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Approximating Common Fixed Points of Two α-Nonexpansive Mappings

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Abstract : In this paper, we introduce and approximating common fixed points of two α-nonexpansive mappings through weak and strong convergence of an iterative sequence in a uniformly convex Babach space.

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1 Introduction

Let E be an ordered Banach space with the partial order $\leq$, K be a nonempty subset of an ordered Banach space E. A mapping $T : K \to K$ is said to be monotone if $Tx \leq Ty$ for all $x, y \in K$ with $x \leq y$ and recall that T is monotone nonexpansive if T is monotone and $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\, y \in K$ with $x \leq y$. Following, Aoyama and Kohsaka [1], a mapping $T : C \to C$ is said to be α-nonexpansive for some $\alpha < 1$ if

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|,\ \forall x,\, y \in C. \quad (1.1)$$

Clearly, nonexpansive mapping is 0-nonexpansive maps. An example of a discontinuous α-nonexpansive mapping (with $\alpha > 0$) has been given in [1]. It is well known that, the concept of nonexpansivity of a map T from a convex set plays an important role in the study of the *Mann iteration* given by

$$x_{n+1} = (1 - s_n)x_n + s_n T x_n,\ x_0 \in K,$$

for each $n \geq 1$, where $s_n \in [0, 1]$ such it was introduced by Mann [2] in 1953.

In 1974, Ishikawa [3] introduced the *Ishikawa iteration* given by

$$x_{n+1} = (1 - a_n)x_n + a_n T(y_n);$$

$$y_n = (1 - b_n)x_n + b_n T x_n.$$

For each $n \geq 1$, where a_n and $b_n \in [0, 1]$. In particular, when all $b_n = 0$, then Ishikawa iteration becomes the standard Mann iteration.

In this paper, we introduce and approximating common fixed points of two α-nonexpansive mappings S and T throught weak and strong convergence of the sequence be defined by we use the following Ishikawa iteration [4–6]

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n S(y_n); \\ y_n &= (1 - b_n)x_n + b_n T x_n, \end{aligned} \quad (1.2)$$

for each $n \geq 1$, where a_n and $b_n \in [0, 1]$, satisfying certain condition.

2 Preliminaries

Next, we state some useful lemmas and definitions as follows.

Lemma 2.1. [7] *Suppose that E is a uniformly convex Banach space and $0 < p \leq q < 1$ for all $n = 1, 2, \cdots$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequence of E such that $\lim_{n\to\infty} \|x_n\| \leq r$, $\lim_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

We recall that a Banach space E is said to satisfy Opial's condition [8] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| \leq \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $x \neq y$. Moreover, we also know that a mapping T is called demiclosed with respect to $y \in K$ if for each sequence $\{x_n\} \in K$ and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \rightharpoonup y$ imply that $x \in K$ and $Tx = y$.

Lemma 2.2. [9] *Let E be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $T : K \longrightarrow K$ be a nonexpansive mapping. Then $I - T$ is demiclosed with respect to zero.*

Definition 2.3. Let K be a nonempty closed convex subset of Banach space E. A mapping $T : K \to K$ is said to be :

(1) *α-nonexpansive* for some $\alpha < 1$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$$

for all $x, y \in K$.

(2) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $p \in F(T)$ and $x \in K$.

Lemma 2.4. *Let K be a nonempty closed convex subset of Banach space E. A mapping $T : K \to K$ be a α-nonexpansive mapping. Then T is a quasi-nonexpansive.*

Proof.

$$\begin{aligned}\|Tx - p\|^2 &= \|Tx - Tp\|^2 \\ &\leq \alpha\|Tx - p\|^2 + \alpha\|Tp - x\|^2 + (1 - 2\alpha)\|x - p\|^2 \\ &= \alpha\|Tx - p\|^2 + (1 - \alpha)\|x - p\|^2 \\ &\leq \|x - p\|\end{aligned}$$

and so T is a quasi-nonexpansive. □

3 Weak and Strongly Convergence Theorems

In this section, first we prove the following Lemma which, in fact, forms a major part of the proofs of both weak and strong convergence theorems.

