Thai Journal of Mathematics
Volume 16 (2018) Number 1 : 121–138
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Tripled Coincidence Point Theorems with M-Invariant Set for a α-ψ-Contractive Mapping in Partially Metric Spaces

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Abstract : In this paper, we introduce the notion M-invariant set for mapping $\alpha : X^3 \times X^3 \to [0, +\infty)$. We show the existence of a tripled coincidence point theorem for a α-ψ-contractive mapping in partially ordered complete metric spaces without the mixed g-monotone property, using the concept of M-invariant set. We also show the uniqueness of a tripled common fixed point for such mappings and give some examples to show the validity of our result.

Keywords : fixed point; tripled coincidence; invariant set; admissible.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

The existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been first considered recently by Ran and Reurings [1] where they established some new results for contractions in partially ordered metric spaces and presented applications to matrix equations. Later, Nieto and lopez [2,3] and Agarwal et al. [4] presented some new results for contractions in partially ordered metric spaces.

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [5]. In 2006, Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone property for contractive operators of the form $F : X \times X \to X$

where X is a partially ordered metric spaces. They also established the coupled fixed point theorems for mappings which satisfy the mixed monotone property. After the publication of this work, many authors conducted research on the coupled fixed point theory in partially ordered metric spaces and different spaces. For example, see [7–22].

In 2011, Berinde and Borcut [23] introduced the concept of tripled fixed point which is a generalization of coupled fixed point theorems for nonlinear mappings in partially ordered complete metric spaces and obtained existence and uniqueness theorems for contractive type mappings. Later in 2012, Berinde and Borcut [24] introduced the concept of tripled coincidence point for a pair of nonlinear contractive mapping $F : X^3 \to X$ and $g : X \to X$ and obtained tripled coincidence point theorems which generalized the results of [23]. After the publication of this work, the tripled fixed point and tripled coincidence point theory has been generalized in different directions in several spaces with applications by many mathematicians over the year (see [12, 13, 25–40]).

Some authors studied the coupled fixed point and coupled coincidence fixed point without mixed monotone and mixed g monotone property ,in 2012, Sintunavarat, Kumam and Cho [16, 17] proved the existence and uniqueness of coupled fixed point theorems for nonlinear contractions without mixed monotone property and extended some coupled fixed point theorems of Bhaskar and Lakshmikantham [6] by using the concept of F-invariant set due to Samet and Vetro [10]. Later, Batra and Vashistha [18] introduced the concept of (F, g)-invariant set which is a generalization of an F-invariant set and proved the existence coupled fixed point theorems for nonlinear contractions under c-distance in cone metric spaces having an (F, g)-invariant subset. Recently, Charoensawan and Klanarong [37] proved the existence and uniqueness of coupled fixed point theorems for nonlinear contractions without mixed g-monotone property by using the concept of (F, g)-invariant set.

In the case of a tripled coincidence point theory without mixed g-monotone property, in 2013, Charoensawan [38] established some tripled coincidence point theorems without a mixed g-monotone property by using the concept of (F, g)-invariant set in a complete metric space which are generalizations of the results of Aydi, Karapinar and Postolache [27].

In this work, we introduce M-invariant set for mapping $\alpha : X^3 \times X^3 \to [0, +\infty)$ and prove the existence of a tripled coincidence point theorem and a tripled common fixed point theorem for a α-ψ-contractive mapping in partially ordered complete metric spaces.

2 Preliminaries

Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leqslant (x, y, z) \;\Leftrightarrow\; x \geqslant u,\; y \leqslant v,\; z \geqslant w.$$

Berinde and Borcut [24] introduced the concept of tripled coincidence point and studied existence and uniqueness theorems in partially ordered complete metric spaces.

Definition 2.1. [24] Let $(X, \leqslant)$ be a partially ordered set and two mappings $F : X \times X \times X \to X$, $g : X \to X$. We say that F has the *mixed g-monotone property* if for any $x, y, z \in X$

$$x_1, x_2 \in X,\ g(x_1) \leqslant g(x_2) \quad \textit{implies} \quad F(x_1, y, z) \leqslant F(x_2, y, z),$$

$$y_1, y_2 \in X,\ g(y_1) \leqslant g(y_2) \quad \textit{implies} \quad F(x, y_1, z) \geqslant F(x, y_2, z).$$

and

$$z_1, z_2 \in X,\ g(z_1) \leqslant g(z_2) \quad \textit{implies} \quad F(x, y, z_1) \leqslant F(x, y, z_2).$$

Definition 2.2. [24] An element $(x, y, z) \in X \times X \times X$ is called a *tripled coincidence point* of a mapping F and g if $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$.

