On Optimization via ϵ-Generalized Weak Subdifferentials

P. Cheraghi[†], A.P. Farajzadeh[†,1] and S. Suantai[‡]

[†]Department of Mathematics, Razi University, Kermanshah 67149, Iran
e-mail : a.sh58@yahoo.com (P. Cheraghi)
farajzadehali@gmail.com (A.P. Farajzadeh)

[‡]Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
e-mail : suthep.s@cmu.ac.th

Abstract : In this paper, we study ϵ-generalized weak subdifferential for functions defined on a real topological vector space. Some necessary and sufficient conditions for having nonempty ϵ-generalized weak subdifferential of a function are presented. The positively homogenous of the ϵ-generalized weak subdifferential operator is proved. A necessary and sufficient conditions for achieving a global minimum of a ϵ-generalized weak subdifferentiable function is stated. A link between sudifferential and Fréchet differential with ϵ-generalized weak subdifferential is established. Finally a result about the equality of the fuzzy sum rule inclusion is investigated.

Keywords : subgradient; weak subgradient; ϵ-generalized subdifferential; ϵ-generalized weak subdifferential; Fréchet differentiable function; locally Lipchitz function.
2010 Mathematics Subject Classification : 47H09; 47H10.

1   Introduction

The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [1]. It uses explicitly defined

---

[1]Corresponding author.

supporting conic surfaces instead of supporting hyperplanes. Recall that, a convex set has a supporting hyperplane at each boundary point. This leads to one of the central notions in convex analysis, that of a subgradient of a possible nonsmooth even extended real valued function [2]. The main reason of difficulties arising when passing from the convex analysis to the nonconvex one is that, the nonconvex cases may arise in many different forms and each case may require a special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradients plays an important role in deriving of optimality conditions and duality theorems. The first canonical generalized gradient introduced by Clarke [3,4]. He applied this generalized gradient systematically to nonsmooth problems in a variety of problems. Since a nonconvex set has no supporting hyperline at each boundary point, the notion of subgradient have been generalized by most researchers on optimality conditions for nonconvex problems [5,6]. By using the notion of subgradients, a collection of zero duality gap conditions for a wise class of nonconvex optimization problems was derived [1]. In this study we give some important properties of the ϵ-generalized weak subdifferentials. By using the definition and properties of the weak subdifferential which are described in [1,2,5,7], we prove some theorems connecting ϵ-generalized weak subdifferential in nonsmooth and nonconvex analysis. It is also obtained sufficient optimality condition by using the ϵ-generalized weak subdifferential. The paper is organized as follows. The definition and preliminaries of ϵ-generalized weak subdifferential is provided in the following sections. In Section 4, we prove some theorems connecting operations on ϵ-generalized weak subdifferential in nonsmooth and nonconvex analysis. sequently, we state sufficient conditions that with them a function obtains a global minimum.

2 Preliminaries

In this section, we give some basic definitions and results. Let Y be a real vector space and C be a convex cone in Y, then the binary relation

$$\leq_C := \{(x,y) \in Y \times Y; y - x \in C\}$$

is a partial ordering on Y. If, in addition, C is pointed, i.e, $C \cap -C = 0$, then $\leq_C$ is antisymmetric. we denote $\leq_C$ as a partial ordering induced by a convex cone C. It is clear that $\leq_C$ is a partial order on Y, and so $(Y, \leq_C)$ is a partially ordered vector space.

Definition 2.1. [8] Let X and Y be real vector spaces and let C be a convex cone in Y. A function $||| \, . \, |||: X \longrightarrow C$ is called *vectorial norm* on X, if for all $x, z \in X$ and all $\lambda \in \mathbb{R}$, then the following conditions are satisfied:

(i) $||| \, x \, ||| = 0_Y \Longleftrightarrow x = 0_X$,

(ii) $||| \, \lambda x \, ||| = | \lambda | \, ||| \, x \, |||$,

(iii) $||| \, x + z \, ||| \leq_C \, ||| \, x \, ||| + \, ||| \, z \, |||$.

If, in addition, $Y = \mathbb{R}, C = \mathbb{R}_+$, the set of nonnegative real numbers, the function $||| \, . \, |||$ is called a *norm* on X and it is denoted by $\| \, . \, \|$.

Let $(Y, \leq_C)$ be an ordered locally convex topological vector space. The topology that is induced by vertical norm on X is the topology induced by the neighborhood base $\{X(a, U) : U \in B(0)\}$, where

$$X(a, U) = \{x \in X : ||| \, x - a \, ||| \in U\},$$

with B(0) is a neighborhood base of the origin in Y and a running over X.

Definition 2.2. [8] Let $(X, \| \, . \, \|_X)$ and $(Y, \| \, . \, \|_Y)$ be real normed spaces and S be a nonempty open subset of X. Let $f : S \longrightarrow Y$ be a given function and $\bar{x} \in S$. If there is a continuous linear function $\acute{f}(\bar{x}) : X \longrightarrow Y$ with the property

$$\lim_{\|h\|_X \longrightarrow 0} \frac{\| \, f(\bar{x} + h) - f(\bar{x}) - \acute{f}(\bar{x})(h) \, \|_Y}{\| \, h \, \|_X} = 0,$$

then $\acute{f}(\bar{x}) : X \longrightarrow Y$ is called the *Fréchet derivative of f at $\bar{x}$* and f is called the *Fréchet differentiable at $\bar{x}$*.

The next definition is a generalization of the usual convexity for the real functions.

Definition 2.3. [8] Let X and Y be real vector spaces and C be a convex cone in Y. Let S be a nonempty convex subset of X. A function $f : S \longrightarrow Y$ is called *C-convex* if for all $x, y \in S$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) \in C. \tag{2.1}$$

If $\leq_C$ is the partial ordering in Y induced by C, then the condition (2.1) can also be written as

$$f(\lambda x + (1 - \lambda)y) \leq_C \lambda f(x) + (1 - \lambda)f(y).$$

A function $f : S \longrightarrow Y$ is called *C-concave*, if $-f$ is *C*-convex.

Theorem 2.4. [8] *Let $(X, \| \, . \, \|_X)$ and $(Y, \| \, . \, \|_Y)$ be real normed spaces and S be a nonempty open convex subset of X and C be a closed convex cone in Y and let $f : S \longrightarrow Y$ be a Fréchet differentiable at every point $x \in S$. Then f is C-convex if and only if*

$$\acute{f}(y)(x - y) \leq_C f(x) - f(y) \qquad (\forall x, y \in S).$$

Definition 2.5. [9] Let X and Y be real topological vector spaces. Let S be an open subset of X and $f : S \longrightarrow Y$ be a given function. If for $\bar{x} \in S$ and $u \in X$ the limit

$$\acute{f}(\bar{x}, u) = \lim_{t \to o^+} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}$$

exists, then $\acute{f}(\bar{x}, u)$ is called the *directional derivative of f at $\bar{x}$ in the direction u*. If this limit exists for all $u \in X$, then f is called *directionally differentiable at $\bar{x}$*.

