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Convergence Theorems of Iterative Methods for System of Strongly Nonlinear Nonconvex Variational Inequalities

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Abstract : Now we known that many authors have studied and introduced the result of nonconvex variational inequalities. By relying on the prox-regularity notion, we introduce and establish the convergence of a modified algorithm of a system of strongly nonlinear nonconvex variational inequalities problems. The presented result is a particular case of some known results in this field.

Keywords : μ-Lipschitz continuous; strongly monotone mapping; nonconvex; uniformly prox-regular; strongly nonlinear nonconvex variational inequalities.
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1 Introduction

The theory of variational inequalities be first introduced by Stampacchia [1], provides simple and unified framwork to study a large number of problem arising in finance, economics, transportation, network and structural analysis, elasticity and optimization. Many research papers have been written lately, both on the theory and applications of this field, see for example [2–4] and the references cited therein.

The existence and iterative scheme of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly

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based on the properties of the projection operator are convex sets. Recently, the concept of convex sets has been generalized in many differnt ways. It is known that the uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

Bounkhel [5, 2003], Noor [6, 2004], Pang et al. [8, 2007] and Moudafi [7, 2009] considered the variational inequality probelm over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

Recently, in 2009, Noor [9] introduced and studied some new classes of variational and the Wiener-Hiof equations and established the equivalent between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equation, by using the projection technique. Noor also presented some new projection methods for solving the nonconvex variational inequalities and proved the convergence of iterative method under suitable conditions.

In the same year, Moudafi [7] introduced the convergence of two-step projection methods for a system of nonconvex variational inequalities problems for a mapping T is γ-strongly monotone and L-Lipschitz continuous.

Very recently, in 2013, Al-Shemas [10] introduced the strongly nonlinear general nonconvex variational inequalities and proved the convergence of the predictor-corrector method only requires pseudomonotonicity which is weaker condition than monotonicity.

Motivated by [7] and [10], we introduce and study the convergence of a modified algorithm for the system of strongly nonlinear nonconvex variational inequalities problems for two mappings satisfying strongly monotone and Lipschitz continuous. This work extends and improves some known results.

2 Preliminaries

Let C be a closed subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called *closest point* or *projection* of u on C if $d_C(u) = \|u-v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$, that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of H. The *proximal normal cone* to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [11].

Lemma 2.3. *Let C be a closed subset of a Hilbert space H. Then*

$$z \in N_C^P(x) \Longleftrightarrow \exists \sigma > 0, \langle z, y - x\rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [12] and Poliquin et al. [13] have introduced and studied a new class of nonconvex sets which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be *uniformly r-prox-regular* with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x}\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets, see [12, 13].

In this work, let C be a closed subset of a real Hilbert space H with is uniformly r-prox-regular (nonconvex), set $C_r := \{x \in H : d(x, C) < r\}$. For given nonlinear mappings $T_1, T_2 : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq \langle Ay^*, x - x^*\rangle, \forall x \in C_r, \rho > 0, \\ &\langle \eta T_2 x^* + y^* - x^*, y - y^*\rangle + \lambda\|y - y^*\|^2 \geq \langle Ax^*, y - y^*\rangle, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the *system of strongly nonlinear nonconvex variational inequalities* (SSNNVI).

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0$ and $T_1 = T_2 = T$, then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq 0, \forall x \in C_r, \rho > 0, \\ &\langle \eta T x^* + y^* - x^*, y - y^*\rangle + \lambda\|y - y^*\|^2 \geq 0, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.6}$$

which is called the *system of nonconvex variational inequalities* (SNVI). We known that the inequalities (2.6) is equivalent as follows:

$$\begin{aligned} y^* - x^* - \rho T y^* &\in N_{C_r}^P x^*, \\ x^* - y^* - \eta T x^* &\in N_{C_r}^P y^*, \end{aligned} \tag{2.7}$$

which is introduced by Moudafi [7].