Definition 2.3. [24] Let X be a non-empty set and $F : X \times X \times X \to X$, $g : X \to X$ two mappings. We say F and g are *commutative* if $g(F(x, y, z)) = F(g(x), g(y), g(z))$, for all $x, y, z \in X$.

Later, Aydi, Karapinar and Postolache [27] extended the tripled coincidence point theorems for mixed g-monotone operator obtained by Berinde and Borcut [24].

Let Φ be a set of functions defined by

$$\Phi = \{\varphi : [0, +\infty) \to [0, +\infty) : \varphi(t) < t \text{ and } \lim_{r \to t^+} \varphi(r) < t,\ t > 0\}.$$

Theorem 2.4. [27] *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$. Assume there is a function $\varphi \in \Phi$ such that*

$$\begin{aligned} &d(F(x, y, z), F(u, v, w)) + d(F(y, x, y), F(v, u, v)) + d(F(z, y, x), F(w, v, u)) \\ &\qquad \leqslant 3\varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))}{3}\right) \end{aligned} \tag{2.1}$$

for all $x, y, , z, u, v, w \in X$ with $g(x) \geqslant g(u)$, $g(y) \leqslant g(v)$ and $g(z) \geqslant g(w)$. Assume that F is continuous, g is continuous and commutes with F. If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0, z_0),\ g(y_0) \geqslant F(y_0, x_0, y_0) \text{ and } g(z_0) \leqslant F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Definition 2.5. [27] Let $(X, \leqslant)$ be a partially ordered set and d be a metric on X. We say that $(X, d, \leqslant)$ is *regular* if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X, then $x_n \leqslant x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$ in X, then $y \leqslant y_n$ for all n.

Theorem 2.6. [27] *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that $(X, d, \leqslant)$ is regular. Suppose that there exist $\varphi \in \Phi$ and mapping $F : X \times X \times X \to X$ and $g : X \to X$ are such that* (2.1) *holds for any $x, y, z, u, v, w \in X$ with $g(x) \geqslant g(u)$, $g(y) \leqslant g(v)$ and $g(z) \geqslant g(w)$. Suppose also that $(g(X), d)$ is complete, F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$. If there exist $x_0, y_0, z_0 \in X$ such that*

$$g(x_0) \leqslant F(x_0, y_0, z_0),\ g(y_0) \geqslant F(y_0, x_0, y_0) \text{ and } g(z_0) \leqslant F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Kaushik and et al. [41] introduced the notion of (α, ψ)-contractive and (α)-admissible for a pair of mapping and prove coupled coincidence point theorem which generalization of the result of Mursaleen et al. [8] as follow.

Let Ψ denote the family non-decreasing and right continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ satisfying the following conditions:

1. $\psi^{-1}(\{0\}) = \{0\}$,
2. $\psi(t) < t$ for all $t > 0$,
3. $\lim_{r \to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 2.7. *If $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and right continuous, then $\psi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Definition 2.8. [41] Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ and $g : X \to X$ be two mapping. Then F and g are said to be (α, ψ)-*contractive* if there exist two functions $\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha\big((gx, gy), (gu, gv)\big) d(F(x, y), F(u, v)) \ \leq \ \psi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$.

Definition 2.9. [41] Let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, \infty)$. We say that F and g are (α)-*admissible* if for all $x, y, u, v \in X$, we have

$$\alpha\big((gx, gy), (gu, gv)\big) \geq 1 \ \Rightarrow \ \alpha\big((F(x, y), F(y, x)), (F(u, v), F(v, u))\big) \geq 1.$$

Now, we give the notion of (α)-admissible and M-invariant which is useful for our main results.