Definition 2.6. [9] Let X be a real topological vector space and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$. Let S be an open subset of X and $f : S \longrightarrow Y$ be a given function and $\epsilon \in \mathbb{R}_+$. If for $\bar{x} \in S$ and $u \in X$ the infimum

$$\acute{f}_\epsilon(\bar{x}, u) = \inf_{t>0} \frac{f(\bar{x} + tu) - f(\bar{x}) + \epsilon 1}{t}$$

exists in Y, then $\acute{f}_\epsilon(\bar{x}, u)$ is called the ϵ-*directional derivative of* f *at* $\bar{x}$ *in the direction u.* If this infimum exists in Y for all $u \in X$, then f is called ϵ-*directionally differentiable at* $\bar{x}$. Note that $1 \in intC$.

3 ϵ-Generalized Weak Subgradient

In this section, the definition of subgradient, ϵ-subgradient, ϵ-weak subgradient and ϵ-generalized weak subgradient are given. Also in the sequel the relation between them is considered.

Definition 3.1. [8] Let X and Y be real topological vector spaces and C be a convex cone in Y. Assume that $f : X \longrightarrow Y$ is a given function and $\bar{x} \in X$. The set

$$\partial f(\bar{x}) = \{T \in B(X, Y) : f(x) - f(\bar{x}) - T(x - \bar{x}) \in C \quad \forall x \in X\},$$

where $B(X, Y)$ denotes the vector space of all continuous linear functions from X to Y is called the *subdifferential of* f *at* $\bar{x}$. Every $T \in B(X, Y)$ is called a *subgradient of* f *at* $\bar{x}$. Also if $\partial f(\bar{x}) \neq \emptyset$, then f is called *subdifferentiable at* $\bar{x}$.

Definition 3.2. [9] Let X be a real topological vector space and C be a pointed closed convex cone in Y that $(Y, \leq_C)$ be a real ordered topological vector space. Assume that $intC \neq \emptyset$, $f : X \longrightarrow Y$ is a given function and $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Then a point $T \in B(X, Y)$ is called ϵ-*subgradient of* f *at* $\bar{x}$ if

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \epsilon 1 \in C \quad (\forall x \in X)$$

where $1 \in intC$. The set of all ϵ-subgradients of f at $\bar{x}$ is called the ϵ-subdifferntial of f at $\bar{x}$ and denoted by

$$\partial_\epsilon f(\bar{x}) = \{T \in B(X, Y) : T \text{ is an } \epsilon\text{-subgradient of } f \text{ at } \bar{x}\},$$

where $B(X, Y)$ is the vector space of all continuous linear functions from X to Y. Also if $\partial_\epsilon f(\bar{x}) \neq \emptyset$, then f is called ϵ-*subdifferentiable at* $\bar{x}$.

Remark 3.3. *If* $f : X \longrightarrow Y$ *is subdifferentiable at* $\bar{x}$, *then there exists* $T \in B(X, Y)$, *such that*

$$f(x) - f(\bar{x}) - T(x - \bar{x}) \in C \quad (\forall x \in X),$$

since $1 \in intC$ *and* $\epsilon \in \mathbb{R}_+$, *then we have* $\epsilon 1 \in C$ *and therefore*

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \epsilon 1 \in C \quad (\forall x \in X),$$

this means that f *is* ϵ-*subdifferentiable at* $\bar{x}$ *for all* $\epsilon \in \mathbb{R}_+$.

The following example shows that the converse may failed.

Example 3.4. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, and let

$$f(x) = \begin{cases} \sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\partial f(0) = \emptyset$ and $\partial_2 f(0) = \{0\}$.

Definition 3.5. [9] Let X be a real normed space, $f : X \longrightarrow \bar{\mathbb{R}}$ be a proper function, $\bar{x} \in X$ be such that $f(\bar{x}) \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_+$. Then $(x^*, c) \in X^* \times \mathbb{R}_+$ is called an ϵ-*weak subgradient of f at $\bar{x}$* if

$$f(x) - f(\bar{x}) + \epsilon \ge \langle x^*, x - \bar{x} \rangle - c \| x - \bar{x} \| \quad (\forall\ x \in X).$$

The set of all ϵ-weak subgradients of f at $\bar{x}$ is called the ϵ-*weak subdifferntial of f at $\bar{x}$* and denoted by

$$\partial_\epsilon^w f(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}_+ : (x^*, c)\}$$

is an ϵ-weak subgradient of f at $\bar{x}$. Also if $\partial_\epsilon^w f(\bar{x}) \ne \emptyset$, then f is called ϵ-*weak subdifferentiable at $\bar{x}$.*

Definition 3.6. [9] Let X be a real topological vector space and C be a pointed closed convex cone in Y that $(Y, \le_C)$ be a real ordered topological vector space with $intC \ne \emptyset$. Let $f : X \longrightarrow Y$ be a function and $||| . |||: X \longrightarrow C$ be a vectorial norm on X and, let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$ be arbitrary. A point $(T, c) \in B(X, Y) \times \mathbb{R}_+$ is called an ϵ-*generalized weak subgradient of f at $\bar{x}$* if

$$f(x) - f(\bar{x}) - T(x - \bar{x}) - \{c\} |||\ x - \bar{x}\ ||| + \epsilon 1 \in C \quad (\forall x \in X),$$

where $1 \in intC$. The set of all ϵ-generalized weak subgradients of f at $\bar{x}$ is called the ϵ-*generalized weak subdifferntial of f at $\bar{x}$* and denoted by

$$\partial_\epsilon^{gw} f(\bar{x}) = \{(T, c) \in B(X, Y) \times \mathbb{R}_+ : (T, c) \text{ is an } \epsilon\text{-weak subgradients of } f \text{ at } \bar{x}\}.$$

Also if $\partial_\epsilon^{gw} f(\bar{x}) \ne \emptyset$, then f is called ϵ-*generalized weak subdifferentiable at $\bar{x}$.*

In the above definition, if $\epsilon = 0$, then ϵ-generalized weak subdifferntial is called *generalized weak subdifferntial.*

Remark 3.7. *If f is generalized weak subdifferentiable at $\bar{x}$, then f is also ϵ-generalized weak subdifferentiable at $\bar{x}$, for all $\epsilon \in \mathbb{R}^+$, that is, if $(T, c) \in \partial^w f(\bar{x})$, then by definition of ϵ-generalized weak subgradient $(T, c) \in \partial_\epsilon^{gw} f(\bar{x})$, for all $\epsilon \in \mathbb{R}^+$. But the converse may failed as the following example.*

Example 3.8. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, and let f be a function defined by Example 3.4. Then $\partial^w f(0) = \emptyset$ and $\partial_2^{gw} f(0) = \{(0, c) : c \ge 0\}$.

