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Convergence Analysis for Relaxed Extragradient Method and Variational Inequality Problem with Numerical Example

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Abstract : The purpose of this paper is to introduce an iterative method for finding a common element of fixed point of nonexpansive mapping which is generated by the general system of variational inequalities with inverse strongly monotone mappings and the set of the solution of variational inequality. By using our main result, we obtain the strong convergence theorem of the proposed iterative method and another corollary in a real Hilbert space.

Keywords : fixed point; nonexpansive mappings; variational inequalities.
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1 Introduction

Throughout this paper, let H be a real Hilbert space with the inner product $\langle ., . \rangle$ and the norm $\| . \|$ and let C be a nonempty closed and convex subset of H. We call $A : H \to H$ a *strongly positive bounded linear operator* if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2,$$

for all $x, y \in C$.

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a

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positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2,$$

for all $x, y \in C$.

Let $T: C \to C$ be a mapping. A point x is called fixed point of T if and only if $Tx = x$. We denote the set of solutions of fixed point of T by $Fix(T)$. It is well known that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory, by Goebel and Kirk [1]. Recall the following mappings:

A mapping $f: H \to H$ is called *contraction* if there exists $\alpha \in (0,1)$ such that

$$\|fx-fy\| \leq \alpha\,\|x-y\|\,,$$

for all $x, y \in H$.

A mapping $T: C \to C$ is called *nonexpansive* if

$$\|Tx-Ty\| \leq \|x-y\|\,,$$

for all $x, y \in C$.

Let $A: C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of the solutions of (1.1) is denoted by $VI(C, A)$.

The variational inequality problem, which were introduced by Lions and Stampacchia [2] in 1964. It has been widely studied in the literature, see [3–6].

In 1953, Mann [7] introduced the following iteration to find a fixed point of nonexpansive mapping T, which referred as the Mann iteration,

$$x_{n+1} = \beta_n Tx_n + (1-\beta_n)x_n, \tag{1.2}$$

for each $n \geq 1$ and $x_1 \in C$ where $\{\beta_n\}$ in $[0,1]$.

In 2006, Marino and Xu [8] introduced the general iterative method and proved the following theorem:

Theorem 1.1. *Let $T: H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $A: H \to H$ be a strongly positive bounded linear operator and $f: H \to H$ be a contraction mapping and let $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in H \\ x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n\gamma f(x_n), \;\; n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of* T.

Let $A, B : C \to H$ be two different mappings. In 2008, Ceng et al. [9] introduced the *general system of variational inequalities* to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \;\; \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \;\; \forall x \in C, \end{aligned} \tag{1.4}$$

where $\lambda, \mu > 0$ are two constants. In particular, if $A = B$, then problem (1.4) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \;\; \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \;\; \forall x \in C, \end{aligned} \tag{1.5}$$

which is called *the new system of variational inequalities* introduced by Verma [10], in 1999. Moreover, if we put $x^* = y^*$, then problem (1.5) reduces to the variational inequality problem.

In order to find the common element of the solutions of the general system of variational inequalities problem (1.4) and the set of fixed point of a nonexpansive mapping, Ceng et.al [9] proved the strong convergence theorem by a relaxed extragradient method as follow:

Theorem 1.2. *Let the mappings* $A, B : C \to H$ *be* α, β *inverse strongly monotone mappings, respectively. Let* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \cap F(G) \neq \emptyset$, *where a mapping* $G : C \to C$ *is defined by* $G(x) = P_C\left[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)\right]$, $\forall x \in C$. *Suppose that* $x_1 = u \in C$ *and* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.6}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to some point* $x^* \in C$ *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (1.4), *where* $y^* = P_C(x^* - \mu Bx^*)$.

In this paper, motivated and inspired by the iterative scheme in Mann [7], Marino and Xu [8] and Ceng et.al [9], we introduce an iterative scheme for finding the solution of the problem (1.4). Then, we prove a strong convergence theorem, that the iterative sequence $\{x_n\}$ converges strongly to some point $x^* \in C$ and (x^*, y^*) is the solution of (1.4) under some proper conditions in a real Hilbert space.

2 Preliminaries

In this section, we collect some lemmas which will be needed to prove our main theorem in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let P_C be the metric projection of H onto C, i.e., for $x \in H$, P_C satisfies the property

$$\|x - P_C x\| \leq \|x - y\|,$$

for all $y \in C$.