Definition 2.10. [41] Let $F : X \times X \times X \to X$, $g : X \to X$ and $\alpha : X^3 \times X^3 \to [0,\infty)$. We say that F and g are (α)-*admissible* if for all $x,y,z,u,v,w \in X$, we have

$$\alpha\big((gx,gy,gz),(gu,gv,gw)\big) \geq 1$$
$$\Rightarrow \alpha\big((F(x,y,z),F(y,x,y),F(z,y,x)),(F(u,v,w),F(v,u,v),F(w,v,u))\big) \geq 1.$$

Definition 2.11. Let (X,d) be a metric space and M be a nonempty subset of X^6 and $\alpha : X^3 \times X^3 \to [0,\infty)$. We say that α is M-*invariant* if for all $x,y,z,u,v,w \in X$,

$$\alpha\big((x,y,z),(u,v,w)\big) \geq 1 \;\Rightarrow\; (x,y,z,u,v,w) \in M.$$

Example 2.12. Let $F : X \times X \times X \to X$, $g : X \to X$. Consider a mapping $\alpha : X^3 \times X^3 \to [0,\infty)$ be such that

$$\alpha\big((x,y,z),(u,v,w)\big) = 1,$$

for all $x,y,z,u,v,w \in X$. It is easy to see that α is M-invariant where $M = X^6$ and F is (α)-admissible, also F and g are (α)-admissible.

In this article, we establish tripled coincidence point theorems for $F : X \times X \times X \to X$ and $g : X \to X$ satisfying nonlinear contractive conditions without mixed g-monotone property by using the concept of α is M-invariant in complete metric spaces.

3 Main Results

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and M be a nonempty subset of X^6 and let d be a metric on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^3 \times X^3 \to [0,\infty)$ such that the following holds:*

$$\alpha\big((gx,gy,gz),(gu,gv,gw)\big)\left[\frac{d(F(x,y,z),F(u,v,w)) + d(F(y,x,y),F(v,u,v))}{3}\right.$$
$$\left. + \frac{d(F(z,y,x),F(w,v,u))}{3}\right] \leq \psi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}\right) \tag{3.1}$$

for all $x,y,z,u,v,w \in X$ with $(gx,gy,gz,gu,gv,gw) \in M$. Suppose also that

(i) *F and g are (α)-admissible,*

(ii) *F is continuous,*

(iii) $F(X \times X \times X) \subseteq g(X)$ *and* g *is continuous and commutes with* F,

(iv) *there exist* $x_0, y_0, z_0 \in X$ *such that*

$$\alpha\Big(\big(gx_0, gy_0, gz_0\big), \big(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)\big)\Big) \geq 1,$$

(v) α *is* M*-invariant.*

Then there exist $(x, y, z) \in X \times X$ *such that* $gx = F(x, y, z)$, $gy = F(y, x, y)$ *and* $gz = F(z, y, x)$, *that is,* F *has a tripled coincidence point.*

Proof. Let $(x_0, y_0, z_0) \in X \times X$. Since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, x_0, y_0) \quad \text{and} \quad gz_1 = F(z_0, y_0, x_0).$$

Again from $F(X \times X \times X) \subseteq g(X)$, we can choose $x_2, y_2, z_2 \in X$ such that

$$gx_2 = F(x_1, y_1, z_1), \quad gy_2 = F(y_1, x_1, y_1) \quad \text{and} \quad gz_2 = F(z_1, y_1, x_1).$$

Continuing this process we can construct sequences $\{(gx_n)\}$, $\{(gy_n)\}$ and $\{(gz_n)\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n),\ gy_{n+1} = F(y_n, x_n, y_n) \text{ and } gz_{n+1} = F(z_n, y_n, x_n) \tag{3.2}$$

for all $n \geq 1$. Since

$$\begin{aligned}&\alpha((gx_0, gy_0, gz_0), (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)))\\ &= \alpha((gx_0, gy_0, gz_0), (gx_1, gy_1, gz_1)) \geq 1\end{aligned}$$

and F and g are (α)-admissible, we have

$$\begin{aligned}&\alpha((F(x_0,y_0,z_0), F(y_0,x_0,y_0), F(z_0,y_0,x_0)), (F(x_1,y_1,z_1), F(y_1,x_1,y_1), F(z_1,y_1,x_1)))\\ &= \alpha((gx_1, gy_1, gz_1), (gx_2, gy_2, gz_2)) \geq 1.\end{aligned}$$

Again, using the fact that F and g are (α)-admissible, we have

$$\begin{aligned}&\alpha((F(x_1,y_1,z_1), F(y_1,x_1,y_1), F(z_1,y_1,x_1)), (F(x_2,y_2,z_2), F(y_2,x_2,y_2), F(z_2,y_2,x_2)))\\ &= \alpha((gx_2, gy_2, gz_2), (gx_3, gy_3, gz_3)) \geq 1.\end{aligned}$$

