Thai Journal of Mathematics
Volume 16 (2018) Number 1 : 173–182

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Existence of Best Proximity Points for a Class of Generalized Cyclic Contraction Mappings

Somboon Niyom[†], **Pornpimon Boriwan**[‡] **and Narin Petrot**[‡,1]

[†]Department of Mathematics and Statistic, Faculty of Science and Technology, Nakhon Sawan Rajabhat University
Nakhon Sawan 60000, Thailand
e-mail : somboon.n@nsru.ac.th (S. Niyom)

[‡]Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
e-mail : boriwanp_1990@hotmail.com (P. Boriwan)
narinp@nu.ac.th (N. Petrot)

Abstract : In this paper, by using a concept of $\mathcal{R}-$function, we introduce a new class of cyclic contraction mappings and consider the best proximity points theorems in the context of complete metric spaces. Some related results and examples are also discussed and provided.

Keywords : cyclic map; best proximity point; $\mathcal{R}-$function; generalized cyclic $\mathcal{R}-$contraction.
2010 Mathematics Subject Classification : 41A17; 47H09.

---

1 Introduction and Preliminaries

The fact that fixed point theory furnishes a unified treatment and is a vital tool for solving equations of form $Tx = x$, where T is a self-mapping defined on a subset of a normed linear space, metric space, topological vector space or some suitable space, leads to the significance of this subject. However, almost all such

[1]Corresponding author.

results dilate upon the existence of a fixed point for self-mappings. In fact, if T is a non-self-mapping, then it is probable that the equation $Tx = x$ has no solution. In this case, Fan [1] introduced the concept of the best approximation theorems which is a concept that explore the existence of an approximate solution, that is, if A is a nonempty subset of a considered space X and $T : A \to X$ then we find a point $x \in A$ such that $d(x, Tx) = d(Tx, A)$.

On the other hand, we also considered the best proximity point theorems which is a concept that analyze the existence of an approximate solution that is optimal, that is, if A and B are a nonempty subset of a considered space and $T : A\cup B \to A\cup B$ then we find a point $x \in A\cup B$ such that $d(x, Tx) = dist(A, B)$, where $dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. An interesting class of mappings for considering the best proximity point theorems is a concept so-called cyclic map, which is defined by following: let A and B be nonempty subsets of a nonempty set X, a map $T : A \cup B \to A \cup B$ is a *cyclic* map if $T(A) \subset B$ and $T(B) \subset A$.

In [2], Eldred and Veeramani introduced and proved the following interesting best proximity point theorem.

Definition 1.1. [2] Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T : A\cup B \to A\cup B$ is called a *cyclic contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$$

for all $x \in A$, $y \in B$.

Theorem 1.2. [2] *Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that*

$$d(x, Tx) = dist(A, B).$$

Remark 1.3. *If A and B are nonempty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic contraction, and $A \cap B \neq \emptyset$, then $dist(A, B) = 0$, subsequently, T is a contraction on the complete metric space $(A \cap B, d)$. Hence, applying the Banach contraction principle, by Theorem 1.2 we known that T has a unique fixed point in $A \cap B$.*

For more examples of best proximity point theorems, the readers may consult [3–16].

On the other hand, let us recall that a function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{R}-$*function* if

$$\limsup_{s\to t^+} \varphi(s) < 1 \text{ for all } t \in [0, \infty).$$

By using $\mathcal{R}$-function, Du and Lakzian [17] introduced the following concept.

Definition 1.4. Let A and B be nonempty subsets of a metric space (X, d). A cyclic mapping $T : A \cup B \to A \cup B$ is called an *cyclic* $\mathcal{R}-$*contraction* if there exists an $\mathcal{R}$-function φ such that

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))dist(A, B)$$

for all $x \in A, y \in B$.

Under some suitable proposed conditions, they provided a following best proximity point theorem.

Theorem 1.5. *Let A and B be nonempty subsets of a metric space (X, d). Let $T : A \cup B \to A \cup B$ be an $\mathcal{R}$-cyclic contraction. Let $x_1 \in A$ be given. Define an iterative sequence $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that*

$$d(x, Tx) = dist(A, B).$$

Note that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{R}$-function. So the set of $\mathcal{R}-$function is a rich class, and subsequently, the results those present in [17] are of interesting. Thus, motivated by the presented results above, the main objective of this paper is to consider a wider class of mappings which was considered by Du and Lakzian [17]. Also, some new existence theorems of such introduced mappings will be considered. In order to do that, the following well know result on a class of $\mathcal{R}$-function is needed.