The following lemmas characterizes the projection P_C.

Lemma 2.1. [11] *For a given $x \in H$ and $z \in C$,*

$$x = P_C y \;\Leftrightarrow\; \langle x - y, z - x\rangle \geq 0, \quad \forall z \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \; \forall x, y \in H.$$

Lemma 2.2. [12] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\displaystyle\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. [11] *Let H be a Hilbert space, let C be a nonempty closed convex of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.4. [13] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $\{x_n\} \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2.5. [14] *Let C be a nonempty closed and convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\overline{\gamma} = \min_{i=1,2,..,N} \gamma_i$. Let $\{a_i\}_{i=1}^N \subseteq (0, 1)$, with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:*

(i) $\left\| I - \rho \sum_{i=1}^N a_i A_i \right\| \le 1 - \rho\overline{\gamma}$ *and* $I - \rho \sum_{i=1}^N a_i A_i$ *is a nonexpansive mapping for every* $0 < \rho < \|A_i\|^{-1} (i = 1, 2, ..., N)$.

(ii) $VI(C, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N VI(C, A_i)$.

Lemma 2.6. [9] *For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of problem (1.4) if and only if x^* is a point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \ \ \forall x \in C,$$

where $y^* = P_C(x - \mu Bx)$.

Lemma 2.7. [15] *In a real Hilbert spaces H, the following inequalities hold: for all $x, y \in H$ and $\alpha \in [0, 1]$,*

(i) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$,

(ii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1, D_2 : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C (I - \lambda_1 D_1) P_C (I - \lambda_2 D_2) x$, for all $x \in C$ and $a \in [0, 1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^N c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)x_n + \beta_n P_C(I - \lambda D)y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) Gx_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in \left(0, 2\overline{d}\right)$ with $\overline{d} = \min \{d_1, d_2\}$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \le \beta_n \le c < 1$;

(iii) $0 < c_k < 1$ *and* $\sum\limits_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty.$

Then, $\{x_n\}$ *convergence strongly to* $x_0 = P_{\Omega}(I-\overline{A}+\gamma f)x_0$ *and* (x_0, y_0) *is a solution of* (1.4) *where* $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1, 2, ..., N$. Let $x, y \in C$. Since D is d-inverse strongly monotone mapping with $\lambda \in (0, 2d)$, we obtain

$$\begin{aligned}\|(I-\lambda D)x-(I-\lambda D)y\|^2 &= \|x-y-\lambda(Dx-Dy)\|^2\\ &= \|x-y\|^2 - 2\lambda\langle x-y, Dx-Dy\rangle - \lambda^2\|Dx-Dy\|^2\\ &\le \|x-y\|^2 - 2\lambda d\|Dx-Dy\|^2 - \lambda^2\|Dx-Dy\|^2\\ &\le \|x-y\|^2 - \lambda(2d-\lambda)\|Dx-Dy\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

This implie that

$$\|(I-\lambda D)x-(I-\lambda D)y\| \le \|x-y\|, \tag{3.2}$$

that is, $(I-\lambda D)$ is a nonexpansive mapping. Then, we have $P_C(I-\lambda D)$ is a nonexpansive mapping. By using the same method as (3.2), we have $P_C(I-\lambda_1 D_1)$ and $P_C(I-\lambda_2 D_2)$ are nonexpansive mappings. Then G is a nonexpansive mapping.

