Thai Journal of Mathematics
Volume 16 (2018) Number 1 : 165–171
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Regular Elements of the Variant Semigroups of Transformations Preserving Double Direction Equivalences

Winita Yonthanthum

Department of Mathematics and Statistics, Faculty of Science
Prince of Songkla University, Hat Yai
Songkhla 90110, Thailand
e-mail : winita.m@psu.ac.th

Abstract : Let $T(X)$ denote the full transformation semigroup on a set X. For an equivalence relation E on X, let

$$T_{E^*}(X) = \{\alpha \in T(X) \mid \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_{E^*}(X)$ is a subsemigroup of $T(X)$. For $\theta \in T_{E^*}(X)$, we define a sandwich operation $*$ on $T_{E^*}(X)$ by $\alpha * \beta = \alpha\theta\beta$ where $\alpha\theta\beta$ is the composition of functions α, θ and β. Under this operation, $T_{E^*}(X)$ is a semigroup which is called the variant semigroup of $T_{E^*}(X)$ with the sandwich function θ, and denoted by $(T_{E^*}(X), \theta)$. In this paper, we give a necessary and sufficient condition for an element of $(T_{E^*}(X), \theta)$ to be regular and determine when $(T_{E^*}(X), \theta)$ is a regular semigroup.

Keywords : regular elements; transformation semigroups; variant semigroups.
2010 Mathematics Subject Classification : 20M20.

1 Introduction

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$. The semigroup S is said to be *regular* if all of its elements are regular. The set of all regular elements of S is denoted by $\mathrm{Reg}(S)$.

The domain and the range of a mapping α will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For an element $x \in \operatorname{dom}\alpha$, the image of x under α is written as $x\alpha$. Notice that $\operatorname{dom}\alpha = \dot{\bigcup\limits_{x\in\operatorname{ran}\alpha}} x\alpha^{-1}$ where the notation $\dot{\bigcup}$ stands for a disjoint union. For $A \subseteq \operatorname{dom}\alpha$, denote by $\alpha_{|A}$ the restriction of α to A. The identity mapping on a nonempty set A is denoted by 1_A.

For convenience, we write a mapping by using a bracket notation. For example, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stands for a mapping α with $\operatorname{dom}\alpha = \{a, b\}, \operatorname{ran}\alpha = \{c, d\}$, $a\alpha = c$ and $b\alpha = d$, $\begin{pmatrix} A_i \\ a_i \end{pmatrix}_{i\in I}$ stands for a mapping β with $\operatorname{dom}\beta = \bigcup\limits_{i\in I} A_i$, $\operatorname{ran}\beta = \{a_i \mid i \in I\}$ and $x\beta = a_i$ for all $x \in A_i$.

For a nonempty set X, let $P(X)$ and $T(X)$ be the partial transformation semigroup on X and the full transformation semigroup on X, respectively. It is well known that $P(X)$ and $T(X)$ are regular semigroups.

In [1], Fernandes and Sanwong introduced the partial transformation semigroup with restricted range $P(X, Y)$ defined by

$$P(X, Y) = \{\alpha \in P(X) \mid X\alpha \subseteq Y\}$$

where $\emptyset \neq Y \subseteq X$. They proved that $\{\alpha \in P(X, Y) \mid X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $P(X, Y)$. Later, Sangkhanan and Sanwong [2] defined the partial linear transformation semigroup with restricted range $P(V, W)$ where W is a subspace of a vector space V, and also described the largest regular subsemigroup of $P(V, W)$.

For a nonempty subset Y of X, let

$$T(X, Y) = \{\alpha \in T(X) \mid X\alpha \subseteq Y\},$$
$$\overline{T}(X, Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Then $\overline{T}(X, Y) \subseteq T(X, Y)$ and both are subsemigroups of $T(X)$. The semigroup $T(X, Y)$ was introduced and studied by Symons [3] in 1975, while Magill [4] introduced and studied the semigroup $\overline{T}(X, Y)$ in 1966. Regular elements of these semigroups are discussed in [5]. Recently, subsemigroups of $\overline{T}(X, Y)$ were studied by Sanwong [6] and Laysirikul [7].

In [8], Anantayasethi and Koppitz introduced the semigroup $T_P(X, Y)$ of all nonempty subsets of the semigroup $T(X, Y), \emptyset \neq Y \subseteq X$, under the operation $\mathcal{AB} := \{\alpha\beta \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$ where $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of $T(X, Y)$. They determined regular elements in $T_P(X, Y)$ for the case $|Y| = 2$.