If $A(x^*) \equiv 0, A(y^*) \equiv 0$, $T_1 = T_2 = T$ and $\lambda = 0$, then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0, \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle \geq 0, \forall y \in C_r, \eta > 0. \end{aligned} \tag{2.8}$$

Which is called *system of variational inequalities* (SVI), which $C_r = C$ introduced by Verma [14].

If $T_1 = T_2 = T$, $x^* = y^*$ and $\rho = \eta = 1$, then the problem (2.5) is equivalent to finding $x^* \in C_r$ such that

$$\langle Tx^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ax^*, x - x^* \rangle, \forall x \in C_r \tag{2.9}$$

which is known as the strongly nonlinear nonconvex variational inequality and studied by Noor [15].

In inequalities (2.9), if we let $A(x^*) \equiv 0$, the problem is to finding $x^* \in C_r$ such that

$$\langle Tx^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r \tag{2.10}$$

which is called the *nonconvex variational inequalities* (NVI), introduced and studied by Bounkhel et. al. [5] and Noor [6, 16].

It is worth mentioning that if $C_r = C$ is convex set, then problem (2.10) is equivalent to finding $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \forall x \in C, \tag{2.11}$$

which is known as *variational inequalities*, introduced and studied by Stamphacia [1].

Now, if C_r is a nonconvex (uniform r-prox regular) set, then problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &0 \in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r} x^*, \\ &0 \in \eta T_2 x^* + y^* - x^* - Ax^* + N^P_{C_r} y^*, \end{aligned} \tag{2.12}$$

where $N^P_{C_r} u$ is the normal cone of C_r at u. The problem (2.12) is called the *the system of nonconvex variational inclusion problem associated with nonconvex variational inequalities.*

We now recall the well-known lemmas of the uniform prox-regular sets.

Lemma 2.5. *Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r := \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) *for all $x \in C_r, P_C(x) \neq \emptyset$,*

(2) *for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,*

(3) *the proximal normal cone is closed as a set-valued mapping.*

Let H be a real Hilbert space. A mapping $T : H \to H$ is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|x - y\|^2, \tag{2.13}$$

for all $x, y \in H$. A mapping T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \tag{2.14}$$

for all $x, y \in H$.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality:*

(1) $\|x+y\|^2 \leq \|x\|^2+2\langle y, x+y\rangle \quad x, y \in H$ *and* $\|x-y\|^2 = \|x\|^2-2\langle x, y\rangle+\|y\|^2$,

(2) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$.

3 Main Results

In this section, we first establish the equivalent between the system of nonconvex variational inequalities (2.5) with the projection technique.

Lemma 3.1. *For given $x^*, y^* \in C_r$ are solution of system of strongly nonlinear general nonconvex variational inequalities* (2.5)*, if and only if*

$$\begin{aligned} x^* &= P_{C_r}[y^* - \rho T_1 y^* + Ay^*],\\ y^* &= P_{C_r}[x^* - \eta T_2 x^* + Ax^*], \end{aligned} \tag{3.1}$$

where $P_{C_r} = (I + N^P_{C_r})^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5), for a constant $\rho > 0$, we have

$$\langle \rho T_1 y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq \langle Ay^*, x - x^*\rangle$$

if and only if

$$\langle Ay^* - \rho T_1 y^* - x^* + y^*, x - x^*\rangle \leq \lambda\|x - x^*\|^2.$$

Then $Ay^* - \rho T_1 y^* - x^* + y^* \in N^P_{C_r} x^*$ and it implies that

$$\begin{aligned} 0 \quad &\in \quad \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r} x^* = (I + N^P_{C_r})x^* - (y^* - \rho T_1 y^* + Ay^*)\\ &\Leftrightarrow \quad (I + N^P_{C_r})x^* = (y^* - \rho T_1 y^* + Ay^*)\\ &\Leftrightarrow \quad x^* = P_{C_r}[y^* - \rho T_1 y^* + Ay^*], \end{aligned}$$

where we have used the well-known fact that $P_{C_r} = (I + N^P_{C_r})^{-1}$. Similarly, we obtain $y^* = P_{C_r}[x^* - \eta T_2 x^* + Ax^*]$. This prove our assertions. $\square$

Algorithm 3.2. *For arbitrarily chosen initial points $x_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:*

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T_2 x_n + A x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_{C_r}[y_n - \rho T_1 y_n + A y_n], \quad \rho > 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Now, we suggest and analyze the following explicit projection method (3.2) for solving the system of nonconvex variational inequalities (2.5).