By repeating this argument, we get

$$\begin{aligned}&\alpha\Big(\big(F(x_n, y_n, z_n), F(y_n, x_n, y_n), F(z_n, y_n, x_n)\big), \big(F(x_{n+1}, y_{n+1}, z_{n+1}),\\ &\qquad F(y_{n+1}, x_{n+1}, y_{n+1}), F(z_{n+1}, y_{n+1}, x_{n+1})\big)\Big)\\ &= \alpha((gx_{n-1}, gy_{n-1}, gz_{n-1}), (gx_n, gy_n, gz_n)) \geq 1.\end{aligned} \tag{3.3}$$

Since α is M-invariant, we have

$$(gx_{n-1}, gy_{n-1}, gz_{n-1}, gx_n, gy_n, gz_n) \in M. \tag{3.4}$$

If there exists $k \in N$ such that $(gx_{k+1}, gy_{k+1}, gz_{k+1}) = (gx_k, gy_k, gz_k)$ then $gx_k = gx_{k+1} = F(x_k, y_k, z_k)$, $gy_k = gy_{k+1} = F(y_k, x_k, y_k)$ and $gz_k = gz_{k+1} = F(z_k, y_k, x_k)$. Thus, (x_k, y_k, z_k) is a tripled coincidence point of F. This finishes the proof. Now we assume that $(gx_{k+1}, gy_{k+1}, gz_{k+1}) \neq (gx_k, gy_k, gz_k)$ for all $n \geq 0$. Thus, we have either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$ or $gz_{n+1} = F(z_n, y_n, x_n) \neq gz_n$ for all $n \geq 0$. From (3.1), (3.3) and (3.4), we have

$$\begin{aligned}
&\left[\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})}{3}\right] \\
&\leq \alpha\Big((gx_{n-1}, gy_{n-1}, gz_{n-1}), (gx_n, gy_n, gz_n)\Big) \\
&\quad \cdot \left[\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})}{3}\right] \\
&= \alpha\Big((gx_{n-1}, gy_{n-1}, gz_{n-1}), (gx_n, gy_n, gz_n)\Big) \left[\frac{d\big(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)\big)}{3}\right. \\
&\left. + \frac{d\big(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)\big) + d\big(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n)\big)}{3}\right] \\
&\leq \psi\left(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)}{3}\right).
\end{aligned} \tag{3.5}$$

From (3.5), we get

$$\begin{aligned}
&\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})}{3} \\
&\leq \psi\left(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)}{3}\right).
\end{aligned} \tag{3.6}$$

Since $\psi(t) < t$ for all $t > 0$, by repeating (3.6), we get

$$\begin{aligned}
&\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})}{3} \\
&\leq \psi^n\left(\frac{d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)}{3}\right).
\end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$. For $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n\left(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)}{3}\right) < \frac{\epsilon}{3}.$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\epsilon)$. Then, by using the triangle inequality, we have

$$\begin{aligned}
&\frac{d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)}{3} \\
&\qquad \le \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})}{3} \\
&\qquad\quad + \frac{d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) + d(z_{n+1}, z_{n+2})}{3} \\
&\qquad\quad + \frac{d(x_{n+2}, x_{n+3}) + d(y_{n+2}, y_{n+3}) + d(z_{n+2}, z_{n+3})}{3} \\
&\qquad\quad + \dots + \frac{d(x_{m-1}, x_m) + d(y_{m-1}, y_m) + d(z_{m-1}, z_m)}{3} \\
&\qquad = \sum_{k=n}^{m-1} \frac{d(gx_k, gx_{k+1}) + d(gy_k, gy_{k+1}) + d(gz_k, gz_{k+1})}{3} \\
&\qquad \le \sum_{k=n}^{m-1} \psi^k \left(\frac{d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)}{3} \right) \\
&\qquad \le \sum_{n \ge n(\epsilon)} \psi^n \left(\frac{d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)}{3} \right) \\
&\qquad < \frac{\epsilon}{3}.
\end{aligned} \tag{3.8}$$

This implies that $d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) < \epsilon$. This shows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequence in the metric space (X, d). Since (X, d) is complete, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are convergent, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \quad \lim_{n\to\infty} gy_n = y \quad \text{and} \quad \lim_{n\to\infty} gz_n = z.$$