Remark 3.9. *If f is ϵ-subdifferentiable at $\bar{x}$, then f is also ϵ-generalized weak subdifferentiable at $\bar{x}$, that is, if $T \in \partial_\epsilon f(\bar{x})$, then by definition of ϵ-generalized weak subgradient $(T, c) \in \partial_\epsilon{}^{gw} f(\bar{x})$, for every $c \geq 0$. But the converse may failed as the following example.*

Example 3.10. Let $X = Y = \mathbb{R}, C = \mathbb{R}_+$, and let $f(x) = - \mid x \mid, \epsilon \in \mathbb{R}_+$. Then it follows from definition of ϵ-generalized weak subgradient that

$$(a, c) \in \partial_\epsilon{}^{gw} f(0) \iff (a, c) \in \mathbb{R} \times \mathbb{R}_+ \text{ and } - \mid x \mid + \epsilon \geq ax - c \mid x \mid, \text{ for all } x \in X.$$

Hence the ϵ-generalized weak subdifferential can explicity be written as

$$\partial_\epsilon{}^{gw} f(0) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : \mid a \mid \leq c - 1\}.$$

On the other hand, it follows from the definition of the ϵ-subdifferential that $\partial_\epsilon f(0) = \emptyset$.

Remark 3.11. *From definition of ϵ-generalized weak subgradient that if $\partial_\epsilon{}^{gw} f(\bar{x})$ is a nonempty set, then it has uncountable members. Because if $(T, \bar{c}) \in \partial_\epsilon{}^{gw} f(\bar{x})$, then we have*

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + \bar{c} \mid\mid\mid x - \bar{x} \mid\mid\mid + \epsilon 1 \in C \quad \forall\, x \in X,$$

where $1 \in intC$. Since $\acute{c} \mid\mid\mid x - \bar{x} \mid\mid\mid \in C$, for all $\acute{c} \in \mathbb{R}^+$, C is a convex cone, it follows that

$$f(x) - f(\bar{x}) - T(x - \bar{x}) - (\bar{c} + \acute{c}) \mid\mid\mid x - \bar{x} \mid\mid\mid + \epsilon 1 \in C \quad \forall\, x \in X,$$

that is, $(T, \bar{c} + \acute{c}) \in \partial_\epsilon{}^{gw} f(\bar{x})$, for all $\acute{c} \in \mathbb{R}_+$. This completes proof of the assertion. Similarly, if $(T, \bar{c}) \in \partial_\epsilon{}^{gw} f(\bar{x})$, then $(T, \bar{c}) \in \partial_{\acute{\epsilon}}{}^{gw} f(\bar{x})$, for all $\acute{\epsilon} \geq \epsilon$. This means that

$$\partial_\epsilon{}^{gw} f(\bar{x}) \neq \emptyset \implies \partial_{\acute{\epsilon}}{}^{gw} f(\bar{x}) \neq \emptyset \qquad \forall \acute{\epsilon} \geq \epsilon.$$

Remark 3.12. *We have the following notices:*

$$(T, c) \in \partial^w f(\bar{x}) \iff T \in \partial(f + c \mid\mid\mid . - \bar{x}) \mid\mid\mid)(\bar{x}),$$

$$(T, c) \in \partial_\epsilon{}^{gw} f(\bar{x}) \iff T \in \partial_\epsilon(f + c \mid\mid\mid . - \bar{x} \mid\mid\mid)(\bar{x}).$$

Example 3.13. Let X be a real topological vector space and C be a pointed closed convex cone in Y that $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $\mid\mid\mid . \mid\mid\mid: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. If vectorial norm is $\mid\mid\mid . \mid\mid\mid$ is ϵ-generalized weak subdifferentiable, then

$$\left\{(T, c) \in B(X, Y) \times \mathbb{R}_+ : T(\bar{x}) - c \mid\mid\mid \bar{x} \mid\mid\mid = \mid\mid\mid \bar{x} \mid\mid\mid,\ T(x) - c \mid\mid\mid x \mid\mid\mid \leq_C \mid\mid\mid x \mid\mid\mid, \forall x \in X\right\}$$

$$\subset \partial_\epsilon{}^{gw} \mid\mid\mid \bar{x} \mid\mid\mid.$$

Remark 3.14. *It follows from Definition 3.5 that the pair $(T, c) \in B(X, Y) \times \mathbb{R}_+$ is a ϵ-generalized weak subdifferential of f at $\bar{x} \in X$, iff the continuous (super linear) C-concave function*

$$g(x) = f(\bar{x}) + T(x - \bar{x}) - \bar{c} \, ||| \, x - \bar{x} \, |||$$

satisfy the following conditions:

(i) $g(x) \leq_C f(x) + \epsilon 1 \quad (\forall x \in X)$,

(ii) $g(\bar{x}) = f(\bar{x})$.

Theorem 3.15. [9] *Let the ϵ-generalized weak subdifferential of $f : X \longrightarrow Y$ at $\bar{x}$ is nonempty set. Then the set $\partial_\epsilon{}^{gw} f(\bar{x})$ is closed and convex.*

In the sequel, we need the following definition that generalizes the notion of lower Lipschitz functions.

Definition 3.16. [9] Let X be a real topological vector space and C be a pointed closed convex cone in Y that $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. A function $f : X \longrightarrow Y$ is called ϵ-*generalized lower locally Lipschitz at $\bar{x} \in X$* if there exist a nonnegative real number L (Lipschitz constant) and a neighborhood N($\bar{x}$) of $\bar{x}$ such that

$$-L \, ||| \, x - \bar{x} \, ||| \, \leq_C f(x) - f(\bar{x}) + \epsilon 1 \quad (\forall \, x \in N(\bar{x})).$$

If the above inequality holds for all $x \in X$, then f is called ϵ-*generalized lower Lipschitz at $\bar{x}$ with the Lipschitz constant L.*

If $\epsilon = 0$, then the definition of ϵ-generalized lower Lipschitz function and the definition of generalized lower Lipschitz function concide with each other.

4 Main results

In this section by recall the results in [9], we present some result in ϵ-generalized weak subdifferential. The first result states the link between ϵ-generalized lower Lipschitz functions and ϵ-generalized lower locally Lipschitz functions.

Proposition 4.1. *Let X and Y be real topological vector spaces and C be a pointed closed convex cone in Y, that induces a totally order on Y, $intC \neq \emptyset$ and $||| \, . \, |||$: $X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function. If f is ϵ-generalized lower locally Lipschitz at $\bar{x}$ and there exists $p \geq 0$ and $q \in Y$ such that*

$$q - p \, ||| \, x \, ||| \, \leq_C f(x) \qquad (\forall x \in X),$$

then f is ϵ-generalized lower Lipschitz at $\bar{x}$.