Theorem 3.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous. If there exists constant $\rho, \eta > 0$ such that*

$$\begin{aligned} |\rho - \frac{\gamma_1}{\mu_1^2}| &< \frac{\sqrt{\gamma_1^2 t_s^2 - t_s \mu_1^2 (t_s + 2\beta - 1)}}{t_s \mu_1^2} \\ |\eta - \frac{\gamma_2}{\mu_2^2}| &< \frac{\sqrt{\gamma_2^2 t_s^2 - t_s \mu_2^2 (t_s + 2\beta - 1)}}{t_s \mu_2^2} \end{aligned} \tag{3.3}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0,r)$ with $t_s\sqrt{t_s^2 - 1} < \frac{\gamma_1}{\mu_1}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converge to a solution of the system of nonconvex variational inequalities (2.5).

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5) and from Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T_1 y_n + A y_n] - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) \\ &\quad + \alpha_n (P_C[y_n - \rho T_1 y_n + A y_n] - P_C[y^* - \rho T_1 y^* + A y^*])\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s \|(y_n - \rho T_1 y_n + A y_n) - (y^* - \rho T_1 y^* + A y^*)\| \\ &= (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s \|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*) + (A y_n - A y^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s [\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \|A y_n - A y^*\|] \\ &\leq (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n t_s [\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \beta \|y_n - y^*\|]. \end{aligned} \tag{3.4}$$

Since T_1 are both μ_1-Lipschitz continuous and γ_1-strongly monotone mapping and from Lemma 2.6, we consider

$$\begin{aligned}\|(y_n - y^*) - \rho(T_1y_n - T_1y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1y_n - T_1y^*\rangle \\ &\quad +\rho^2\|T_1y_n - T_1y^*\|^2 \\ &\leq \|y_n - y^*\|^2 - 2\rho\gamma_1\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= (1 - 2\rho\gamma_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2.\end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1y_n - T_1y^*)\| \leq \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.5}$$

Substituting (3.5) into (3.4), we have

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2})\|y_n - y^*\|. \tag{3.6}$$

On the other hand, we can compute that

$$\begin{aligned}\|y_n - y^*\| &= \|P_C[x_n - \eta T_2x_n + Ax_n] - y^*\| \\ &= \|P_C[x_n - \eta T_2x_n + Ax_n] - P_C[x^* - \eta T_2x^* + Ax^*]\| \\ &\leq t_s\|(x_n - \eta T_2x_n + Ax_n) - (x^* - \eta T_2x^* + Ax^*)\| \\ &\leq t_s[\|(x_n - x^*) - \eta(T_2x_n - T_2x^*)\| + \|Ax_n - Ax^*\|] \\ &\leq t_s[\|(x_n - x^*) - \eta(T_2x_n - T_2x^*)\| + \beta\|x_n - x^*\|].\end{aligned} \tag{3.7}$$

Similarly, from T_2 are both μ_2-Lipschitz continuous and γ_2-strongly monotone mapping, we have

$$\begin{aligned}\|(x_n - x^*) - \eta(T_2x_n - T_2x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_2x_n - T_2x^*\rangle \\ &\quad +\eta^2\|T_2x_n - T_2x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\eta\gamma_2\|x_n - x^*\|^2 + \eta^2\mu_2^2\|x_n - x^*\|^2 \\ &= (1 - 2\eta\gamma_2 + \eta^2\mu_2^2)\|x_n - x^*\|^2.\end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_2x_n - T_2x^*)\| \leq \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2}\|x_n - x^*\|. \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\|y_n - y^*\| \leq t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})\|x_n - x^*\|. \tag{3.9}$$