From continuity of g, we get

$$\lim_{n\to\infty} g(gx_n) = gx, \quad \lim_{n\to\infty} g(gy_n) = gy \quad \text{and} \quad \lim_{n\to\infty} g(gz_n) = gz.$$

From (3.3) and commutativity of F and g, we have

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n), \tag{3.9}$$
$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n), \tag{3.10}$$
$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n). \tag{3.11}$$

We now show that $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$. Taking the limit as $n \to +\infty$ in (3.9), (3.10) and (3.11), by continuity of F, we get

$$\begin{aligned}
g(x) &= g(\lim_{n\to\infty} gx_{n+1}) = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n, z_n)) \\
&= \lim_{n\to\infty} F(gx_n, gy_n, gz_n) = F(x, y, z),
\end{aligned}$$

$$
\begin{aligned}
g(y) &= g(\lim_{n\to\infty} gy_{n+1}) = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n, y_n)) \\
&= \lim_{n\to\infty} F(gy_n, gx_n, gy_n) = F(y, x, y),
\end{aligned}
$$

and

$$
\begin{aligned}
g(z) &= g(\lim_{n\to\infty} gz_{n+1}) = \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} g(F(z_n, y_n, x_n)) \\
&= \lim_{n\to\infty} F(gz_n, gy_n, gx_n) = F(z, y, x).
\end{aligned}
$$

Thus we prove that $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$. $\square$

In the next theorem, we omit the continuity hypothesis of F.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and M be a nonempty subset of X^6 and let d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^3 \times X^3 \to [0, \infty)$, such that the following holds:*

$$
\alpha\big((gx, gy, gz), (gu, gv, gw)\big)\left[\frac{d(F(x, y, z), F(u, v, w)) + d(F(y, x, y), F(v, u, v))}{3}\right.
$$
$$
\left. + \frac{d(F(z, y, x), F(w, v, u))}{3}\right] \leq \psi\left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gw)}{3}\right)
$$

for all $x, y, z, u, v, w \in X$ with $(gx, gy, gz, gu, gv, gw) \in M$. Suppose also that

(i) *F and g are (α)-admissible,*

(ii) *For any three sequences (x_n), (y_n) and (z_n) in X with*

$$\alpha\big((x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1})\big) \geq 1$$

for all n, if there is $x, y, z \in X$ such that $lim_{n\to\infty} x_n = x$, $lim_{n\to\infty} y_n = y$ and $lim_{n\to\infty} z_n = z$, then $\alpha((x, y, z), (x_n, y_n, z_n)) \geq 1$ for all n,

(iii) *$F(X \times X \times X) \subseteq g(X)$, g is continuous and commutes with F and $(g(x), d)$ is complete,*

(iv) *there exist $x_0, y_0, z_0 \in X$ such that*

$$\alpha\Big(\big(gx_0, gy_0, gz_0\big), \big(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)\big)\Big) \geq 1,$$

(v) *α is M-invariant.*

Then there exist $(x, y, z) \in X \times X$ such that $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$, that is, F has a tripled coincidence point.

Proof. Proceeding exactly as in Theorem 3.1, we have that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences in the complete metric space $(g(X), d)$ and

$$\alpha\big((x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1})\big) \geq 1.$$

Then, there exist $gx, gy, gz \in g(X)$ such that $\{gx_n\} \to gx$, $\{gy_n\} \to gy$ and $\{gz_n\} \to gz$. By assumption (ii), we have