Proof. Assume to the contrary that f is not ϵ-generalized lower Lipschitz at $\bar{x}$. Then for every $k \in \mathbb{N}$, there exists $x_k \in X$ such that

$$-k \; ||| \; x_k - \bar{x} \; ||| \; \nleq_C f(x_k) - f(\bar{x}) + \epsilon 1$$

i.e.

$$f(x_k) - f(\bar{x}) + \epsilon 1 + k \; ||| \; x_k - \bar{x} \; ||| \; \notin C,$$

therefore,

$$f(x_k) - f(\bar{x}) + k \; ||| \; x_k - \bar{x} \; ||| \; \notin C.$$

Since $(Y, \leq_C)$ is a real totally ordered topological vector space, we have

$$- f(x_k) + f(\bar{x}) - k \; ||| \; x_k - \bar{x} \; ||| \; \in C. \tag{4.1}$$

As $q - p \; ||| \; x \; ||| \; \leq_C f(x)$ for all $x \in X$ and from triangle inequality, we have

$$-p \; ||| \; x_k - \bar{x} \; ||| \; -p \; ||| \; \bar{x} \; ||| \; +q \; \leq_C \; - p \; ||| \; x_k \; ||| \; +q \; \leq_C f(x_k),$$

i.e.,

$$f(x_k) + p \; ||| \; x_k - \bar{x} \; ||| \; +p \; ||| \; \bar{x} \; ||| \; -q \in C. \tag{4.2}$$

From the inequalities (4.1), (4.2) we obtain

$$(p - k) \; ||| \; x_k - \bar{x} \; ||| \; +f(\bar{x}) + p \; ||| \; \bar{x} \; ||| \; -q \in C.$$

Therefore,

$$(k - p) \; ||| \; x_k - \bar{x} \; ||| \; \leq_C f(\bar{x}) + p \; ||| \; \bar{x} \; ||| \; -q.$$

As $k - p > 0$ for a sufficiently large k and C is a convex cone, then we have

$$||| \; x_k - \bar{x} \; ||| \; \leq_C \frac{f(\bar{x}) + p \; ||| \; \bar{x} \; ||| \; -q}{k - p}$$

as $k \longrightarrow \infty$, then we obtain $\frac{f(\bar{x}) + p |||\bar{x}|||-q}{k-p} \longrightarrow 0$. Hence, we have $x_k \longrightarrow x$ with respect to vectorial norm. Now since f is ϵ-generalized lower locally Lipschitz at $\bar{x}$, there exist a nonnegative real number L (Lipschitz constant) and a neighborhood N$(\bar{x})$ of $\bar{x}$ such that

$$-L \; ||| \; x - \bar{x} \; ||| \; \leq_C f(x) - f(\bar{x}) + \epsilon 1 \quad (\forall \; x \in N(\bar{x})).$$

As $x_k \longrightarrow x$, there exists $k_0 \in N$ such that $x_k \in N(\bar{x})$ for all $k \geq k_0$. Therefore,

$$- L \; ||| \; x_k - \bar{x} \; ||| \; \leq_C f(x_k) - f(\bar{x}) + \epsilon 1. \tag{4.3}$$

From the inequalities (4.1) and (4.3), we obtain

$$k \; ||| \; x_k - \bar{x} \; ||| \; \leq_C L \; ||| \; x_k - \bar{x} \; ||| \qquad (\forall k \geq k_0)$$

which is a contradiction. So f is ϵ-generalized lower Lipschitz at $\bar{x}$. $\qquad\square$

In the sequel, we recall the sufficient condition for ϵ-generalized weak subdifferentiability of f at $\bar{x}$ and then we present the interesting neccesary condition for ϵ-generalized weak differentiability of f at $\bar{x}$.

Theorem 4.2. [9] *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $int C \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function. If f is ϵ-generalized lower Lipschitz at $\bar{x}$, then f is ϵ-generalized weak subdifferentiable at $\bar{x}$.*

Proposition 4.3. *Let X be a real topological vector space and C be a closed convex cone in Y and $(Y, \leq_C)$ be a real ordered topological vector space with $int C \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function that $(T, c) \in \partial_\epsilon^{gw} f(\bar{x})$ and there exists L such that*

$$L \, ||| \, x - \bar{x} \, ||| \, \leq_c T(x - \bar{x}) \quad (\forall x \in X).$$

Then f is ϵ-generalized lower Lipschitz at $\bar{x}$.

Proof. Suppose that $(T, c) \in \partial_\epsilon^{gw} f(\bar{x})$, then we have

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + c \, ||| \, x - \bar{x} \, ||| + \epsilon 1 \, \in C \quad (\forall \, x \in X).$$

Therefore,

$$T(x - \bar{x}) - c \, ||| \, x - \bar{x} \, ||| \leq_c f(x) - f(\bar{x}) + \epsilon 1 \quad (\forall \, x \in X).$$

Since $L \, ||| \, x - \bar{x} \, ||| \leq_c T(x - \bar{x}) \quad (\forall x \in X)$, we get

$$(L - c) \, ||| \, x - \bar{x} \, ||| \leq_c f(x) - f(\bar{x}) + \epsilon 1 \quad (\forall \, x \in X).$$

This means that f is ϵ-generalized lower Lipschitz at $\bar{x}$. $\qquad \square$

Remark 4.4. *If $L \geq 0$, then $T = 0$, and result is obvious. But the case $L < 0$ is nontrivial and considerable.*

Theorem 4.5. [9] *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $int C \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function. If f is ϵ-generalized lower Lipschitz at $\bar{x}$, then there exists $p \geq 0$ and $q \in Y$ such that*

$$q - p \, ||| \, x \, ||| \leq_c f(x) + \epsilon 1 \quad (\forall \, x \in X),$$

where $1 \in int C$.

Proposition 4.6. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $int C \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and*

$\epsilon \in \mathbb{R}_+$. *Let* $f : X \longrightarrow Y$ *be a function that* $(T, c) \in \partial_\epsilon{}^{gw} f(\bar{x})$ *and there exists* L *such that*

$$L \parallel\mid x - \bar{x} \mid\parallel \ \leq_c T(x - \bar{x}) \qquad (\forall x \in X),$$

then there exists $p \geq 0$ *and* $q \in Y$ *such that*

$$q - p \parallel\mid x \mid\parallel \ \leq_c f(x) + \epsilon 1 \qquad (\forall\ x \in X),$$

where $1 \in intC$.

Proof. From $(T, c) \in \partial_\epsilon{}^{gw} f(\bar{x})$, then we have

$$f(x) - f(\bar{x}) - T(x - \bar{x}) + c \parallel\mid x - \bar{x} \mid\parallel + \epsilon 1 \in C \quad (\forall\ x \in X),$$

therefore

$$f(\bar{x}) + T(x - \bar{x}) - c \parallel\mid x - \bar{x} \mid\parallel \ \leq_c f(x) + \epsilon 1 \quad (\forall\ x \in X).$$

Since $L \parallel\mid x - \bar{x} \mid\parallel \ \leq_c T(x - \bar{x}) \quad (\forall\ x \in X)$, we obtain

$$(L - c) \parallel\mid x - \bar{x} \mid\parallel + f(\bar{x}) \ \leq_c f(x) + \epsilon 1 \quad (\forall\ x \in X).$$

For $L \geq c$, we have,

$$(L - c) \parallel\mid x \mid\parallel - (L - c) \parallel\mid \bar{x} \mid\parallel + f(\bar{x}) \ \leq_c f(x) + \epsilon 1 \quad (\forall\ x \in X),$$

and it is enough to set that $p = -(L - c), \quad q = f(\bar{x}) - (L - c) \parallel\mid \bar{x} \mid\parallel$. Similarly, for $L \leq c$, we have

$$(L - c) \parallel\mid x \mid\parallel + (L - c) \parallel\mid \bar{x} \mid\parallel + f(\bar{x}) \ \leq_c f(x) + \epsilon 1 \quad (\forall\ x \in X),$$

and it is enough to set that $p = -(L - c), \quad q = f(\bar{x}) + (L - c) \parallel\mid \bar{x} \mid\parallel$. $\square$

The following proposition states the sufficient conditions for ϵ-generalized weak subdifferentiability of f at $\bar{x}$. This results generalizes the results in [9].

