q\|$, we have

$$\lim_{n\to\infty}\|x_{n_j} - q\| = 0.$$

By Lemma 3.2, $\lim_{n\to\infty}\|x_n - q\|$ exists; hence it is zero. □

Theorem 3.5. *Suppose that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$*, one of* T_1*,*T_2 *and* T_3 *is semi-compact and* $(x_0, p), (p, x_0) \in E(G)$ *for all* $p \in F$*. Then* $\{x_n\}$ *converges strongly to some common fixed point of* T_1*,* T_2 *and* T_3*.*

Proof. By Lemma 3.3, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2, 3$. By the semi-compactness of any of T_1, T_2, and T_3, there exist $q \in C$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - q\|$ approaches 0 as j tends to ∞. For each $i = 1, 2, 3$, by Lemma 3.1 and the G-nonexpansiveness of T_i, we obtain

$$\begin{aligned} \|q - T_i q\| &\leq \|q - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|T_i x_{n_j} - T_i q\| \\ &\leq \|q - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|x_{n_j} - q\|. \end{aligned}$$

Letting j tend to ∞, we have $T_i q = q$ for each $i = 1, 2, 3$ and hence $q \in F$. Since the subsequence $\{x_{n_j}\}$ converges to $q \in F$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Repeating the same argument as in the proof of Theorem 3.4, we derive that $\{x_n\}$ converges strongly to some fixed point. □

Theorem 3.6. *Suppose that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$*. If* X *satisfies Opial's property,* $I - T_i$ *is demiclosed at zero for* $i = 1, 2, 3$ *and* $(x_0, p), (p, x_0) \in E(G)$ *for all* $p \in F$*, then* $\{x_n\}$ *converges weakly to some common fixed point of* T_1*,* T_2 *and* T_3*.*

Proof. Note that, by Lemma 3.2, $\{x_n\}$ is bounded. Since X is uniformly convex, there exist $q \in X$ and a subsequence $\{x_{n_k}\}$ which converges weakly to q. Suppose the sequence $\{x_n\}$ does not converge weakly to q. Then there exist $f \in X^*$, $\varepsilon > 0$ and a subsequence $\{x_{n_p}\}$ such that

$$|f(x_{n_p}) - f(q)| \geq \varepsilon \text{ for all } p \in \mathbb{N}.$$

Since $\{x_{n_p}\}$ is bounded, there exist $q' \in F$ and a subsequence $\{x_{n_{p_j}}\}$ of $\{x_{n_p}\}$ such that $\{x_{n_{p_j}}\}$ converges weakly to q'. By Lemma 3.3,

$$\lim_{k\to\infty} ||x_{n_k} - T_i x_{n_k}|| = 0 \quad \text{and} \quad \lim_{j\to\infty} ||x_{n_{p_j}} - T_i x_{n_{p_j}}|| = 0.$$

Since $I - T_i$ is demiclosed at zero, $T_i q = q$ and $T_i q' = q'$ for all $i = 1, 2, 3$. Then $q, q' \in F$ and, by Lemma 2.9, $q = q'$. Then $f(x_{n_{p_j}}) \to f(q)$ as $j \to \infty$, which is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point in F. □

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