Lemma 1.6. [18] *Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then, φ is an $\mathcal{R}-$ function if an only if for any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.*

2 Main Results

Here we introduce a following new subclass of cyclic mappings.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is called *generalized cyclic* $\mathcal{R}-$*contraction mapping* if there exists an $\mathcal{R}-$function φ such that

$$\begin{aligned} d(Tx, Ty) \quad &\leq \quad \varphi(d(x, y)) \max\Big\{d(x, y), \frac{1}{2}\Big[d(Tx, x) + d(Ty, y) \\ &\qquad + \min\{d(x, Ty), d(y, Tx)\}\Big]\Big\} + (1 - \varphi(d(x, y)))dist(A, B) \end{aligned}$$

for all $x \in A$, $y \in B$.

First, we establish a following theorem related to generalized cyclic $\mathcal{R}-$contraction mappings.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X,d) and $T : A \cup B \longrightarrow A \cup B$ be a generalized $\mathcal{R}-$cyclic contraction. For each $x_1 \in A$ define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by*

$$x_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Then,

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A,B).$$

Proof. We first show that $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence. Now, let us fix $n \in \mathbb{N}$. Since T is a generalized $\mathcal{R}-$cyclic contraction mapping type, we see that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(Tx_{n+1}, Tx_n) \\ &\leq \varphi(d(x_{n+1}, x_n)) \max\Big\{d(x_{n+1}, x_n), \frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + \min\{d(x_{n+1}, x_{n+1}), d(x_n, x_{n+2})\}\Big]\Big\} \\ &\quad + (1 - \varphi(d(x_{n+1}, x_n)))dist(A,B) \\ &= \varphi(d(x_{n+1}, x_n)) \max\Big\{d(x_{n+1}, x_n), \frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + d(x_{n+1}, x_{n+1})\Big]\Big\} + \\ &\quad (1 - \varphi(d(x_{n+1}, x_n)))dist(A,B) \\ &= \varphi(d(x_{n+1}, x_n)) \max\Big\{d(x_{n+1}, x_n), \frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) \\ &\quad + d(x_{n+1}, x_n)\Big]\Big\} + (1 - \varphi(d(x_{n+1}, x_n)))dist(A,B). \end{aligned} \tag{2.2}$$

Suppose that $d(x_{n+1}, x_n) < \frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\Big]$, then $d(x_{n+1}, x_n) < d(x_{n+2}, x_{n+1})$. This would implies,

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq \varphi(d(x_{n+1}, x_n))\frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\Big] \\ &\quad + (1 - \varphi(d(x_{n+1}, x_n)))dist(A,B) \\ &< \varphi(d(x_{n+1}, x_n))\frac{1}{2}\Big[d(x_{n+2}, x_{n+1}) + d(x_{n+2}, x_{n+1})\Big] \\ &\quad + (1 - \varphi(d(x_{n+1}, x_n)))dist(A,B) \\ &= \varphi(d(x_{n+1}, x_n))d(x_{n+2}, x_{n+1}) + (1-\varphi(d(x_{n+1}, x_n)))dist(A,B) \\ &\leq \varphi(d(x_{n+1}, x_n))d(x_{n+2}, x_{n+1}) + \\ &\quad (1 - \varphi(d(x_{n+1}, x_n)))d(x_{n+2}, x_{n+1}) \\ &= d(x_{n+2}, x_{n+1}), \end{aligned}$$

which is a contradiction. Subsequently, by (2.2), we must have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq \varphi(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + (1 - \varphi(d(x_{n+1}, x_n)))dist(A, B) \\ &\leq \varphi(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + (1-\varphi(d(x_{n+1}, x_n)))d(x_{n+1}, x_n) \\ &= d(x_{n+1}, x_n). \end{aligned}$$

This shows that $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence.