Lemma 3.1. *Let C be a bounded, closed and convex subset of a uniformly convex ordered Banach space $(E, \leq)$. Let $S, T : C \to C$ be monotone α-nonexpansive mappings. Assume there exists $x_1 \in C$ such that $x_1 \leq Sx_1$, $x_1 \leq Tx_1$ and there exists $p \in F(S) \cap F(T)$ such that x_1 and p are comparable. Consider the sequences $\{x_n\}$ be defined by Ishikawa's iteration. Then*

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|.$$

Proof. Let $p \in F(S) \cap F(T)$. By Lemma 2.4, we consider

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-a_n)x_n + a_n S y_n - p\| \\
&= \|(1-a_n)x_n + a_n S((1-b_n)x_n + b_n T x_n) - p\| \\
&\le \|(1-a_n)(x_n - p)\| + \|a_n S((1-b_n)x_n + b_n T(x_n)) - p\| \\
&\le \|(1-a_n)(x_n - p)\| + \|a_n((1-b_n)x_n + b_n T(x_n)) - p\| \\
&\le \|(1-a_n)(x_n - p)\| + \|a_n(1-b_n)(x_n - p)\| + \|a_n b_n (T x_n - p)\| \\
&\le (1-a_n)\|x_n - p\| + a_n(1-b_n)\|x_n - p\| + a_n b_n \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = r$ where $r \ge 0$ is a real number. By T is quasi-nonexpansive mapping then we have $\|Tx_n - p\| \le \|x_n - p\|$ for all $n = 1, 2, 3, \ldots$, so

$$\limsup_{n\to\infty} \|Tx_n - p\| \le r.$$

Also

$$\begin{aligned}
\|y_n - p\| &= \|(1-b_n)x_n + b_n T x_n - p\| \\
&\le \|(1-b_n)(x_n - p)\| + \|b_n T x_n - p\| \\
&\le (1-b_n)\|(x_n - p)\| + b_n\|x_n - p\| \\
&= \|x_n - p\|
\end{aligned}$$

and we get

$$\limsup_{n\to\infty} \|y_n - p\| \le r. \tag{3.1}$$

By S is quasi-nonexpansive mapping then we have

$$\limsup_{n\to\infty} \|Sy_n - p\| \le r.$$

Moreover, $\lim_{n\to\infty} \|x_n - p\| = r$ means that

$$\lim_{n\to\infty} \|(1-a_n)(x_n - p) + a_n(Sy_n - p)\| = r.$$

By Lemma 2.1 , we get

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{3.2}$$

Now

$$\|x_n - p\| \le \|x_n - Sy_n\| + \|Sy_n - p\| \le \|x_n - Sy_n\| + \|y_n - p\|,$$

then we get

$$r \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.3}$$

By (3.1) and (3.3), we get

$$\lim_{n\to\infty} \|y_n - p\| = r. \tag{3.4}$$

That is

$$\lim_{n\to\infty} \|(1-b_n)(x_n - p) + b_n(Tx_n - p)\| = r.$$

By Lemma 2.1, we get

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.5}$$

And we consider,

$$\begin{aligned}\|Tx_n - y_n\| &= \|Tx_n - (1-b_n)x_n - b_nT(x_n)\|\\ &= \|(1-b_n)Tx_n - (1-b_n)x_n\|\\ &= (1-b_n)\|(Tx_n - x_n)\|,\end{aligned}$$

