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On the Delta-Hedging of the Option Price on Future from the Black-Scholes Equation

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Abstract : At present the option on future is popular one for trading but there are some problems of risk. So the minimizing of risk which is call the hedging is needed. In this paper we studied such hedging by using the Delta-hedging which is popular at present. We found the new results which having the interesting properties. We hope that such results may be useful in the research area the Financial Mathematics.

Keywords : Delta-hedging; Black-Scholes equation; option price.
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1 Introduction

Back to the year 1973 that F. Black and M. Scholes has first introduced the Black-Scholes formula which is the solution of Black-Scholes Equation, see [1]. Such Black-Scholes formula is the option price which is fair price for trading in European Options. Now the Black-Scholes Equation is given by

$$\frac{\partial}{\partial t}u(s,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) - ru(s,t) = 0, \tag{1.1}$$

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with the terminal condition or the call payoff

$$u(s_T, T) = (s_T - p)^+ \tag{1.2}$$

and denote $(s_T - p)^+ \equiv \max(s_T - p, 0)$ where $u(s,t)$ is the option price at time t for $0 \leq t \leq T$, T is the expiration date, s is the stock price at time t, r is the interest rate, σ is the volatility of stock and p is strike price.

The well known solution of (1.1) that satisfies (1.2) which call the Black-Scholes formula is given by

$$u(s,t) = sN(d_1) - pe^{-r(T-t)}N(d_2) \tag{1.3}$$

see [2], where

$$d_1 = \frac{\ln\left(\frac{s}{p}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{s}{p}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and denote $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$.

At present the option price on future is the popular one for trading. Let $F = se^{r(T-t)}$ be the stock price on future and write $u(s,t) = C(F,t)$. Substitute F and $C(F,t)$ into (1.1) then (1.1) is transformed to the equation

$$\frac{\partial}{\partial t}C(F,t) + \frac{1}{2}\sigma^2F^2\frac{\partial^2}{\partial F^2}C(F,t) - rC(F,t) = 0 \tag{1.4}$$

with the call payoff

$$C(F_T, T) = (F_T - p)^+, \tag{1.5}$$

where $C(F,t)$ is the option price on future, see [3] and F_T is the stock price at the expiration date T. Thus $F_T = s_T$ where s_T is the stock price at time T. Now in this paper we studied the Delta-hedging of $C(F,t)$ from (1.4). Such Delta-hedging is defined by $\Delta_F = \frac{\partial}{\partial F}C(F,t)$. In fact we obtain the Black-Scholes formula which is the the solution of (1.4) and is similar to (1.3) of the form

$$C(F,t) = e^{-r(T-t)}\left(FN(d_1) - pN(d_2)\right). \tag{1.6}$$

Now from (1.6) we obtain

$$\Delta_F = \frac{\partial}{\partial F}C(F,t) = e^{-r(T-t)}N(d_1), \tag{1.7}$$

see [3]. Now let $R = \ln F$ and $\tau = T - t$ and write $C(F,t) = V(R,\tau)$ and substitute into (1.4). Then (1.4) is transformed to the equation

$$\frac{\partial}{\partial \tau}V(R,\tau) + \frac{1}{2}\sigma^2\frac{\partial}{\partial R}V(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,\tau) + RV(R,\tau) = 0. \tag{1.8}$$

with the call payoff or the initial condition

$$V(R,0) = C(F_T,T) = (F_T - p)^+ = (e^R - p)^+$$

where $\tau = 0$ correspond to to $t = T$. Let

$$V(R,0) = (e^R - p)^+ = f(R) \tag{1.9}$$

where f is the continuous function of R. Now we take the Fourier transform with respect to R to (1.7) and (1.8), we obtain

$$V(R,\tau) = \exp\left[-r\tau + \sigma^2\tau + R\right] - e^{-r\tau}p \tag{1.10}$$

as the solution of (1.7). Since

$$\begin{aligned} C(F,t) &= V(R,\tau) = V(\ln F, T-t) \\ &= \exp\left[-r(T-t) + \sigma^2(T-t) + \ln F\right] - e^{r(T-t)}p \\ &= F\exp\left[-r(T-t) + \sigma^2(T-t)\right] - e^{-r(\tau-t)}p. \end{aligned}$$

Thus we have

$$\Delta_F = \frac{\partial}{\partial F}C(F,t) = \exp\left[-r(T-t) + \sigma^2(T-t)\right]. \tag{1.11}$$

Now (1.11) is the results of this paper which is different from the well known in (1.7).

2 Preliminaries

The following some definitions and lemmas are needed.