$$\alpha\big((gx, gy, gz), (gx_n, gy_n, gz_n)\big) \geq 1$$

for all n. From α is M-invariant, we have $(gx, gy, gz, gx_n, gy_n, gz_n) \in M$. Now by (3.1), the triangle inequality and $\psi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}
&\frac{d(F(x,y,z),gx) + d(F(y,x,y),gy) + d(F(z,y,x),gz)}{3} \\
&\leq \frac{d(F(x,y,z),gx_{n+1}) + d(gx_{n+1},gx) + d(F(y,x,y),gy_{n+1}) + d(gy_{n+1},gy)}{3} \\
&\quad + \frac{d(F(z,y,x),gz_{n+1}) + d(gz_{n+1},gz)}{3} \\
&= \frac{d(F(x,y,z),F(x_n,y_n,z_n)) + d(F(y,x,y),F(y_n,x_n,y_n))}{3} \\
&\quad + \frac{d(F(z,y,x),F(z_n,y_n,x_n)) + d(gx_{n+1},gx) + d(gy_{n+1},gy) + d(gz_{n+1},gz)}{3} \\
&\leq \alpha\big((gx,gy,gz),(gx_n,gy_n,gz_n)\big)\Bigg[\frac{d(F(x,y,z),F(x_n,y_n,z_n))}{3} \\
&\quad + \frac{d(F(y,x,y),F(y_n,x_n,z_n)) + d(F(z,y,x),F(z_n,y_n,x_n))}{3}\Bigg] \\
&\quad + \frac{d(gx_{n+1},gx) + d(gy_{n+1},gy) + d(gz_{n+1},gz)}{3} \\
&\leq \psi\left(\frac{d(gx,gx_n) + d(gy,gy_n) + d(gz,gz_n)}{3}\right) \\
&\quad + \frac{d(gx_{n+1},gx) + d(gy_{n+1},gy) + d(gz_{n+1},gz)}{3} \\
&< \frac{d(gx,gx_n)+d(gy,gy_n)+d(gz,gz_n)}{3} + \frac{d(gx_{n+1},gx)+d(gy_{n+1},gy)+d(gz_{n+1},gz)}{3}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$d(F(x,y,z),gx) + d(F(y,x,y),gy) + d(F(z,y,x),gz) = 0,$$

which implies that $F(x,y,z) = gx$, $F(y,x,y) = gy$ and $F(z,y,x) = gz$. Thus (x,y,z) is a tripled coincidence point of F and g. □

The following example is valid for Theorem 3.1.

Example 3.3. Let $X=\mathbb{R}$, $d(x,y)=|x-y|$ and $F: X\times X\times X\to X$ be defined by

$$F(x,y,z)=\frac{x+2y+3z}{8},\quad (x,y,z)\in X^3,$$

and $g: X\to X$ by $g(x)=\frac{3x}{2}$, clearly, F has no mixed g-monotone property.

Consider the mapping $\alpha: X^3\times X^3\to(0,+\infty]$ such that

$$\alpha\big((x,y,z),(u,v,w)\big)=\begin{cases}1 & \text{if } x\geq u,\ y\geq v \text{ and } z\geq w,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that F is α-admissible. Now, we claim that F satisfies Condition (3.1). If $\alpha((gx,gy,gz),(gu,gv,gw))=0$, then the result is straightforward. Let $x,y,z,u,v,w\in X$ and M=$\{(x,y,z,u,v,w)\in X^6 : x\geq u, y\geq v \quad \text{and } z\geq w\}$. Without loss of generality, assume that $gx\geq gu$, $gy\geq gv$ and $gz\geq gw$, we have $\alpha((gx,gy,gz),(gu,gv,gw))=1$ and $(gx,gy,gz,gu,gv,gw)\in M$. Then we have

$$\begin{aligned}
&\alpha\Big((gx,gy,gz),(gu,gv,gw)\Big)\\
&\quad\cdot\left[\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3}\right]\\
&=\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3}\\
&=\left|\frac{x+2y+3z}{24}-\frac{u+2v+3w}{24}\right|+\left|\frac{y+2x+3y}{24}-\frac{v+2u+3v}{24}\right|\\
&\quad+\left|\frac{z+2y+3x}{24}-\frac{w+2v+3u}{24}\right|\\
&\leq 6\left|\frac{x-u}{24}\right|+8\left|\frac{y-v}{24}\right|+4\left|\frac{z-w}{24}\right|\leq\frac{1}{3}(|x-u|+|y-v|+|z-w|).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}&=\frac{d\left(\frac{3x}{2},\frac{3u}{2}\right)+d\left(\frac{3y}{2},\frac{3v}{2}\right)+d\left(\frac{3z}{2},\frac{3w}{2}\right)}{3}\\
&=\frac{1}{2}(|x-u|+|y-v|+|z-w|).
\end{aligned}$$

Now, choose $\psi\in\Psi$ such that $\psi(t)=2t/3$, then

$$\begin{aligned}
&\alpha\Big((gx,gy,gz),(gu,gv,gw)\Big)\\
&\quad\cdot\left[\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3}\right]\\
&\leq\psi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}\right).
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold, we know that F has a tripled coincidence point $(0,0,0)$.