The proof will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.
Let $x^* \in \Im$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - x^*\|\\ &\le (1-\beta_n)\|x_n-x^*\| + \beta_n\|P_C(I-\lambda D)y_n - x^*\|\\ &\le (1-\beta_n)\|x_n-x^*\| + \beta_n\|y_n-x^*\|\\ &= (1-\beta_n)\|x_n-x^*\| + \beta_n\left\|\alpha_n\gamma f(x_n) + \left(I-\alpha_n\bar{A}\right)Gx_n - x^*\right\|\\ &\le (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}x^*\right\|\\ &\quad + \beta_n\left\|I-\alpha_n\bar{A}\right\|\|Gx_n-x^*\|\\ &\le (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n\gamma\|f(x_n)-f(x^*)\|\\ &\quad + \beta_n\alpha_n\left\|\gamma f(x^*)-\bar{A}x^*\right\| + \beta_n(1-\alpha_n\bar{\gamma})\|Gx_n-x^*\|\\ &\le (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n\gamma\alpha\|x_n-x^*\| + \beta_n\alpha_n\left\|\gamma f(x^*)-\bar{A}x^*\right\|\\ &\quad + \beta_n(1-\alpha_n\bar{\gamma})\|x_n-x^*\|\\ &= (1-\beta_n+\beta_n(\alpha_n\gamma\alpha+1-\alpha_n\bar{\gamma}))\|x_n-x^*\| + \beta_n\alpha_n\left\|\gamma f(x^*)-\bar{A}x^*\right\|\\ &= (1-\beta_n+\beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha)))\|x_n-x^*\| + \beta_n\alpha_n\left\|\gamma f(x^*)-\bar{A}x^*\right\|\\ &= (1-\beta_n\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x^*\| + \beta_n\alpha_n\left\|\gamma f(x^*)-\bar{A}x^*\right\|\\ &\le \max\left\{\|x_1-x^*\|, \frac{\left\|\gamma f(x^*)+\bar{A}x^*\right\|}{\bar{\gamma}-\gamma\alpha}\right\}.\end{aligned}$$

By induction, we have $\|x_n - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*)+\overline{A}x^*\|}{\overline{\gamma}-\gamma\alpha}\right\}, \forall n \in \mathbb{N}$.
Hence $\{x_n\}$ is bounded and so is $\{y_n\}$.
Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|\alpha_{n+1}\gamma f(x_{n+1})+\left(I-\alpha_{n+1}\overline{A}\right)Gx_{n+1}-\alpha_n\gamma f(x_n)-\left(I-\alpha_n\overline{A}\right)Gx_n\right\| \\
&\le \alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+\gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad+\left\|I-\alpha_{n+1}\overline{A}\right\|\|Gx_{n+1}-Gx_n\|+\left\|\left(I-\alpha_{n+1}\overline{A}\right)Gx_n-\left(I-\alpha_n\overline{A}\right)Gx_n\right\| \\
&\le \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+\gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad+(1-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\left\|\overline{A}Gx_n\right\| \\
&= (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\|+\gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad+|\alpha_{n+1}-\alpha_n|\left\|\overline{A}Gx_n\right\|. \qquad (3.3)
\end{aligned}$$

From the definition of $\{x_n\}$ and (3.3), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\beta_n)x_n+\beta_nP_C(I-\lambda D)y_n-(1-\beta_{n-1})x_{n-1} \\
&\quad-\beta_{n-1}P_C(I-\lambda D)y_{n-1}\| \\
&= \|(1-\beta_n)x_n+\beta_nP_C(I-\lambda D)y_n-(1-\beta_n)x_{n-1}+(1-\beta_n)x_{n-1} \\
&\quad-(1-\beta_{n-1})x_{n-1}-\beta_nP_C(I-\lambda D)y_{n-1}+\beta_nP_C(I-\lambda D)y_{n-1} \\
&\quad-\beta_{n-1}P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n-x_{n-1}\|+\beta_n\|P_C(I-\lambda D)y_n-P_C(I-\lambda D)y_{n-1}\| \\
&\quad+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n-x_{n-1}\|+\beta_n\|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n-x_{n-1}\|+\beta_n((1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| \\
&\quad+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\left\|\bar{A}Gx_{n-1}\right\|) \\
&\quad+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n)\|x_n-x_{n-1}\|+\beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| \\
&\quad+\beta_n\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n|\alpha_n-\alpha_{n-1}|\left\|\bar{A}Gx_{n-1}\right\| \\
&\quad+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n+\beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha)))\|x_n-x_{n-1}\| \\
&\quad+\beta_n\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n|\alpha_n-\alpha_{n-1}|\left\|\bar{A}Gx_{n-1}\right\| \\
&\quad+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad+|\alpha_n-\alpha_{n-1}|\left\|\bar{A}Gx_{n-1}\right\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\|.
\end{aligned}$$

This together with conditions $(i), (iii)$ and Lemma 2.2 , we get that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