Definition 2.1. Let f be locally integrable function then the *Fourier transform* of f is defined by

$$\mathfrak{F}f(x) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}f(x)dx \tag{2.1}$$

and the *inverse Fourier transform* is also defined by

$$f(x) = \mathfrak{F}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega \tag{2.2}$$

Definition 2.2. Let $C(s,t)$ be the call option and s is the stock price at time t. Then the *Delta-hedging* denoted by Δ is defined by $\Delta = \frac{\partial}{\partial s}C(s,t)$ or $\partial C(s,t) = \Delta \partial s$. In fact Δ is the number of shares of stock times the change of stock price.

Now the concepts of Delta-hedging is that if we want to hedge the sale of one call option we need to buy Δ shares of stock.

Lemma 2.3. *Recall the equation* (1.8) *and the call payoff* (1.9) *that*

$$\frac{\partial}{\partial \tau}V(R,\tau) + \frac{1}{2}\sigma^2\frac{\partial}{\partial R}V(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,\tau) + rV(R,\tau) = 0 \tag{2.3}$$

and the call payofff or the initial condition

$$V(R,0) = f(R). \tag{2.4}$$

Then

$$V(R,\tau) = \exp[-r\tau + \sigma^2\tau + R] - e^{-r\tau}p \tag{2.5}$$

as the the solution of (2.3) *and the Delta-hedging*

$$\Delta_R = \frac{\partial}{\partial R}V(R,\tau) = \exp[-r\tau + \sigma^2\tau + R]. \tag{2.6}$$

Proof. Take the Fourier transform defined by (2.1) with respect to R to (2.3). Then we obtain

$$\frac{\partial}{\partial \tau}\widehat{V}(\omega,\tau) - \frac{1}{2}\sigma^2 i\omega\widehat{V}(\omega,\tau) + \frac{1}{2}\sigma^2\omega^2\widehat{V}(\omega,\tau) + r\widehat{V}(\omega,\tau) = 0. \tag{2.7}$$

Thus we have

$$\widehat{V}(\omega,\tau) = C(\omega)\exp\left[(-\frac{1}{2}\sigma^2\omega^2 + \frac{1}{2}\sigma^2 i\omega - r)\tau\right]$$

as the solution of (2.6). Now from (2.4),

$$\widehat{V}(\omega,0) = \widehat{f}(\omega).$$

Thus

$$C(\omega) = \widehat{f}(\omega).$$

Since

$$V(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{V}(\omega,\tau)d\omega$$

from (2.2). Thus

$$\begin{aligned} V(R,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{f}(\omega)\exp\left[(-\frac{1}{2}\sigma^2\omega^2+\frac{1}{2}\sigma^2 i\omega - r)\tau\right]d\omega \\ &= \frac{e^{r\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega R}e^{-i\omega y}\exp\left[(-\frac{1}{2}\sigma^2\omega^2+\frac{i}{2}\sigma^2\omega)\tau\right]f(y)fyd\omega \end{aligned}$$

where $\widehat{f}(\omega) = \mathfrak{F}f(y) = \int_{-\infty}^{\infty} e^{-i\omega y}f(y)dy$. Thus

$$\begin{aligned} V(R,\tau) &= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega^2 - 2i(\frac{\sigma^2\frac{\tau}{2}+R-y}{\sigma^2\tau})\omega)\right]d\omega)f(y)dy \\ &= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega - i(\frac{\sigma^2\frac{\tau}{2}+R-y}{\sigma^2\tau}))^2\right]d\omega) \\ &\qquad \exp\left[-(\frac{\sigma^2\frac{\tau}{2}+R-y}{2\sigma^2\tau})^2\right]f(y)dy. \end{aligned}$$

Put $u = \sigma\sqrt{\frac{\tau}{2}}(\omega - i(\frac{\sigma^2\tau+R-y}{\sigma^2\tau}))$. Then $d\omega = \frac{1}{\sigma}\sqrt{\frac{2}{\tau}}du$. Thus we have

$$\begin{aligned} V(R,\tau) &= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-u^2}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}du)\exp\left[\frac{-(\sigma^2\frac{\tau}{2}+R-y)^2}{2\sigma^2\tau}\right]f(y)dy \\ &= \frac{e^{-rT}}{2\pi}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left[\frac{-(\sigma^2\frac{\tau}{2}+R-y)^2}{2\sigma^2\tau}\right]f(y)dy. \end{aligned}$$

(Note that $\int_{-\infty}^{\infty} e^{u^2}du = \sqrt{\pi}$). It follows that

$$\begin{aligned} V(R,\tau) &= \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma^2}}\int_{-\infty}^{\infty}\exp\left[\frac{-(\sigma^2\frac{\tau}{2}+R-y)^2}{2\sigma^2\tau}\right]f(y)dy \\ &= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{(\sigma^2\frac{\tau}{2}+R-y)^2}{2\sigma^2\tau}\right](e^y - p)dy \end{aligned}$$

since $f(y) = e^y - p$, (1.9) and (2.4). Thus

$$V(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(\frac{\sigma^2}{2}\tau + R - y)^2}{2\sigma^2\tau}\right] e^y dy$$

$$- \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} p \int_{-\infty}^{\infty} \exp\left[-\frac{(\frac{\sigma^2}{2}\tau + R - y)^2}{2\sigma^2\tau}\right] dy.$$