Example 3.4. Let $X = [0,1]$, $d(x,y) = |x-y|$ and $F : X \times X \times X \to X$ be defined by

$$F(x,y,z) = \frac{xyz}{4}, \quad (x,y,z) \in X^3,$$

and $g : X \to X$ by $g(x) = \frac{3x}{2}$, clearly, F has no a mixed g-monotone property.

Consider the mapping $\alpha : X^3 \times X^3 \to (0,+\infty]$ such that

$$\alpha\big((x,y,z),(u,v,w)\big) = \begin{cases} 1 & \text{if } x \le u,\ y \le v \text{ and } z \le w, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that F is α-admissible. Now, we claim that F satisfies Condition (3.1). If $\alpha((gx,gy,gz),(gu,gv,gw)) = 0$, then the result is straightforward. Let $x,y,z,u,v,w \in X$ and M=$\{(x,y,z,u,v,w) \in X^6 : x \le u, y \le v \quad \text{and } z \le w\}$. Without loss of generality, assume that $gx \le gu$, $gy \le gv$ and $gz \le gw$, we have $\alpha((gx,gy,gz),(gu,gv,gw)) = 1$ and $(gx,gy,gz,gu,gv,gw) \in M$. Then we have

$$\begin{aligned}
&\alpha\Big((gx,gy,gz),(gu,gv,gw)\Big) \\
&\quad \cdot \left[\frac{d(F(x,y,z),F(u,v,w)) + d(F(y,x,y),F(v,u,v)) + d(F(z,y,x),F(w,v,u))}{3}\right] \\
&= \frac{d(F(x,y,z),F(u,v,w)) + d(F(y,x,y),F(v,u,v)) + d(F(z,y,x),F(w,v,u))}{3} \\
&= \left|\frac{xyz}{12} - \frac{uvw}{12}\right| + \left|\frac{yxy}{12} - \frac{vuv}{12}\right| + \left|\frac{zyx}{12} - \frac{wvu}{12}\right| \\
&\le 3\left|\frac{x-u}{12}\right| + 4\left|\frac{y-v}{12}\right| + 2\left|\frac{z-w}{12}\right| \le \frac{1}{3}(|x-u| + |y-v| + |z-w|).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{d(gx,gu) + d(gy,gv) + d(gz,gw)}{3} &= \frac{d\left(\frac{3x}{2},\frac{3u}{2}\right) + d\left(\frac{3y}{2},\frac{3v}{2}\right) + d\left(\frac{3z}{2},\frac{3w}{2}\right)}{3} \\
&= \frac{1}{2}(|x-u| + |y-v| + |z-w|).
\end{aligned}$$

Now, choose $\psi \in \Psi$ such that $\psi(t) = 2t/3$, then

$$\begin{aligned}
&\alpha\Big((gx,gy,gz),(gu,gv,gw)\Big) \\
&\quad \cdot \left[\frac{d(F(x,y,z),F(u,v,w)) + d(F(y,x,y),F(v,u,v)) + d(F(z,y,x),F(w,v,u))}{3}\right] \\
&\le \psi\left(\frac{d(gx,gu) + d(gy,gv) + d(gz,gw)}{3}\right).
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold, we know that F has a tripled coincidence point $(0,0,0)$.

Next, we give a sufficient condition for the uniqueness of the coupled coincidence point in Theorem 3.1.

Theorem 3.5. *In addition to the hypotheses of Theorem* 3.1*, suppose that for every* $(x,y,z),(x^*,y^*,z^*) \in X \times X \times X$ *there exists* $(u,v,w) \in X \times X \times X$ *such that*

$$\alpha((gu,gv,gw),(gx,gy,gz)) \geq 1 \quad \textit{and} \quad \alpha((gu,gv,gw),(gx^*,gy^*,gz^*)) \geq 1.$$

Then F *and* g *have a unique tripled common fixed point, that is, there exists a unique* $(x,y,z) \in X \times X \times X$ *such that* $x = gx = F(x,y,z)$*,* $y = gy = F(y,x,y)$ *and* $z = gz = F(z,y,x)$*.*