From conditions $(i), (iii)$, (3.3), and (3.4), we obtain $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.
From the definition of y_n, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \left\|\alpha_n \gamma f(x_n) + \left(I - \alpha_n \bar{A}\right) G x_n - x^*\right\|^2 \\
&= \left\|(Gx_n - x^*) + \left(\alpha_n \gamma f(x_n) - \alpha_n \bar{A} G x_n\right)\right\|^2 \\
&\leq \|Gx_n - x^*\|^2 + 2\alpha_n \left\langle \gamma f(x_n) - \bar{A} G x_n, y_n - x^*\right\rangle \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \|y_n - x^*\|.
\end{aligned} \tag{3.5}$$

From nonexpansiveness of P_C and (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|P_C(I - \lambda D) y_n - x^*\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|(I - \lambda D) y_n - (I - \lambda D) x^*\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|y_n - x^* - \lambda(Dy_n - Dx^*)\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|y_n - x^*\|^2 \\
&\quad - 2\lambda\beta_n \langle y_n - x^*, Dy_n - Dx^*\rangle + \beta_n \lambda^2 \|Dy_n - Dx^*\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + 2\beta_n \alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \\
&\quad \times \|y_n - x^*\| - 2\lambda d \beta_n \|Dy_n - Dx^*\|^2 + \beta_n \lambda^2 \|Dy_n - Dx^*\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + 2\beta_n \alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \\
&\quad \times \|y_n - x^*\| - \lambda\beta_n (2d - \lambda)\|Dy_n - Dx^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\beta_n \alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \|y_n - x^*\| \\
&\quad - \lambda\beta_n (2d - \lambda)\|Dy_n - Dx^*\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\lambda\beta_n (2d - \lambda)\|Dy_n - Dx^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\beta_n \alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \|y_n - x^*\| \\
&\leq (\|x_n - x^*\| - \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\
&\quad + 2\beta_n \alpha_n \left\|\gamma f(x_n) - \bar{A} G x_n\right\| \|y_n - x^*\|.
\end{aligned} \tag{3.6}$$

Form conditions $(i), (ii)$, (3.4) and (3.6), we have

$$\lim_{n\to\infty} \|Dy_n - Dx^*\| = 0. \tag{3.7}$$

Step 3. Show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, $\lim_{n\to\infty} \|y_n - Ty_n\|$

$= 0, \lim_{n\to\infty} \|y_n - Gy_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
From the definition of x_n and (3.5), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - x^*\|^2 \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 \\
&\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|y_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-P_C(I-\lambda D)y_n\|^2 \\
&\le (1-\beta_n)\|x_n-x^*\|^2 + \beta_n\|x_n-x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 \\
&\le \|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\le (\|x_n - x^*\| - \|x_{n+1} - x^*\|)\, \|x_{n+1} - x_n\| \\
&\quad + 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\|. \qquad (3.8)
\end{aligned}$$

Form conditions $(i), (ii)$, (3.4) and (3.8), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda D)y_n\| = 0. \qquad (3.9)$$

By Lemma 2.1 and (3.5), we obtain

$$\begin{aligned}
\|P_C(I-\lambda D)y_n - x^*\|^2 &\le \langle (I-\lambda D)y_n - (I-\lambda D)x^*, P_C(I-\lambda D)y_n - x^*\rangle \\
&= \frac{1}{2}\Big(\|(I-\lambda D)y_n - (I-\lambda D)x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad -\|(I-\lambda D)y_n - (I-\lambda D)x^* - (P_C(I-\lambda D)y_n - x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|y_n - x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad -\|y_n - P_C(I-\lambda D)y_n - \lambda(Dy_n - Dx^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad + 2\lambda\langle y_n - P_C(I-\lambda D)y_n, Dy_n - Dx^*\rangle - \lambda^2\|Dy_n - Dx^*\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\|\Big).
\end{aligned}$$

It follow that

$$\begin{aligned}\|P_C(I-\lambda D)y_n-x^*\|^2 \le{}& \|x_n-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\|y_n-x^*\|\\ &-\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\\ &\times\|Dy_n-Dx^*\|.\end{aligned} \tag{3.10}$$