For nonempty sets X and Y, let $T(X, Y)$ be the set of all mappings from X into Y. For $\theta \in T(Y, X)$, we define a *sandwich* operation $*$ on $T(X, Y)$ by

$$\alpha * \beta = \alpha\theta\beta \text{ for all } \alpha, \beta \in T(X, Y).$$

Then $(T(X, Y), *)$ is a semigroup which we denote by $(T(X, Y), \theta)$. This semigroup is called a *sandwich semigroup* with the sandwich function θ. Sandwich semigroups have been studied by many authors, such as Magill and Subbiah [9,10], Magill et al. [11], Symons [12] and Hickey [13].

If $X = Y$, then the sandwich semigroup $(T(X,Y),\theta)$ is written as $(T(X),\theta)$ and called the *variant semigroup* of $T(X)$ with the sandwich function $\theta \in T(X)$.

Let E be an equivalence relation on a set X, A, B be subsets of X and α be a mapping from A into B. α is said to be *E-preserving* if for any $x, y \in A$, $(x, y) \in E$ implies that $(x\alpha, y\alpha) \in E$. If α satisfies the condition that $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$, then α is called *E^*-preserving*. Denote by X/E the set of all equivalence classes determined by E. Let

$$T_E(X) = \{\alpha \in T(X) \mid \alpha \text{ is } E\text{-preserving}\}.$$

Then $T_E(X)$ is a subsemigroup of $T(X)$ and its regular elements are investigated in [14]. The regular elements of the variant semigroup $(T_E(X),\theta)$ of $T_E(X)$ where $\theta \in T_E(X)$ are characterized in [15]. Denote

$$T_{E^*}(X) = \{\alpha \in T(X) \mid \alpha \text{ is } E^*\text{-preserving}\}.$$

Then $T_{E^*}(X)$ is a subsemigroup of $T_E(X)$. It is obvious that if $E = X \times X$, then $T_{E^*}(X) = T_E(X) = T(X)$. The characterizations of the regular elements in $T_{E^*}(X)$ and the regularity of $T_{E^*}(X)$ are given in [16] as follows:

Theorem 1.1. [16] *Let $\alpha \in T_{E^*}(X)$. Then α is regular if and only if $A \cap \operatorname{ran}\alpha \neq \emptyset$ for every $A \in X/E$.*

Theorem 1.2. [16] *$T_{E^*}(X)$ is regular if and only if $|X/E|$ is finite.*

For a fixed element $\theta \in T_{E^*}(X)$, the variant semigroup of $T_{E^*}(X)$ with the sandwich function θ will be denoted by $(T_{E^*}(X),\theta)$. The purpose of this paper is to characterize the regular elements of the variant semigroup $(T_{E^*}(X),\theta)$. This characterization is then applied to determine when the variant semigroup $(T_{E^*}(X),\theta)$ and the semigroup $T_{E^*}(X)$ have the same set of regular elements. In addition, we give a necessary and sufficient condition for the semigroup $(T_{E^*}(X),\theta)$ to be regular.

2 Main Results

We first give a characterization of the regular elements of the variant semigroup $(T_{E^*}(X),\theta)$.

Theorem 2.1. *For $\theta, \alpha \in T_{E^*}(X)$, α is regular in $(T_{E^*}(X),\theta)$ if and only if the following conditions hold:*

(i) *$A \cap \operatorname{ran}(\alpha\theta) \neq \emptyset$ for any $A \in X/E$,*

(ii) *$\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$,*

(iii) *$\theta_{|\operatorname{ran}\alpha}$ is injective.*

Proof. Assume that α is regular in $(T_{E^*}(X),\theta)$. Then $\alpha=\alpha\theta\beta\theta\alpha$ for some $\beta\in(T_{E^*}(X),\theta)$. It follows that $\alpha\theta,\beta\theta\in T_{E^*}(X)$ and $\alpha\theta=(\alpha\theta)(\beta\theta)(\alpha\theta)$. Thus $\alpha\theta$ is regular in $T_{E^*}(X)$. By Theorem 1.1, $A\cap\operatorname{ran}(\alpha\theta)\neq\emptyset$ for any $A\in X/E$. This verifies (i). Now, since

$$\operatorname{ran}\alpha=X\alpha=X\alpha\theta\beta\theta\alpha\subseteq X\theta\alpha\subseteq X\alpha=\operatorname{ran}\alpha,$$

we get $\operatorname{ran}\alpha=\operatorname{ran}(\theta\alpha)$. Hence (ii) holds. Finally, since $\alpha=\alpha\theta\beta\theta\alpha$, we get

$$z=z\theta\beta\theta\alpha\ \text{ for all } z\in\operatorname{ran}\alpha. \tag{2.1}$$