then by (3.5), we get

$$\lim_{n\to\infty} \|Tx_n - y_n\| = 0. \tag{3.6}$$

By Definition 2.3, we consider

$$\begin{aligned}\|Sx_n - x_n\|^2 &\le [\|Sx_n - Sy_n\| + \|Sy_n - x_n\|]^2\\ &= \|Sx_n - Sy_n\|^2 + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\| + \|Sy_n - x_n\|^2\\ &\le \alpha\|Sx_n - y_n\|^2 + \alpha\|Sy_n - x_n\|^2 + (1-2\alpha)\|x_n - y_n\|^2\\ &\quad + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\| + \|Sy_n - x_n\|^2\\ &= \alpha\|Sx_n - y_n\|^2 + (1-2\alpha)\|x_n - y_n\|^2 + (1+\alpha)\|Sy_n - x_n\|^2\\ &\quad + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\|\\ &\le \alpha[\|Sx_n - x_n\| + \|x_n - y_n\|]^2 + (1-2\alpha)\|x_n - y_n\|^2\\ &\quad + (1+\alpha)\|Sy_n - x_n\|^2 + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\|\\ &= \alpha\|Sx_n - x_n\|^2 + 2\alpha\|Sx_n - x_n\|\|x_n - y_n\| + \alpha\|x_n - y_n\|^2\\ &\quad + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\| + (1-2\alpha)\|x_n - y_n\|^2\\ &\quad + (1+\alpha)\|Sy_n - x_n\|^2\end{aligned}$$

then

$$\begin{aligned}(1-\alpha)\|Sx_n - x_n\|^2 &\le (1-\alpha)\|x_n - y_n\|^2 + 2\alpha\|Sx_n - x_n\|\|x_n - y_n\|\\ &\quad + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\| + (1+\alpha)\|Sy_n - x_n\|^2\\ &\le (1-\alpha)[\|x_n - Tx_n\| + \|Tx_n - y_n\|]^2\\ &\quad + 2\alpha\|Sx_n - x_n\|[\|x_n - Tx_n\| + \|Tx_n - y_n\|]\\ &\quad + 2\|Sx_n - Sy_n\|\|Sy_n - x_n\| + (1+\alpha)\|Sy_n - x_n\|^2.\end{aligned}$$

By (3.2), (3.5) and (3.6), we can conclude that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|. \tag{3.7}$$

□

Theorem 3.2. *Let C be a bounded, closed and convex subset of a uniformly convex ordered Banach space $(E, \leq)$. Let $S, T : C \to C$ be monotone α-nonexpansive mappings. Assume E satisfies Opial's condition and the sequence $\{x_n\}$ be defined by Ishikawa's iteration with $x_1 \leq Sx_1$, $x_1 \leq Tx_1$. If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$ converges weakly to a unique common fixed point of S and T.*

Proof. From we let p be a common fixed point of S and T and $\lim_{n\to\infty} \|x_n - p\|$ exists. Next we will prove that $\{x_n\}$ has a uniqua weak subsequential limit in $F(S) \cap F(T)$. Let u and v be weak limit of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.1, we have $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$ and $I - S$ is demiclosed with respect to zero, respectively. Therefore, we obtain $Su = u$. Similarly, $Tu = u$. Again in the same fashion, we can prove that $v \in F(S) \cap F(T)$.

Next, we will prove the uniqueness by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \lim_{n\to\infty} \|x_{n_i} - u\| \\
&\leq \lim_{i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&= \lim_{j\to\infty} \|x_{n_j} - v\| \\
&\leq \lim_{j\to\infty} \|x_{n_j} - u\| \\
&= \lim_{n\to\infty} \|x_n - u\|.
\end{aligned}$$

This is a contradiction, then $u = v$. □

Theorem 3.3. *Let C be a compact, closed and convex subset of a uniformly convex ordered Banach space $(E, \leq)$. Let $S, T : C \to C$ be monotone α-nonexpansive mappings. Assume E satisfies Opial's condition and the sequence $\{x_n\}$ be defined by Ishikawa's iteration with $x_1 \leq Sx_1$, $x_1 \leq Tx_1$. If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$ converges strongly to a unique common fixed point of S and T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$. Since K is compact so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q$. Continuity of S and T gives $Sx_{n_i} \to Sq$ and $Tx_{n_i} \to Tq$ as $n_i \to \infty$. Then we get

$$\|Sq - q\| = 0 = \|Tq - q\|.$$

This results $q \in F(S) \cap F(T)$ so that $\{x_{n_i}\}$ converges strongly to q in $\in F(S) \cap F(T)$. But again by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T)$ therefore $\{x_n\}$ must itself converge to $q \in F(S) \cap F(T)$. This completes the proof. □

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