From (3.10), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|P_C(I-\lambda D)y_n-x^*\|^2\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\Big(\|x_n-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\\ &\quad\times\|y_n-x^*\|-\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\\ &\quad\times\|Dy_n-Dx^*\|\Big)\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\\ &\quad\times\|y_n-x^*\|-\beta_n\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\\ &\quad\times\|Dy_n-Dx^*\|\\ &= \|x_n-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|\\ &\quad-\beta_n\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\|Dy_n-Dx^*\|.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n\|y_n-P_C(I-\lambda D)y_n\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\\ &\quad\times\|y_n-x^*\|+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\|Dy_n-Dx^*\|\\ &\le (\|x_n-x^*\|-\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\ &\quad+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|\\ &\quad+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\|Dy_n-Dx^*\|.\end{aligned} \tag{3.11}$$

From conditions $(i),(ii)$, (3.4), (3.7) and (3.11), we get

$$\lim_{n\to\infty}\|y_n-P_C(I-\lambda D)y_n\|=0. \tag{3.12}$$

Consider,

$$\|x_n-y_n\|\le\|x_n-P_C(I-\lambda D)y_n\|+\|P_C(I-\lambda D)y_n-y_n\|.$$

From (3.9) and (3.12), we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.13}$$

From definition of y_n and condition (i), we have

$$\begin{aligned}\|y_n-Gx_n\| &= \left\|\alpha_n\gamma f(x_n)+(I-\alpha_n\overline{A})Gx_n-Gx_n\right\|\\ &= \alpha_n\left\|\gamma f(x_n)+\overline{A}Gx_n\right\|\to 0 \;\; as\; n\to\infty.\end{aligned} \tag{3.14}$$

Consider,

$$\|x_n - Gx_n\| \leq \|x_n - y_n\| + \|y_n - Gx_n\|.$$

By (3.13) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.15}$$

From (3.13), (3.15) and

$$\begin{aligned}\|y_n - Gy_n\| &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|Gx_n - Gy_n\| \\ &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|x_n - y_n\|,\end{aligned}$$

we get that

$$\lim_{n\to\infty} \|y_n - Gy_n\| = 0. \tag{3.16}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle \leq 0$, where $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$.

To show this, choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle = \lim_{k\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_{n_k} - x_0 \rangle. \tag{3.17}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup q$ as $k \to \infty$, where $q \in C$. Then, from (3.13) and $x_{n_k} \rightharpoonup q$, we obtain $y_{n_k} \rightharpoonup q$ as $k \to \infty$. From (3.17) and $y_{n_k} \rightharpoonup q$ as $k \to \infty$, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \rangle. \tag{3.18}$$

In order to show $\langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \rangle \leq 0$, we need to show that $q \in \Im = F(G) \cap VI(C, D)$. Assume that $q \notin F(G)$. It implies that $q \neq Gq$. From Lemma 2.4 and (3.16), we have

$$\begin{aligned}\liminf_{n\to\infty} \|y_{n_k} - q\| &< \liminf_{n\to\infty} \|y_{n_k} - Gq\| \\ &\leq \liminf_{n\to\infty} (\|y_{n_k} - Gy_{n_k}\| + \|Gy_{n_k} - Gq\|) \\ &\leq \liminf_{n\to\infty} (\|y_{n_k} - Gy_{n_k}\| + \|y_{n_k} - q\|) \\ &\leq \liminf_{n\to\infty} \|y_{n_k} - q\|.\end{aligned}$$

This is a contraction, that is,

$$q \in F(G). \tag{3.19}$$

Next, we will show that $q \in VI(C, D)$.
Assume that $q \notin VI(C, D)$. Since $VI(C, D) = F(P_C(I - \lambda D))$, we have $q \neq$

$P_C(I-\lambda D)q$. From Lemma 2.4 and (3.12), we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \|y_{n_k}-q\| &< \liminf_{n\to\infty} \|y_{n_k} - P_C(I-\lambda D)q\| \\
&\leq \liminf_{n\to\infty} (\|y_{n_k}-P_C(I-\lambda D)y_{n_k}\|+\|P_C(I-\lambda D)y_{n_k}-P_C(I-\lambda D)q\|) \\
&\leq \liminf_{n\to\infty} (\|y_{n_k} - P_C(I-\lambda D)y_{n_k}\| + \|y_{n_k} - q\|) \\
&\leq \liminf_{n\to\infty} \|y_{n_k} - q\|.
\end{aligned}$$