Proposition 4.7. *Let* X *be a real topological vector space and* C *be a pointed closed convex cone in* Y, *that induces a totally order on* Y, *with* $intC \neq \emptyset$ *and* $\parallel\mid . \mid\parallel : X \longrightarrow C$ *be a vectorial norm on* X, *and let* $\bar{x} \in X$ *and* $\epsilon \in \mathbb{R}_+$. *Let* $f : X \longrightarrow Y$ *be a function. If f is* ϵ-*generalized lower locally Lipschitz at* $\bar{x}$ *and either one of the following two statements holds:*

(1) f *is bounded from below, i.e., there exists* $y \in Y$ *such that* $y \leq_c f(x)$ *for all* $x \in X$;

(2) *There is a point* $x_0 \in X$ *where* $T \in \partial_\epsilon f(x_0)$, *and there exists* L *such that*

$$L \parallel\mid x - \bar{x} \mid\parallel \ \leq_c T(x - \bar{x}) \ \forall x \in X,$$

then f is ϵ-*generalized weak subdifferential at* $\bar{x}$.

Proof. Let the statement (1) holds. Then there exists $y \in Y$ such that $y \leq_c f(x)$ for all $x \in X$. By choosing $p = 0, q = y$ and from last inequality we have

$$-p \mid\mid\mid x \mid\mid\mid +q \ \leq_c f(x) \qquad \forall x \in X.$$

From Proposition 4.1 and Theorem 4.2, we obtain f is ϵ-generalized weak subdifferential at $\bar{x}$.

If the statement (2) holds, then there exists $T \in B(X, Y)$ such that

$$f(x) - f(x_0) - T(x - x_0) + \epsilon 1 \in C.$$

By assumptions, we obtain,

$$f(x_0) + L \mid\mid\mid x \mid\mid -L \mid\mid\mid \bar{x} \mid\mid\mid \leq_C f(x_0) + L \mid\mid\mid x - x_0 \mid\mid\mid \leq_C f(x),$$

therefore,

$$f(x_0) + L \mid\mid\mid x \mid\mid -L \mid\mid\mid \bar{x} \mid\mid\mid \leq_C f(x).$$

If $L < 0$, by choosing $p = -L, q = f(x_0) - L \mid\mid\mid \bar{x} \mid\mid\mid$ similarly to the first part of the proof. From Proposition 4.1 and Theorem 4.2, we obtain f is ϵ-generalized weak subdifferential at $\bar{x}$. Now if $L \geq 0$, we can write

$$f(x_0) - L \mid\mid\mid x \mid\mid -L \mid\mid\mid \bar{x} \mid\mid\mid \leq_C f(x_0) + L \mid\mid\mid x \mid\mid\mid -L \mid\mid\mid \bar{x} \mid\mid\mid \leq_C f(x).$$

Therefore,

$$f(x_0) - L \mid\mid\mid x \mid\mid\mid -L \mid\mid\mid \bar{x} \mid\mid\mid \leq_C f(x),$$

by choosing $p = L, q = f(x_0) - L \mid\mid\mid \bar{x} \mid\mid\mid$, we obtain f is ϵ-generalized weak subdifferential at $\bar{x}$. $\square$

The following conclusion is a generalization of Proposition 3.5 in [10], that provides a link between Fréchet differentiability and ϵ-generalized weak subdifferentiability of a function.

Proposition 4.8. *Let X and Y be real normed spaces and C be a pointed closed convex cone in Y, $f : X \longrightarrow Y$ be a given function and $\bar{x} \in X$. Assume f is subdifferentiable and Fréchet differentiable at $\bar{x}$. Then*

$$\{(\acute{f}(\bar{x}), c) : c \geq 0\} \subset \partial_\epsilon^{gw} f(\bar{x}) \quad (\forall \epsilon \geq 0).$$

Proof. Since f is subdifferentiable at $\bar{x} \in X$, then from definition of subdifferentiable of f at $\bar{x}$, there is $T \in B(X, Y)$ such that

$$T(x - \bar{x}) \ \leq_c f(x) - f(\bar{x}) \qquad \forall \ x \in X.$$

By taking $x = \bar{x} + te$ such that $t \geq 0, e \in X, \parallel e \parallel = 1$, this leads to the inequality

$$T(te) \leq_c f(\bar{x} + te) - f(\bar{x}),$$

therefore,

$$T(e) \leq_c \frac{f(\bar{x} + te) - f(\bar{x})}{t}.$$

By Fréchet differentiability of f at $\bar{x}$ and by letting $t \longrightarrow 0^+$, we have

$$(T - \acute{f})(\bar{x})(e) \leq_c 0.$$

Note that C is pointed and closed convex cone, hence we have $T = \acute{f}(\bar{x})$ and $\acute{f}(\bar{x}) \in \partial f(\bar{x})$. By Remark 3.3 and Remark 3.9, f is ϵ-generalized weakly subdifferentiable at $\bar{x} \in X$ for all $\epsilon \geq 0$ and hence $\{(\acute{f}(\bar{x}), c); c \geq 0\} \subset \partial_\epsilon{}^{gw} f(\bar{x})$ for all $\epsilon \geq 0$. $\square$

The following example shows that subdifferentiability of f at $\bar{x}$ in above Proposition is essential.

Example 4.9. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, and let $f(x) = -x^2$, $\bar{x} = 0$. Then it is easy to verify that $\partial f(0) = \emptyset$, $\partial_0{}^{gw} f(0) = \emptyset$ and $\acute{f}(0) = 0$.

Remark 4.10. *If f is C-convex function and Fréchet differentiable at $\bar{x}$ with a Fréchet dirivative $\acute{f}(\bar{x})$, then $\partial f(\bar{x}) = \{\acute{f}(\bar{x})\}$. That is, convexity is a sufficient condition for subdifferentiability and therefore ϵ-generalized weak subdifferentiability of a Fréchet differentiable function.*

The next result generalizes the Proposition 3.9 in [10], that gives a characterization of having global minimum for a ϵ-generalized weakly subdifferentiable function.

Proposition 4.11. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function such that f is ϵ-generalized weak subdifferentiable at $\bar{x}$. Then f has a global minimizer at $\bar{x}$ iff $(0, c) \in \partial_\epsilon{}^{gw} f(\bar{x})$, for all $\epsilon \in \mathbb{R}_+$, for all $c \in \mathbb{R}_+$.*

Proof. The proof follows from the definition of ϵ-generalized weak subdifferentiability of f at $\bar{x} \in X$. $\square$

In fact, the function f defined by Example 3.4. has not a global minimizer at $\bar{x} = 0$. So ϵ-generalized weakly subdifferentiability of f at $\bar{x} \in X$ for all $\epsilon \in \mathbb{R}_+$ is a essential conditiion.

The next result asserts the relation between ϵ-generalized weak subdifferential f and ϵ-generalized weak subdifferential λf for all $\lambda > 0$.