Now put $u = \frac{1}{\sigma\sqrt{2\pi}}(y - \frac{\sigma^2}{2}\tau - R)$, then $dy = \sigma\sqrt{2\tau}du$. By computing directly the same as before, we obtain

$$V(R,\tau) = \exp[(-r\tau + \sigma^2\tau + R)] - e^{-r\tau}p$$

as the solution of (2.3) and by Definition 2.2, we also obtain

$$\Delta_R = \frac{\partial}{\partial R} V(R,\tau) = \exp[-r\tau + \sigma^2 + R].$$

Thus we obtain (2.6) as required. □

3 Main Results

Theorem 3.1. *Recall the equation* (1.4) *and The call payoff* (1.5) *that*

$$\frac{\partial}{\partial t} C(F,t) + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2}{\partial F^2} C(F,t) - rC(F,t) = 0 \tag{3.1}$$

with the call payoff

$$C(F_t,T) = (F_t,T) = (F_T - p)^+ \tag{3.2}$$

Then (3.1) *has the solution*

$$C(F,t) = F\exp\left[(-r(T-t) + \sigma^2(T-t)\right] - e^{-r(\tau-t)}p \tag{3.3}$$

with the Delta-hedging

$$\Delta_F = \exp\left[(-r(T-t) + \sigma^2(T-t)\right] \tag{3.4}$$

and

$$0 \le \Delta_F \le 1 \tag{3.5}$$

Proof. By Lemma (2.3), we obtain

$$V(R,\tau) = \exp\left[-(r\tau + \sigma^2\tau + R)\right] - e^{-r\tau}p$$

where $\tau = T - t$ and $R = \ln F$. Since we let $C(F,t) = V(R,\tau) = V(\ln F, T-t)$. Thus we have

$$\begin{aligned} C(F,t) &= \exp\left[(-r(T-t) + \sigma^2(T-t) + \ln F)\right] - e^{-r(T-t)}p \\ &= F\exp\left[(-r(T-t) + \sigma^2(T-t))\right] - e^{-r(T-t)}p \end{aligned}$$

as the solution of (3.1). By Definition 2.2, $\Delta_F = \frac{\partial}{\partial F}C(F,t)$. It follows that $\Delta_F = \exp\left[(-r(T-t) + \sigma^2(T-t))\right]$. Thus we obtain (3.3) and (3.4) as required. Now at $t = 0$ we have $\Delta_F = \exp\left[(-rT + \sigma^2 T)\right] > 0$ and at $t = T$. We have $\Delta_F = 1$. Since $0 \leq t \leq T$. It follows that $0 < \Delta_F \leq 1$. Thus we obtain (3.5).

Moreover, from (1.6), $C(F,t) = e^{-r(T-t)}\left[FN(d_1) - pN(d_2)\right]$ which is the Black-Scholes formula for the option price on future and from (1.7),

$$\Delta_F = \frac{\partial}{\partial F}C(F,t) = e^{-r(T-t)}N(d_1).$$

Now for $t = 0$, $\Delta_F = e^{-rT}N(d_1) > 0$ and for $t = T$ we have from (1.3) that $d_1 = \infty$ and $N(\infty) = 1$. It follows that $\Delta_F = N(\infty) = 1$ since $0 \leq t \leq T$, thus we have $0 < \Delta_F \leq 1$. We see that the option price on future which is the Black-Scholes formula and the option price given by (3.3) of this paper has the same condition of Δ_F which is $0 < \Delta_F \leq 1$. □

4 Conclusion

Consider the option price on future given by (3.3). That is

$$C(F,t) = F\exp\left[((-r(T-t) + \sigma^2(T-t))\right] - e^{-r(T-t)p}$$

and the call payoff

$$C(F_T,T) = F_T e^0 - e^0 p = (F_T - p)^+$$

and the option price on future which is Black-Scholes formula given by (1.6)

$$C(F,t) = e^{-r(T-t)}\left[FN(d_1) - pN(d_2)\right].$$

The call payoff at $t = T$,

$$C(F_T,T) = e^0\left[F_T N(\infty) - pN(\infty)\right] = (F_T - p)^+$$

where $d_1 = d_2 = \infty$ at $t = T$ and $N(\infty) = 1$. We see that both options price on future has different forms but has the same call payoff. Moreover we have the same condition of the Delta-hedging which is $0 < \Delta_F \leq 1$.

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