This is a contraction, that is,

$$q \in VI(C, D). \tag{3.20}$$

From (3.19) and (3.20), we have $q \in \Im = F(G) \cap VI(C, D)$.By (3.18) and Lemma 2.1, we have

$$\limsup_{n\to\infty} \left\langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \right\rangle = \left\langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \right\rangle \leq 0.$$

Step 5. Finally, We will show that $\{x_n\}$ converges strongly to x_0, where $x_0 = P_\Im(I - \overline{A} + \gamma f)x_0$.
From the definition of x_n and $x_0 = P_\Im(I - \overline{A} + \gamma f)x_0$, we have

$$\begin{aligned}
\|x_{n+1}-x_0\|^2 &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - x_0\|^2 \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x_0\|^2 \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n\|y_n - x_0\|^2 \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n\left\|\alpha_n\gamma f(x_n) + \left(I - \alpha_n\bar{A}\right) Gx_n - x_0\right\|^2 \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n \Big(\left\|\left(I - \alpha_n\bar{A}\right)(Gx_n - x_0)\right\|^2 \\
&\quad +2\alpha_n \left\langle\gamma f(x_n) - \bar{A}x_0, y_n - x_0\right\rangle\Big) \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n \Big((1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad +2\alpha_n\gamma \langle f(x_n) - f(x_0), y_n - x_0\rangle + 2\alpha_n \left\langle\gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle\Big) \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n \Big((1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad +2\alpha_n\gamma \|f(x_n)-f(x_0)\| \|y_n - x_0\| + 2\alpha_n \left\langle\gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle\Big) \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n - x_0\| \|y_n - x_0\| + 2\alpha_n\beta_n \left\langle\gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n-x_0\| \left(\alpha_n\left\|\gamma f(x_n)-\bar{A}x_0\right\| + (1-\alpha_n\bar{\gamma})\|Gx_n-x_0\|\right) \\
&\quad + 2\alpha_n\beta_n \left\langle\gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n - x_0\| \left(\alpha_n\gamma\alpha \|x_n - x_0\| + \alpha_n \left\|\gamma f(x_0) - \bar{A}x_0\right\|\right. \\
&\quad \left. + (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|\right) + 2\alpha_n\beta_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle \\
&= (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 + 2\alpha_n^2\gamma^2\alpha^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\gamma\alpha\beta_n \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}\beta_n \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle \\
&= (1-\beta_n + \beta_n - \beta_n\alpha_n\bar{\gamma} + 2\alpha_n\gamma\alpha\beta_n) \|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}\beta_n \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n\beta_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle \\
&= (1-\alpha_n\beta_n (\bar{\gamma} - 2\gamma\alpha)) \|x_n - x_0\|^2 + \alpha_n\beta_n \Big(2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\bar{\gamma} \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle\Big) \\
&= (1-\alpha_n\beta_n (\bar{\gamma} - 2\gamma\alpha)) \|x_n - x_0\|^2 + \alpha_n\beta_n (\bar{\gamma} - 2\gamma\alpha) \left(\frac{2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2}{(\bar{\gamma} - 2\gamma\alpha)}\right. \\
&\quad \left. + \frac{2\alpha_n\bar{\gamma} \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\|}{(\bar{\gamma} - 2\gamma\alpha)} + \frac{2 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0 \right\rangle}{(\bar{\gamma} - 2\gamma\alpha)}\right).
\end{aligned}
$$

By step 4, condition (i) and Lemma 2.2, we can conclude that $\{x_n\}$ converges strongly to $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$. Then, from Lemma 2.6, we have (x_0, y_0) is a solution of the problem (1.4) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$. This completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1 : C \to H$ be d, d_1-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C (I - \lambda_1 D_1) P_C (I - \lambda_2 D_2) x$, for all $x \in C$ and $a \in [0, 1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$
\begin{aligned}
x_{n+1} &= (1-\beta_n)x_n + \beta_n P_C(I - \lambda D)y_n, \\
y_n &= \alpha_n\gamma f(x_n) + \left(I - \alpha_n\overline{A}\right) Gx_n,
\end{aligned}
\tag{3.21}
$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2d_1)$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *convergence strongly to* $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ *and* (x_0, y_0) *is a solution of* (1.5) *where* $y_0 = P_C(x_0 - \lambda_2 D_1 x_0)$.