If there exists $j \in \mathbb{N}$ such that $x_j = x_{j+1} \in A \cap B$, then

$$\lim_{n\longrightarrow\infty} d(x_{n+1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n+1}, x_n) = dist(A, B) = 0,$$

and the proof is completed. So it remains to consider for the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing in $(0, \infty)$, by Lemma1.6, we have

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(d(x_{n+1}, x_n)) < 1.$$

Let $\lambda := \sup_{n\in\mathbb{N}} \varphi(d(x_{n+1}, x_n))$. Then $0 \leq \varphi(d(x_{n+1}, x_n)) \leq \lambda < 1$ for all $n \in \mathbb{N}$. From $x_1 \in A$, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. Since T is a cyclic $\mathcal{R}-$contraction, we have

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \varphi(d(x_1, x_2))d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))dist(A, B) \\ &\leq \lambda d(x_1, x_2) + dist(A, B). \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} d(x_3, x_4) &= d(Tx_2, Tx_3) \\ &\leq \varphi(d(x_2, x_3))d(x_2, x_3) + (1 - \varphi(d(x_2, x_3)))dist(A, B) \\ &\leq \varphi(d(x_2, x_3))[\lambda d(x_1, x_2) + dist(A, B)] \\ &\quad +(1 - \varphi(d(x_2, x_3)))dist(A, B) \\ &= \varphi(d(x_2, x_3))\lambda d(x_1, x_2) + dist(A, B) \\ &\leq \lambda^2 d(x_1, x_2) + dist(A, B). \end{aligned}$$

Continuing this process, we obtain

$$dist(A, B) \leq d(x_{n+1}, x_{n+2}) \leq \lambda^n d(x_1, x_2) + dist(A, B). \tag{2.4}$$

Since $\lambda \in [0, 1)$, $\lim_{n\to\infty} \lambda^n = 0$. Using (2.4) and the decreasingness of $\{d(x_n, x_{n+1})\}$, we obtain

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = dist(A, B).$$

This complete the proof. □

Next, we give a best proximity point theorem for a subclass of cyclic mappings.

Theorem 2.3. *Let A and B be nonempty subsets of a metric (X,d) and $T: A\cup B \longrightarrow A\cup B$ be a cyclic map such that*

$$d(Tx,Ty) \leq \max\left\{d(x,y), \frac{1}{2}\left[d(Tx,x)+d(Ty,y)+\min\{d(x,Ty),d(y,Tx)\}\right]\right\} \tag{2.5}$$

for all $x\in A, y\in B$. Let $x_1\in A$ be given. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. Suppose that

- (i) $\{x_{2n-1}\}$ *has a convergent subsequence in A*
- (ii) $\lim_{n\to\infty} d(x_n,x_{n+1}) = dist(A,B)$.

Then there exists $v\in A$ such that $d(v,Tv)=dist(A,B)$.

Proof. Since T is a cyclic map and $x_1\in A$, it follows that $x_{2n-1}\in A$ and $x_{2n}\in B$, for all $n\in\mathbb{N}$. By (i), there are $v\in A$ and a subsequence $\{x_{2n_k-1}\}$ of $\{x_{2n-1}\}$ such that $x_{2n_k-1}\to v$ as $k\to\infty$. Also, we note that

$$dist(A,B)\leq d(v,x_{2n_k})\leq d(v,x_{2n_k-1})+d(x_{2n_k-1},x_{2n_k}),$$

for all $k\in\mathbb{N}$. Thus, by $\lim\limits_{k\to\infty} d(v,x_{2n_k-1})=0$ and the condition (ii), we know that

$$\lim_{k\longrightarrow\infty} d(v,x_{2n_k})=dist(A,B).$$

On the other hand, by (2.5), we have

$$\begin{aligned} dist(A,B) &\leq d(Tv,x_{2n_k+1}) \\ &\leq \max\Big\{d(v,x_{2n_k}),\frac{1}{2}\Big[d(Tv,v)+d(x_{2n_k+1},x_{2n_k}) \\ &\quad +\min\{d(v,x_{2n_k+1}),d(x_{2n_k+1},Tv)\}\Big]\Big\} \end{aligned}$$

for all $k\in\mathbb{N}$. Letting $k\to\infty$, we obtain

$$\begin{aligned} dist(A,B) &\leq d(Tv,v) \\ &\leq \max\Big\{dist(A,B),\frac{1}{2}\Big[d(Tv,v)+dist(A,B)+ \\ &\quad \min\{d(v,v),d(v,Tv)\}\Big]\Big\} \\ &= \max\Big\{dist(A,B),\frac{1}{2}\Big[d(Tv,v)+dist(A,B)\Big]\Big\}. \end{aligned} \tag{2.6}$$