If $y_1,y_2\in\operatorname{ran}\alpha$ are such that $y_1\theta=y_2\theta$, then from (2.1), we obtain

$$y_1=y_1\theta\beta\theta\alpha=y_2\theta\beta\theta\alpha=y_2.$$

This shows that $\theta_{|\operatorname{ran}\alpha}$ is injective.

Conversely, assume that (i), (ii) and (iii) hold. Since $A\cap\operatorname{ran}(\alpha\theta)\neq\emptyset$ for any $A\in X/E$, by Theorem 1.1, $\alpha\theta$ is regular in $T_{E^*}(X)$. Let $\beta\in T_{E^*}(X)$ be such that $\alpha\theta=(\alpha\theta)\beta(\alpha\theta)$. Then $\alpha(\theta_{|\operatorname{ran}\alpha})=\alpha\theta\beta\alpha(\theta_{|\operatorname{ran}\alpha})$. Since $\theta_{|\operatorname{ran}\alpha}$ is injective, we obtain $\alpha=\alpha\theta\beta\alpha$. Then

$$\operatorname{ran}\alpha=\operatorname{ran}(\alpha\theta\beta\alpha)\subseteq\operatorname{ran}(\beta\alpha)\subseteq\operatorname{ran}\alpha,$$

so $\operatorname{ran}\alpha=\operatorname{ran}(\beta\alpha)$ which implies that $\operatorname{ran}(\beta\alpha)=\operatorname{ran}(\theta\alpha)$.
For each $y\in\operatorname{ran}(\beta\alpha)=\operatorname{ran}(\theta\alpha)$, choose an element $d_y\in y(\theta\alpha)^{-1}$. Then

$$d_y(\theta\alpha)=y\quad\text{for all } y\in\operatorname{ran}(\beta\alpha). \tag{2.2}$$

Note that $X=\dot{\bigcup}_{y\in\operatorname{ran}(\beta\alpha)} y(\beta\alpha)^{-1}$. Define $\beta':X\to X$ by

$$\beta'=\begin{pmatrix} y(\beta\alpha)^{-1}\\ d_y\end{pmatrix}_{y\in\operatorname{ran}(\beta\alpha)}.$$

To show that $\beta\in T_{E^*}(X)$, let $x_1,x_2\in X$. Then $x_1\in(x_1\beta\alpha)(\beta\alpha)^{-1}$ and $x_2\in(x_2\beta\alpha)(\beta\alpha)^{-1}$, so

$$x_1\beta'=d_{x_1\beta\alpha}\ \text{ and }\ x_2\beta'=d_{x_2\beta\alpha}.$$

By (2.2),

$$(d_{x_1\beta\alpha})(\theta\alpha)=x_1\beta\alpha\ \text{ and }\ (d_{x_2\beta\alpha})(\theta\alpha)=x_2\beta\alpha.$$

Since $\beta\alpha,\theta\alpha\in T_{E^*}(X)$, we obtain that

$$\begin{aligned}(x_1,x_2)\in E &\Leftrightarrow (x_1\beta\alpha,x_2\beta\alpha)\in E\\ &\Leftrightarrow \big((d_{x_1\beta\alpha})(\theta\alpha),(d_{x_2\beta\alpha})(\theta\alpha)\big)\in E\\ &\Leftrightarrow (d_{x_1\beta\alpha},d_{x_2\beta\alpha})\in E\\ &\Leftrightarrow (x_1\beta',x_2\beta')\in E.\end{aligned}$$