Remark 4.12. [9] *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a function such that f is $\frac{\epsilon}{\lambda}$-generalized weak subdifferentiable at $\bar{x}$, then*

$$\partial_\epsilon{}^{gw}(\lambda f)(\bar{x}) = \lambda \partial_{\frac{\epsilon}{\lambda}}^{gw} f(\bar{x}) \quad (\forall \lambda > 0).$$

Note that $\partial_\epsilon^{gw}(f(\alpha\bar{x})) = \partial_\epsilon^{gw}\alpha f(\bar{x})$ may failed. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\bar{x} = 1, \alpha = \sqrt{2}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in Q^c \\ 0, & \text{if } x \in Q. \end{cases}$$

Then we have

$$\partial_0^{gw} f(1) = \big\{ (a, c) \in \mathbb{R} \times \mathbb{R}_+ : |a| \leq c \big\}, \ \partial_0^{gw} f(\sqrt{2}) = \emptyset.$$

Next, we investigate a sufficient condition that the following equality holds.

Proposition 4.13. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \,.\, |||: X \longrightarrow C$ be a vectorial norm on X and, let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be ϵ-generalized weak subdifferentiable at $\bar{x}$, $\alpha\bar{x}$ and f is positively homogeneous function. Then we have*

$$\partial_\epsilon^{gw} f(\alpha\bar{x}) \ = \ \partial_{\frac{\epsilon}{\alpha}}^{gw} f(\bar{x}) \qquad (\forall \alpha > 0).$$

Proof. From assumptions we have

$$(T, c) \in \partial_\epsilon^{gw} f(\alpha\bar{x}) \Longleftrightarrow T(\alpha x - \alpha\bar{x}) - c \,|||\, \alpha x - \alpha\bar{x} \,||| \leq_c f(\alpha x) - f(\alpha\bar{x}) + \epsilon 1 \Longleftrightarrow$$

$$\alpha(T(x - \bar{x}) - c \,|||\, x - \bar{x} \,|||) \ \leq_c \ \alpha\big(f(x) - f(\bar{x}) + \frac{\epsilon}{\alpha} 1\big) \Longleftrightarrow (T, c) \in \partial_{\frac{\epsilon}{\alpha}}^{gw} f(\bar{x}). \qquad \square$$

Proposition 4.14. *If all f_i, $i \in I$ (I is a finite nonempty set) and $f(u) = \sup_{i \in I} f_i(u)$, $u \in X$, are finite at $\bar{x}$, then the closure of the convex hull of the set $\bigcup_{i \in I_0(\bar{x})} \partial_{\epsilon_i}^{gw} f_i(\bar{x})$ is a subset of $\partial_\epsilon^{gw} f(\bar{x})$, where $\epsilon = \sum_{i \in I_0(\bar{x})} \epsilon_i$, i.e.,*

$$cl\big(co(\cup_{i \in I_0(\bar{x})} \partial_{\epsilon_i}^{gw} f_i(\bar{x}))\big) \ \subset \ \partial_\epsilon^{gw} f(\bar{x}),$$

where $I_0(\bar{x}) = \big\{ i \in I : f_i(\bar{x}) = f(\bar{x}) \big\}$.

Proof. Suppose that

$$\sum_{i \in I_0(\bar{x})} \alpha_i(T_i, c_i) \ \in \ co\big(\bigcup_{i \in I_0(\bar{x})} \partial_{\epsilon_i}^{gw} f_i(\bar{x}) \big),$$

such that $\sum_{i \in I_0(\bar{x})} \alpha_i = 1$, $\alpha_i \geq 0$, $(T_i, c_i) \in \partial_{\epsilon_i}^{gw} f_i(\bar{x})$. Then we have

$$\big(\forall x \in X, \forall i \in I_0(\bar{x})\big) \quad T_i(x - \bar{x}) \ \leq_C \ f_i(x) - f_i(\bar{x}) + \epsilon_i 1 - c_i \,|||\, x - \bar{x} \,|||.$$

Therefore, $\forall x \in X$,

$$\sum_{i \in I_0(\bar{x})} \alpha_i T_i(x - \bar{x}) - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \,|||\, x - \bar{x} \,||| \ \leq_C \ \sum_{i \in I_0(\bar{x})} \alpha_i f_i(x) - \sum_{i \in I_0(\bar{x})} \alpha_i f_i(\bar{x}) + \epsilon 1.$$

Since $f(x) = \sup_{i \in I} f_i(x)$, $x \in X$, we have $I_0(\bar{x}) = \{i \in I \; : \; f_i(\bar{x}) = f(\bar{x})\}$, so that

$$\sum_{i \in I_0(\bar{x})} \alpha_i T_i(x - \bar{x}) - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \; ||| \, x - \bar{x} \, ||| \; \leq_C \quad f(x) - f(\bar{x}) + \epsilon 1 \; (\forall x \in X)$$

and

$$\sum_{i \in I_0(\bar{x})} \alpha_i(T_i, c_i) \in \partial_\epsilon^{gw} f(\bar{x}).$$

Consequently,

$$co(\bigcup_{i \in I_0(x)} \partial_{\epsilon_i}{}^{gw} f_i(\bar{x})) \subset \partial_\epsilon^{gw} f(\bar{x}).$$

Now the clossedness of the set $\partial_\epsilon^{gw} f(\bar{x})$ completes the proof.   □

Remark 4.15. *If f is Fréchet differentiable at $\bar{x}$, then $f, -f$ are subdifferentiable at $\bar{x}$, if and only if $f(x) - f(\bar{x}) = \acute{f}(\bar{x})(x - \bar{x}) \quad (\forall x \in X)$.*

The following theorem recall the fuzzy sum rule and then we investigate special sufficient condition which the equality holds.

Theorem 4.16. [9] *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}_+$. Let $f, g : X \longrightarrow Y$ are functions such that f is ϵ_1-generalized weak subdifferentiable and g is ϵ_2-generalized weak subdifferentiable at $\bar{x}$. Then*

$$\partial_{\epsilon_1}{}^{gw} f(\bar{x}) + \partial_{\epsilon_2}{}^{gw} g(\bar{x}) \; \subset \; \partial_{\epsilon_1 + \epsilon_2}{}^{gw} (f + g)(\bar{x}).$$

Proposition 4.17. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f : X \longrightarrow Y$ be a Fréchet differentiable at $\bar{x}$ and $f, -f$ be subdifferentiable at $\bar{x}$, $g : X \longrightarrow Y$ be a function such that g is ϵ-generalized weak subdifferentiable at $\bar{x}$. Then*

$$(\partial f(\bar{x}), 0) + \partial_\epsilon^{gw} g(\bar{x}) \; = \; \partial_\epsilon^{gw} (f + g)(\bar{x}).$$