Proof. If we put $D_1 = D_2$ in Theorem 3.1, we have the desired conclusion. □

4 Example and Numerical Results

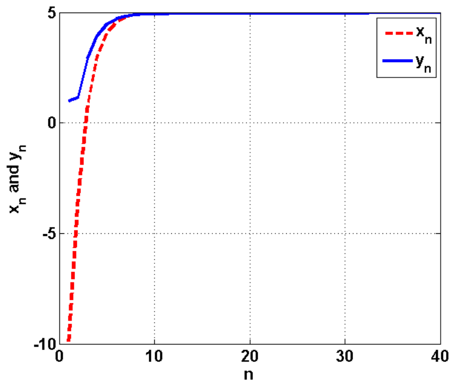
Example 4.1. Let $\mathbb{R}$ be the set of real numbers. Let D, D_1, D_2 be a mapping from $[-50, 50]$ to $\mathbb{R}$ defined by $Dx = \frac{2x-10}{3}, D_1 x = \frac{x-5}{2}$ and $D_2 x = \frac{3x-15}{4}$, for all $x \in [-50, 50]$. Let mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{5x}{9}$, for every $x \in \mathbb{R}$. For $k = 1, 2, ..., N$, let $c_k = \frac{2}{3^k} + \frac{1}{N3^N}$ and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A_k x = \frac{kx}{5}$, for every $x \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ be generated by (3.1) where $\lambda = 1.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \alpha = 1, \gamma = 0.05$, $\alpha_n = \frac{2}{5n}$ and $\beta_n = \frac{5n-1}{9n}$. By the definition of D, D_1, D_2, A and f, we have $5 \in F(G) \cap VI(C, D)$. Then, from Theorem 3.1, the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 5. We can rewritten (3.1) as follow:

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n+1}{8n}\right) x_n + \left(\tfrac{5n-1}{8n}\right) P_{[-50,50]}(I - (1.5)D)y_n, \\ y_n &= \tfrac{0.1}{5n} f(x_n) + \left(I - \left(\tfrac{2}{5n}\right) \bar{A}\right) G x_n. \end{aligned} \tag{4.1}$$

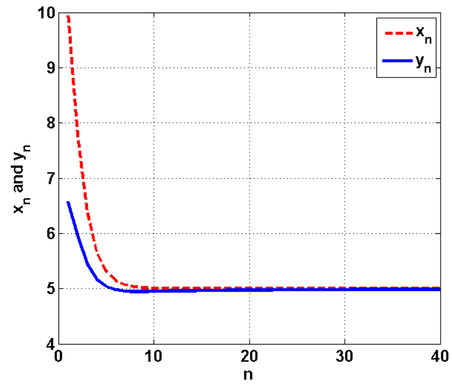
The following table and figure shows the values of the sequence $\{x_n\}$ and $\{y_n\}$ of iterative (4.1), where $x_1 = -10$, $x_1 = 10$ and $n = N = 40$.

	$x_1 = -10$		$x_1 = 10$	
n	x_n	y_n	x_n	y_n
1	−10.000000	0.988889	10.000000	6.573611
2	−3.333333	1.156481	7.777778	5.967168
3	0.833333	2.928086	6.388889	5.448663
⋮	⋮	⋮	⋮	⋮
20	4.999993	4.972774	5.000002	4.972779
⋮	⋮	⋮	⋮	⋮
38	5.000000	4.985673	5.000000	4.985673
39	5.000000	4.986040	5.000000	4.986040
40	5.000000	4.986389	5.000000	4.986389

Table 1: The values of $\{x_n\}$ and $\{y_n\}$ with different initial value x_1



(a) $x_1 = -10$



(b) $x_1 = 10$

Figure 1: The convergence of the sequence $\{x_n\}$ and $\{y_n\}$ with different initial value x_1 and $n = N = 40$.

From Table 1 and Figure 1 (a) and (b), we can observe that $\{x_n\}$ and $\{y_n\}$ converge to 5, where $5 \in F(G) \cap VI(C, D)$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 4.1 can be guaranteed by Theorem 3.1.

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