In order to show that $\alpha = \alpha\theta\beta'\theta\alpha$, we must verify that $x\alpha = x\alpha\theta\beta'\theta\alpha$ for all $x \in X$. Let $x \in X$. Then $x\alpha \in \operatorname{ran}\alpha = \operatorname{ran}(\beta\alpha)$ and $x\alpha = (x\alpha\theta)\beta\alpha$, that is, $x\alpha\theta \in (x\alpha)(\beta\alpha)^{-1}$. Thus $(x\alpha\theta)\beta' = d_{x\alpha}$ and by (2.2), $d_{x\alpha}(\theta\alpha) = x\alpha$. Hence $x\alpha\theta\beta'\theta\alpha = d_{x\alpha}(\theta\alpha) = x\alpha$. Consequently, $\alpha = \alpha\theta\beta'\theta\alpha$. The proof is thereby complete. □

Remark 2.2. It is obvious that all regular elements in $(T_{E^*}(X), \theta)$ are also regular in $T_{E^*}(X)$. However, the converse is not generally true. For example, let $X = \{1, 2, 3\}$ and $E = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$. Then $X/E = \{\{1, 3\}, \{2\}\}$. Define $\alpha, \theta \in T(X)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}.$$

Thus $\alpha, \theta \in T_{E^*}(X)$ and $\operatorname{ran}\alpha = X$. Since $A \cap \operatorname{ran}\alpha \neq \emptyset$ for any $A \in X/E$, by Theorem 1.1, α is regular in $T_{E^*}(X)$. However, by Theorem 2.1, α is not regular in $(T_{E^*}(X), \theta)$ since $\theta_{|\operatorname{ran}\alpha} = \theta$ is not injective.

We have seen that $\operatorname{Reg}((T_{E^*}(X), \theta)) \subseteq \operatorname{Reg}(T_{E^*}(X))$. The following theorem tells us when $\operatorname{Reg}((T_{E^*}(X), \theta)) = \operatorname{Reg}(T_{E^*}(X))$.

Theorem 2.3. *Let $\theta \in T_{E^*}(X)$. Then $\operatorname{Reg}((T_{E^*}(X), \theta)) = \operatorname{Reg}(T_{E^*}(X))$ if and only if θ is a bijection.*

Proof. Assume that $\operatorname{Reg}((T_{E^*}(X), \theta)) = \operatorname{Reg}(T_{E^*}(X))$. Since 1_X is regular in $T_{E^*}(X)$, it follows that 1_X is regular in $(T_{E^*}(X), \theta)$. By Theorem 2.1, we obtain $\operatorname{ran}(\theta) = \operatorname{ran}(\theta 1_X) = \operatorname{ran}(1_X) = X$ and that $\theta = \theta_{|\operatorname{ran}(1_X)}$ is injective. That is, θ is a bijection.

For the converse, suppose that θ is a bijection. Then, so is θ^{-1}. Since $\theta \in T_{E^*}(X)$, we have $\theta^{-1} \in T_{E^*}(X)$. Since $\operatorname{Reg}((T_{E^*}(X), \theta)) \subseteq \operatorname{Reg}(T_{E^*}(X))$, it remains to show that $\operatorname{Reg}(T_{E^*}(X)) \subseteq \operatorname{Reg}((T_{E^*}(X), \theta))$. Let α be a regular element in $T_{E^*}(X)$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{E^*}(X)$. Consequently, $\theta^{-1}\beta\theta^{-1} \in T_{E^*}(X)$ and $\alpha = \alpha\theta(\theta^{-1}\beta\theta^{-1})\theta\alpha$ which implies that $\alpha \in \operatorname{Reg}((T_{E^*}(X), \theta))$. □

In what follows we investigate when the semigroup $(T_{E^*}(X), \theta)$ is regular.

Theorem 2.4. *For $\theta \in T_{E^*}(X)$, $(T_{E^*}(X), \theta)$ is a regular semigroup if and only if*

(i) *θ is a bijection and*

(ii) *$|X/E|$ is finite.*

Proof. Assume that $(T_{E^*}(X), \theta)$ is regular. Then $\operatorname{Reg}((T_{E^*}(X), \theta)) = T_{E^*}(X)$. Since $\operatorname{Reg}((T_{E^*}(X), \theta)) \subseteq \operatorname{Reg}(T_{E^*}(X))$, it follows that

$$\operatorname{Reg}((T_{E^*}(X), \theta)) = \operatorname{Reg}(T_{E^*}(X))$$

and $T_{E^*}(X)$ is regular. By Theorem 2.3 and Theorem 1.2, we obtain that θ is a bijection and $|X/E|$ is finite, respectively.