Now we consider the following two cases:

Case I: If $\max\left\{dist(A,B),\frac{1}{2}\Big[d(Tv,v)+dist(A,B)\Big]\right\}=dist(A,B)$.
In this case, it is easy to very that

$$d(Tv,v)=dist(A,B).$$

Case II: If $\max\Big\{dist(A,B), \frac{1}{2}\Big[d(Tv,v)+dist(A,B)\Big]\Big\} = \frac{1}{2}\Big[d(Tv,v)+dist(A,B)\Big]$. Then, from (2.6), we have

$$d(Tv,v) \leq \frac{1}{2}\Big[d(Tv,v) + dist(A,B)\Big],$$

which is equivalent to

$$d(Tv,v) \leq dist(A,B),$$

and it follows that $d(Tv,v) = dist(A,B)$. Thus, by the two cases above, we reach the desired result. □

Applying Theorems 2.2 and 2.3, we establish the following new best proximity point theorem for a subclass of generalized cyclic $\mathcal{R}-$contraction mappings.

Theorem 2.4. *Let A and B be nonempty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ be an generalized cyclic $\mathcal{R}-$contraction. Let $x_1 \in A$ be given. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$, for $n \in \mathbb{N}$. Suppose that $\{x_{2n-1}\}$ has a convergent subsequence in A, then there exists $v \in A$ such that $d(v,Tv) = dist(A,B)$.*

Proof. Let us observe that every generalized cyclic $\mathcal{R}-$contraction satisfies condition (2.5). Also, by Theorem 2.2, we know that the conditions (i) and (ii) of Theorem 2.3 are always satisfied. Using these observations, in view of Theorem 2.3, we get the desired result immediately. □

Note that Theorem 2.4 contains Theorem 1.5, and then Theorem 1.2, as special cases. We will complete this research by giving an example which shows that Theorem 2.4 is a genuine generalization of Theorem 1.5.

Example 2.5. Let $X = \mathbb{R}$ and d is a usual metric on $\mathbb{R}$. Let us consider for $A = [-2,-1]$, $B = [2,3]$ and a cyclic mapping $T : A \cup B \to A \cup B$ which is defined by

$$Tx := \begin{cases} 2, & \text{if } x \in [-2,-1], \\ -1, & \text{if } x \in [2,3), \\ -2, & \text{if } x = 3. \end{cases}$$

Now, we will show that T is a generalized cyclic $\mathcal{R}-$contraction mapping with respect to an $\mathcal{R}-$ function φ that defined by

$$\varphi(t) = \frac{2}{3} \qquad \text{for all } t \in [0,\infty).$$

We consider the following cases:
Case(i): If $x = -1$ and $y = 3$.

We see that

$$\frac{1}{2}[d(x,Tx) + d(y,Ty) + \min\{d(x,Ty), d(y,Tx)\}] = \frac{1}{2}[3 + 5 + \min\{1,1\}] = 4.5.$$

It follows that,

$$\begin{aligned} d(Tx,Ty) \ = \ 4 \ &= \ \frac{2}{3}(4.5) + \left(1 - \frac{2}{3}\right)3 \\ &\leq \ \varphi(d(x,y)) \max\Big\{d(x,y), \frac{1}{2}\Big[d(Tx,x) + d(Ty,y) \\ &\quad + \min\{d(x,Ty), d(y,Tx)\}\Big]\Big\} + (1 - \varphi(d(x,y)))dist(A,B). \end{aligned}$$

Case(ii): For each $x \in [-2,-1]$ and $y \in [2,3)$.

We note that $d(Tx,Ty) = 3$, and

$$\begin{aligned} d(x,y) &= |\, x - y \,| = y - x, \\ d(x,Tx) &= |\, x - 2 \,| = 2 - x, \\ d(y,Ty) &= |\, y + 1 \,| = y + 1, \\ d(x,Ty) &= |\, x + 1 \,| = -x - 1, \\ d(y,Tx) &= |\, y - 2 \,| = y - 2. \end{aligned}$$

Next, from $y \in [2,3)$ and $x \in [-2,-1]$, we see that $d(x,y) = y - x \geq 3$. This gives

$$\begin{aligned} d(Tx,Ty) \ = \ 3 \ &\leq \ \frac{2}{3}(y-x) + 3 - \frac{2}{3}(3) \\ &= \ \frac{2}{3}(y-x) + \left(1 - \frac{2}{3}\right)3 \\ &\leq \ \varphi(d(x,y)) \max\Big\{d(x,y), \frac{1}{2}\Big[d(Tx,x) + d(Ty,y) \\ &\quad + \min\{d(x,Ty), d(y,Tx)\}\Big]\Big\} + (1 - \varphi(d(x,y)))dist(A,B). \end{aligned}$$

Therefore, T is generalized a cyclic $\mathcal{R}-$contraction on $A \cup B$.