Proof. By Theorem 4.16, $f + g$ is ϵ-generalized weak subdifferentiable at $\bar{x}$. Then there exist (T, c) such that

$$T(x - \bar{x}) - c \; ||| \, x - \bar{x} \, ||| \; \leq_c \; (f + g)(x) - (f + g)(\bar{x}) + \epsilon 1 \quad (\forall \, x \in X),$$

and $f(x) - f(\bar{x}) = \acute{f}(\bar{x})(x - \bar{x})$. Thus,

$$(T - \acute{f})(x - \bar{x}) - c \; ||| \, x - \bar{x} \, ||| \; \leq_c \; g(x) - g(\bar{x}) + \epsilon 1 \quad (\forall \, x \in X),$$

this means that $(T - \acute{f}, c) \in \partial_\epsilon^{gw} g(\bar{x})$. From $(T, c) = (T - \acute{f}, c) + (\acute{f}, 0)$, we obtain

$$\partial_\epsilon^{gw} (f + g)(\bar{x}) \; \subset \; \partial f(\bar{x}) + \partial_\epsilon^{gw} g(\bar{x}).$$

Therefore, the equality is obtained by Theorem 4.16.   □

Proposition 4.18. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$. Let $f : X \longrightarrow Y$ be a Fréchet differentiable at $\bar{x}$ and $f, -f$ be subdifferentiable at $\bar{x}$, $g : X \longrightarrow Y$ be a function. If $f + g$ attains a global minimum at $\bar{x}$, then*

$$(-\acute{f}(\bar{x}), 0) \in \partial_\epsilon^{gw} g(\bar{x}) \quad (\forall \epsilon \geq 0).$$

Proof. Since $f + g$ attains a global minimum at $\bar{x}$, then

$$0 \leq_c (f + g)(x) - (f + g)(\bar{x}) \quad (\forall \, x \in X),$$

and so we can rewrite the inequality as:

$$g(\bar{x}) - g(x) \; \leq_c \; f(x) - f(\bar{x}) \; = \; \acute{f}(\bar{x})(x - \bar{x}) \quad (\forall \, x \in X),$$

this means that $(-\acute{f}(\bar{x}), 0) \in \partial_\epsilon^{gw} g(\bar{x}) \quad (\forall \epsilon \geq 0)$. $\qquad\qquad \square$

Proposition 4.19. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f, g : X \longrightarrow Y$ are functions such that f is ϵ-generalized weak subdifferentiable at $\bar{x}$ and $g - f$ has a global minimizer at $\bar{x}$. Then we have*

$$\partial_\epsilon^{gw} f(\bar{x}) \; \subset \; \partial_\epsilon^{gw} g(\bar{x}).$$

Proof. Since $g - f$ has a global minimizer at $\bar{x}$, then we have

$$0 \; \leq_c \; (g - f)(x) - (g - f)(\bar{x}).$$

Now, assume that $(T, c) \in \partial_\epsilon^{gw} f(\bar{x})$, then we have

$$T(x - \bar{x}) - c \, ||| \, x - \bar{x} \, ||| \leq_c \; f(x) - f(\bar{x}) + \epsilon 1 \quad (\forall x \in X),$$

and so

$$T(x - \bar{x}) - c \, ||| \, x - \bar{x} \, ||| \leq_c \; g(x) - g(\bar{x}) + \epsilon 1 \quad (\forall \, x \in X),$$

that is, $(T, c) \in \partial_\epsilon^{gw} g(\bar{x})$. Therefore, the proof is completed. $\qquad \square$

Corollary 4.20. *Let $f : X \longrightarrow Y$ be a ϵ-generalized weak subdifferentiable function at $\bar{x}$ and f has a global minimizer at $\bar{x}$. Then we have*

$$\partial_\epsilon^{gw} 0(\bar{x}) \; \subset \; \partial_\epsilon^{gw} f(\bar{x}).$$

Proposition 4.21. *Let X be a real topological vector space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $intC \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X, and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. Let $f, g : X \longrightarrow Y$ are functions such that f, g is ϵ-generalized weak subdifferentiable at $\bar{x}$ and $g - f$ is a constant function. Then*

$$\partial_\epsilon^{gw} f(\bar{x}) \; = \; \partial_\epsilon^{gw} g(\bar{x}).$$

Proof. From $0 \leq_c (g-f)(x) - (g-f)(\bar{x})$ with the proposition, we obtain

$$\partial_\epsilon{}^{gw} f(\bar{x}) \subset \partial_\epsilon{}^{gw} g(\bar{x}).$$

Similarly, by $(g-f)(x) - (g-f)(\bar{x}) \leq_c 0$, we have

$$\partial_\epsilon{}^{gw} g(\bar{x}) \subset \partial_\epsilon{}^{gw} f(\bar{x}).$$

So, the desired result is obtained. $\square$

Corollary 4.22. *Let* $f : X \longrightarrow Y$ *be a constant function. Then*

$$\partial_\epsilon{}^{gw} f(\bar{x}) = \partial_\epsilon{}^{gw} 0(\bar{x}) \quad (\forall \epsilon \geq 0).$$

Let X, Y, Z be a linear normed space and C be a pointed closed convex cone in Y, and $(Y, \leq_C)$ be a real ordered topological vector space with $int C \neq \emptyset$ and $||| \, . \, |||: X \longrightarrow C$ be a vectorial norm on X. Let $||| \, . \, |||_1 : Z \longrightarrow C$ be a vectorial norm on Z and let $\bar{x} \in X$ and $\epsilon \in \mathbb{R}_+$. For any $T \in B(Z, Y)$, we consider a function $< T, h >$ defined by the equality

$$< T, h > (u) := < T, h(u) >,$$

where $h : X \longrightarrow Z$ be a function such that there exsists a nonnegative L with the following property

$$||| \, h(x) - h(\bar{x}) \, |||_1 \leq_C L \, ||| \, x - \bar{x} \, ||| \qquad (\forall x \in X).$$

In this case h is called a *Lipschitz function at* $\bar{x}$ with Lipschitz constant L. Let $g : Z \longrightarrow Y$ be finite at $\bar{z} = h(\bar{x})$. We consider a composition $f(u) = g(h(u)) \, u \in X$ and recall that projection operator $\pi : X \times Y \longrightarrow X$ such that $\pi(x, y) = x$ for all $(x, y) \in X \times Y$.

Proposition 4.23. *Let* g *be* ϵ-*weak subdifferentiable at* $\bar{z}$ *and* $< T, h >$ *is* $\bar{\epsilon}$ - *weak subdifferentiable at* $\bar{x}$ *for some* $T \in \pi(\partial_\epsilon{}^{gw} g(\bar{z}))$ *and* h *be a Lipschitz function at* $\bar{x}$ *with nonnegative Lipschitz constant* L. *Then* f *is* $(\epsilon + \bar{\epsilon})$-*weak subdifferentiable at* $\bar{x}$ *and*

$$\partial_{\bar{\epsilon}}{}^{gw} < T, h > (\bar{x}) \subset \partial_{(\epsilon + \bar{\epsilon})}{}^{gw} f(\bar{x}).$$

Proof. If $(w, c) \in \partial_{\bar{\epsilon}}{}^{gw} < T, h > (\bar{x})$, then

$$0 \leq_C < T, h > (x) - < T, h > (\bar{x}) - w(x - \bar{x}) + c \, ||| \, x - \bar{x} \, ||| + \bar{\epsilon} 1, \ (\forall \, x \in X).$$