Conversely, assume that θ is a bijection and $|X/E|$ is finite. Since θ is a bijection, by Theorem 2.3, $\mathrm{Reg}((T_{E^*}(X),\theta)) = \mathrm{Reg}(T_{E^*}(X))$. Also, since $|X/E|$ is finite, we get from Theorem 1.2 that $T_{E^*}(X)$ is regular. Then $\mathrm{Reg}((T_{E^*}(X),\theta)) = \mathrm{Reg}(T_{E^*}(X)) = T_{E^*}(X)$ which implies that $(T_{E^*}(X),\theta)$ is regular. □

Remark 2.5. The condition (i) of Theorem 2.4 cannot be removed. For example, let $\theta = \begin{pmatrix} X \\ a \end{pmatrix}$ where $a \in X$ and $E = X \times X$. It is clear that $\left\{ \begin{pmatrix} X \\ b \end{pmatrix} \mid b \in X \right\} \subseteq \mathrm{Reg}((T_{E^*}(X),\theta))$. If $\alpha \in \mathrm{Reg}((T_{E^*}(X),\theta))$, then $\alpha = \alpha\theta\beta\theta\alpha$ for some $\beta \in T_{E^*}(X)$, so $\alpha = \begin{pmatrix} X \\ a\alpha \end{pmatrix}$. Thus $\mathrm{Reg}((T_{E^*}(X),\theta)) = \left\{ \begin{pmatrix} X \\ b \end{pmatrix} \mid b \in X \right\}$. Hence $(T_{E^*}(X),\theta)$ is not a regular semigroup.

Acknowledgement(s) : I would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] V.H. Fernandes, J. Sanwong, On the ranks of semigroups of transformations on a finite set with restricted range, Algebra Colloq. 21 (3) (2014) 497-510.

[2] K. Sangkhanan, J. Sanwong, Green's relations and partial orders on semigroups of partial linear transformations with restricted range, Thai J. Math. 12 (1) (2014) 81-93.

[3] J.S.V. Symons, Some results concerning a transformation semigroup, J. Aust. Math. Soc. 19 (1975) 413-425.

[4] K.D. Magill Jr., Subsemigroups of $S(X)$, Math. Japon. 11 (1966) 109-115.

[5] S. Nenthein, P. Youngkhong, Y. Kemprasit, Regular elements of some transformation semigroups, Pure. Math. Appl. 16 (2005) 307-314.

[6] J. Sanwong, The regular part of a semigroup of transformations with restricted range, Semigroup Forum 83 (2011) 134-146.

[7] E. Laysirikul, Semigroups of full transformations with the restriction on the fixed set is bijective, Thai J. Math. 14 (2) (2016) 497-503.

[8] A. Anantayasethi, J. Koppitz, On a semigroup of sets of transformations with restricted range, Thai J. Math. 14 (3) (2016) 667-676.

[9] K.D. Magill Jr., S. Subbiah, Green's relations for regular elements of sandwich semigroup, (I) general results, Proc. London Math. Soc. 3 (1975) 194-210.

[10] K.D. Magill Jr., S. Subbiah, Green's relations for regular elements of sandwich semigroup, (II) semigroup of continuous function, J. Austral. Math. Soc. 25 (1978) 45-65.

[11] K.D. Magill Jr., P.R. Misra, U.B. Tewari, Symons' d-congruence on sandwich semigroups, Czec. Math. J. 33 (1983) 221-236.

[12] J.S.V. Symons, On a generalization of the transformation semigroup, J. Aust. Math. Soc. 19 (1975) 47-61.

[13] J.B. Hickey, Semigroup under a sandwich operation, Proc. Edinburgh Math. Soc. 26 (1983) 371-382.

[14] H. Pei, Regularity and Green's relations for semigroups of transformations that preserve an equivalence, Commun. Algebra 33 (2005) 109-118.

[15] H. Pie, L. Sun, H. Zhai, Green's relations for the variants of transformation semigroups preserving an equivalence relation, Commun. Algebra 35 (2007) 1971-1986.

[16] L.Z. Deng, J.W. Zeng, B. Xu, Green's relations and regularity for semigroups of transformations that preserve double direction equivalence, Semigroup Forum 80 (2010) 416-425.

(Received 23 November 2015)
(Accepted 3 March 2018)