On the other hand, let us consider when $x = -1$ and $y = 3$. For any $\mathcal{R}-$function φ, we see that

$$\begin{aligned} d(Tx,Ty) \ = \ 4 \ &> \ 4\varphi(d(x,y)) + 3 - 3\varphi(d(x,y)) \\ &= \ 4\varphi(d(x,y)) + (1 - \varphi(d(x,y)))3 \\ &= \ d(x,y)\varphi(d(x,y)) + (1 - \varphi(d(x,y)))dist(A,B). \end{aligned}$$

This means that T does not satisfy the Definition 1.4.

Further, one can see that $-1 \in A$ and $y = 2 \in B$ are two best proximity points for T.

Acknowledgement(s) : This work is supported by the National Research Council of Thailand (Project No. 2557A13702021).

References

[1] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z. 112 (1969) 234-240.

[2] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006) 1001-1006.

[3] A. Abkar, M. Gabeleh, Generalized cyclic contractions in partially ordered metric spaces, Optim. Lett. 6 (2012) 1819-1830.

[4] M.A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal. 70 (2009) 3665-3671.

[5] S.S. Basha, Best proximity point theorems: an exploration of a common solution to approximation and optimization problems, Appl. Math. Comput. 218 (2012) 9773-9780.

[6] S.S. Basha, N. Shahzed, R. Jeyaraj, Best proxmity point theorems: exposition of a significant non-linear programming problem, J. Glob. Optim. 56 (2013) 1699-1705.

[7] M. De La Sen, R.P. Agarwal, Common fixed points and best proximity points of two cyclic self-mappings, Fixed Point Theory Appl. 2012:136 (2012) doi:10.1186/1687-1812-2012-136.

[8] A.A. Harandi, Best proximity points for proximal generalized contractions in metric spaces, Optim. Lett. 7 (2013) 913-921.

[9] M. Jleli, E. Karapinar, B. Samet, Fixed point results for almost generalized cyclic $(\psi, \varphi)-$weak contractive type mappings with application, Abstr. Appl. Anal. 2012:917831 (2012) doi:10.1155/2012/917831.

[10] E. Karapinar, Best proximity points of cyclic mappings, Appl. Math. Lett. 25 (2012) 1761-1766.

[11] W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4 (2003) 79-89.

[12] C. Mongkolkeha, P. Kuman, Best proximity point theorems for generalized cyclic contractions in ordered metric spaces, J. Optim. Theory Appl. 155 (2012) 215-226.

[13] C. Mongkolkeha, Y.J. Cho, P. Kuman, Best proximity points for generalized proximal C-contraction mappings in metric spsces with partial orders, J. Inequal. Appl. 2013:94 (2013) doi:10.1186/1029-242x-2013-94.

[14] H.K. Nashine, P. Kumam, C. Vetro, Best proximity point theorems for rational proximal contractions, Fixed Point Theory Appl. 2013:95 (2013) doi:10.1186/1687-1812-2013-95.

[15] H.K. Nashine, Existence of best proximity points: Global optimal approximate solution, Sarajevo J. Math. 9 (22) (2013) 231-241.

[16] T. Suzuki, M. Kikkawa, C. Vetro, The existence of the best proximity points in metric spaces with the property UC, Nonlinear Anal. 71 (2009) 2918-2926.

[17] W.S. Du, H. Lakzian, Nonlinear conditions for the existence of best proximity points, J. Inequal. Appl. 2012:206 (2012) doi:10.1186/1029-242X-2012-206.

[18] W.S. Du, On approximate coincidence point properties and their applications to fixed point theory, J. Appl. Math. 2012:302830 (2012) doi:10.1155/2012/302830.

(Received 9 March 2015)
(Accepted 8 February 2016)

Thai J. Math. Online @ http://thaijmath.in.cmu.ac.th