From $(T, \bar{c}) \in \partial_\epsilon{}^{gw} g(\bar{z})$, we have

$$0 \leq_C g(z) - g(\bar{z}) - T(z - \bar{z}) + \bar{c} \, ||| \, z - \bar{z} \, |||_1 + \epsilon 1, \ (\forall \, z \in Z).$$

Specially,

$$0 \leq_C g(h(x)) - g(h(\bar{x})) - T(h(x) - h(\bar{x})) + \bar{c} \, ||| \, h(x) - h(\bar{x}) \, |||_1 + \epsilon 1, \ (\forall \, x \in X).$$

Hence

$$
\begin{aligned}
0 \ &\leq_C \ f(x) - f(\bar{x}) - T(h(x) - h(\bar{x})) + \bar{c} \ ||| \ h(x) - h(\bar{x}) \ |||_1 \ + \epsilon 1 \\
&\leq_C \ f(x) - f(\bar{x}) - (w, x - \bar{x}) + c \ ||| \ x - \bar{x} \ ||| \ + \bar{c}L \ ||| \ x - \bar{x} \ ||| \ + (\epsilon + \bar{\epsilon})1 \\
&= f(x) - f(\bar{x}) - (w, x - \bar{x}) + (c + \bar{c}L) \ ||| \ x - \bar{x} \ ||| \ + (\epsilon + \bar{\epsilon})1.
\end{aligned}
$$

Then $w \in \pi(\partial_{(\epsilon+\bar{\epsilon})}{}^{gw} f(\bar{x}))$. $\qquad\square$

The conclusion of Proposition 4.23 can be rewritten in the following form:

$$
\bigcup \left\{ \partial_{\bar{\epsilon}}{}^{gw} < T, h > (\bar{x}) \ : \ T \in \partial_\epsilon{}^{gw} g(\bar{z}) \right\} \ \subset \ \partial_{(\epsilon+\bar{\epsilon})}{}^{gw} f(\bar{x}).
$$

Proposition 4.24. *Let f be ϵ-weak subdifferentiable at $\bar{x}$, $-g$ be $\bar{\epsilon}$- weak subdifferentiable at $\bar{z}$, h be a Lipschitz function with Lipschitz constant L, then $< -T, h >$ is $(\epsilon + \bar{\epsilon})$-weak subdifferentiable at $\bar{x}$ for any $T \in \pi(\partial_\epsilon{}^{gw}(-g(\bar{z})))$ and*

$$
\partial_\epsilon{}^{gw} f(\bar{x}) \subset \partial_{(\epsilon+\bar{\epsilon})}{}^{gw} < -T, h > (\bar{x}).
$$

Proof. If $(w, c) \in \partial_\epsilon{}^{gw} f(\bar{x})$, then we have

$$
0 \ \leq_C \ f(x) - f(\bar{x}) - w(x - \bar{x}) + c \ ||| \ x - \bar{x} \ ||| \ + \epsilon 1, \quad (\forall \ x \in X).
$$

If $(T, \bar{c}) \in \partial_\epsilon{}^{gw}(-g(\bar{z}))$, then we have

$$
0 \ \leq_C \ -g(z) + g(\bar{z}) - < T, z - \bar{z} > + \bar{c} \ ||| \ z - \bar{z} \ |||_1 \ + \bar{\epsilon}1, \quad (\forall \ z \in Z).
$$

Hence, $\forall \ z \in Z$,

$$
0 \ \leq_C \ - < T, h > (x) + < T, h > (\bar{x}) - g(z) + g(\bar{z}) + \bar{c} \ ||| \ z - \bar{z} \ |||_1 \ + \bar{\epsilon}1,
$$

therefore

$$
0 \ \leq_C \ - < T, h > (x) + < T, h > (\bar{x}) - f(x) + f(\bar{x}) + \bar{c} \ ||| \ h(x) - h(\bar{x}) \ |||_1 \ + \bar{\epsilon}1
$$

and

$$
\begin{aligned}
0 \ \leq_C \ &- < T, h > (x) + < T, h > (\bar{x}) - w(x - \bar{x}) + c \ ||| \ x - \bar{x} \ ||| \ + (\epsilon + \bar{\epsilon})(1) \\
&+ \ \bar{c} ||| \ h(x) - h(\bar{x}) \ |||_1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 \ \leq_C \ &- < T, h > (x) + < T, h > (\bar{x}) - w(x - \bar{x}) + c ||| \ x - \bar{x} \ ||| \\
&+ \ \bar{c}L ||| \ x - \bar{x} \ ||| \ + (\epsilon + \bar{\epsilon})(1) \\
= \ &- < T, h > (x) + < T, h > (\bar{x}) - w(x - \bar{x}) + (c + \bar{c}L) ||| \ x - \bar{x} \ ||| \\
&+ \ (\epsilon + \bar{\epsilon})(1),
\end{aligned}
$$

this means that $w \in \pi(\partial_{(\epsilon+\bar{\epsilon})}{}^{gw} < -T, h > (\bar{x}))$. $\qquad\square$

Corollary 4.25. *Let f be ϵ-weak subdifferentiable at $\bar{x}$, g be Frechet differentiable at $\bar{y}$, $g, -g$ be subdifferentiable, and h be a Lipschitz function, then*

$$
\partial_\epsilon{}^{gw} f(\bar{x}) = \partial_\epsilon{}^{gw} < \acute{g}(\bar{y}), h > (\bar{x}).
$$

Proof. Combining Proposition 4.23 with Proposition 4.24. $\qquad\square$

References

[1] A. Azimov, R.N. Gasimov, On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, Int. J. Appl. Math. 1 (1999) 171-192.

[2] R. Kasimbeyli, G. Inceoglu, The properties of the weak subdifferentials, G.U.J. Science 23 (1) (2010) 49-52.

[3] F. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, New York, 1983.

[4] F. Clarke, A new approach to lagrange multipliers, Math. Oper. Re. 1 (1976) 165-174.

[5] R. Kasimbeyli, M. Mammadov, On weak subdifferentials, directional derivaties and radial epiderivativees for nonconvex functions, SIAM J. Optimization 20 (2) (2009) 841-855.

[6] N. Nimana, A.P. Farajzadeh, N. Petrot, Adaptive subgradient method for the split quasiconvex feasibility problems, Optimization 65 (10) (2016) 1885-1898.

[7] M. Küçük,, R. Urbański, J. Grzybowski, Y. Küçük, İ Atasever Güvenç, D. Tozkan, M. Soyertem, Some relationships among quasidifferential, weak subdifferential and exhausters, Optimization 65 (11) (2016) 1949-1961.

[8] J. Jahn, Vector Optimization: Theory, Applications and Extensions, Second Edition, Springer-Verlag, Berlin Heidelberg, 2011.

[9] A. Mohebi, H. Mohebi, Some relation between ϵ-directional derivative and ϵ-generalized weak subdifferential, Wavelets and linear algebra 2 (1) (2015) 65-80.

[10] P. Cheraghi, A.P. Farajzadeh, G.V. Milovanovic, Some notes on weak subdifferential, Filomat 31 (11) (2017) 3407-3420.