Moreover, from (3.6) and (3.9) we put $\theta_1 = t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}), \theta_2 = t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})$, it follows that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1-\theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\leq \prod_{i=0}^{n}(1 - (1-\theta_1\theta_2)\alpha_i)\|x_0 - x^*\|.\end{aligned} \tag{3.10}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and (3.3), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{3.11}$$

It follows from (3.10) and (3.11), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{3.12}$$

From (3.9) and (3.12), we have $\lim_{n\to\infty}\|y_n - y^*\| = 0$, that is, $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.5). □

Corollary 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho,\ \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \tag{3.13}$$

where $t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.14}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. From Theorem 3.3, if $T_1 = T_2 = T$ we have a result. □

Corollary 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho,\ \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \tag{3.15}$$

where $t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0, \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.16}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. By Theorem 3.3, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for $n \geq 1$, we have a result. □

4 Applications

In this section, we can apply Theorem 3.3 to the system of general nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0, \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{4.1}$$

which is called the *system of general nonconvex variational inequalities.*

If $T_1 = T_2 = T$, then the problem (4.1) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0, \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0. \end{aligned} \tag{4.2}$$

By using the same way as the proof of Lemma 3.1, we have the result.

Lemma 4.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities* (4.1) *if and only if*

$$\begin{aligned} g(x^*) &= P_C[g(y^*) - \rho T_1 g(y^*)], \\ g(y^*) &= P_C[g(x^*) - \eta T_2 g(x^*)], \end{aligned} \tag{4.3}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, g : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and g is continuous. If there exists constant $\rho, \eta > 0$ such that*

$$\begin{aligned} &|\rho - \frac{\gamma_1}{t_s^2\mu_1^2}| < \frac{\sqrt{\gamma_1^2 - t_s^2\mu_1^2(t_s^2-1)}}{t_s^2\mu_1^2} \ \textit{and} \ \ t_s\sqrt{t_s^2-1} < \frac{\gamma_1}{\mu_1} \\ &|\eta - \frac{\gamma_2}{t_s^2\mu_2^2}| < \frac{\sqrt{\gamma_2^2 - t_s^2\mu_2^2(t_s^2-1)}}{t_s^2\mu_2^2} \ \textit{and} \ \ t_s\sqrt{t_s^2-1} < \frac{\gamma_2}{\mu_2}, \end{aligned} \tag{4.4}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T_2 g(x_n)], \ \eta > 0 \\ g(x_{n+1}) &= (1-\alpha_n)g(x_n) + \alpha_n P_C[g(y_n) - \rho T_1 g(y_n)], \ \rho > 0, \end{aligned} \tag{4.5}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. By the similar proof to Theorem 3.3, let $x^*, y^* \in C_r$ be a solution of (4.1) and from Lemma 4.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \leq \prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|, \tag{4.6}$$

where $\theta_1 = t_s\sqrt{1-2\rho\gamma_1+\rho^2\mu_1^2}, \theta_2 = t_s\sqrt{1-2\eta\gamma_2+\eta^2\mu_2^2}$. From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and Conditions (4.4), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{4.7}$$

It follows from (4.6) and (4.7), we have

$$\lim_{n\to\infty}\|g(x_n) - g(x^*)\| = 0.$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \leq \theta_2\|g(x_n) - g(x^*)\|,$$

it follows that

$$\lim_{n\to\infty}\|g(y_n) - g(y^*)\| = 0.$$

From g is a continuous mapping, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and $\lim_{n\to\infty}\|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (4.1). This complete the proof. □

Corollary 4.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T, g : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping and g is continuous mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho,\ \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \tag{4.8}$$

where $t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, t_s = \frac{r}{r-s}$ *for some* $s \in (0, r)$*. Then the sequence* $\{x_n\}$ *and* $\{y_n\}$ *is generated by for* $x_0, y_0 \in C_r$*,*

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)],\ \eta > 0, \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)],\ \rho > 0, \end{aligned} \tag{4.9}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.2).

Proof. From Theorem 4.2, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for $n \geq 0$, we have a result. □

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