Proof. From Theorem 3.1, the set of tripled coincidence point is non-empty. Suppose (x,y,z) and (x^*,y^*,z^*) are tripled coincidence point of F, that is

$$\begin{aligned} &gx = F(x,y,z),\ gy = F(y,x,y),\ gz = F(z,y,x) \quad \text{and} \\ &gx^* = F(x^*,y^*,z^*),\ gy^* = F(y^*,x^*,y^*),\ gz^* = F(z^*,y^*,x^*). \end{aligned}$$

We shall show that

$$gx^* = gx,\ gy^* = gy \text{ and } gz^* = gz. \tag{3.12}$$

Put $u_0 = u,\ v_0 = v,\ w_0 = w$ and choose $u_1,\ v_1,\ w_1 \in X$ such that

$$g(u_1) = F(u_0,v_0,w_0),\ g(v_1) = F(v_0,u_0,v_0) \text{ and } g(w_1) = F(w_0,v_0,u_0).$$

Then similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{g(u_n)\},\ \{g(v_n)\}$ and $\{g(w_n)\}$ such that for all $n \geq 1$,

$$\begin{gathered} g(u_n) = F(u_{n-1},v_{n-1},w_{n-1}),\ g(v_n) = F(v_{n-1},u_{n-1},v_{n-1}) \text{ and} \\ g(w_n) = F(w_{n-1},v_{n-1},u_{n-1}). \end{gathered}$$

By assumption, there is $(u,v,w) \in X \times X$ such that

$$\alpha((gu,gv,gw),(gx,gy,gz)) \geq 1 \quad \text{and} \quad \alpha((gu,gv,gw),(gx^*,gy^*,gz^*)) \geq 1.$$

Since F and g are α-admissible and $\alpha((gu_0,gv_0,gw_0),(gx,gy,gz)) \geq 1$, we have

$$\begin{aligned} &\alpha\big((F(u_0,v_0,w_0),F(v_0,u_0,v_0)F(w_0,v_0,u_0)),(F(x,y,z),F(y,x,y),F(z,y,x))\big) \\ &= \alpha((gu_1,gv_1,gw_1),(gx,gy,gz)) \geq 1. \end{aligned}$$

From this, if we use again the property of F and g are α-admissible, then it follow that

$$\begin{aligned} &\alpha\big((F(u_1,v_1,w_1),F(v_1,u_1,v_1)F(w_1,v_1,u_1)),(F(x,y,z),F(y,x,y),F(z,y,x))\big) \\ &= \alpha((gu_2,gv_2,gw_2),(gx,gy,gz)) \geq 1. \end{aligned}$$

By repeating this process, we get

$$\alpha((gu_n, gv_n, gw_n), (gx, gy, gz)) \geq 1. \quad \text{for all } n \geq 1. \tag{3.13}$$

Using the property of α is M-invariant, we have

$$(gu_n, gv_n, gw_n, gx, gy, gz) \in M. \tag{3.14}$$

From (3.1) , (3.13)and (3.14), we have

$$\begin{aligned}
&\frac{d(gu_{n+1}, gx) + d(gv_{n+1}, gy) + d(gw_{n+1}, gz)}{3} \\
&= \frac{d(F(u_n, v_n, w_n), F(x, y, z)) + d(F(v_n, u_n, v_n), F(y, x, y))}{3} \\
&\quad + \frac{d(F(w_n, v_n, u_n), F(z, y, x))}{3} \\
&\leq \alpha\Big((gu_n, gv_n, gw_n), (gx, gy, gz)\Big)\Bigg[\frac{d(F(u_n, v_n, w_n), F(x, y, z))}{3} \\
&\quad + \frac{d(F(v_n, u_n, v_n), F(y, x, y)) + d(F(w_n, v_n, u_n), F(z, y, x))}{3}\Bigg] \\
&\leq \psi\left(\frac{d(gu_n, gx) + d(gv_n, gy) + d(gw_n, gz)}{3}\right).
\end{aligned} \tag{3.15}$$

From(3.15), we have

$$\begin{aligned}
&\frac{d(gu_{n+1}, gx) + d(gv_{n+1}, gy) + d(gw_{n+1}, gz)}{3} \\
&\leq \psi\left(\frac{d(gu_n, gx) + d(gv_n, gy) + d(gw_n, gz)}{3}\right).
\end{aligned} \tag{3.16}$$

Since ψ is non-decreasing, from (3.16), we get

$$\begin{aligned}
&\frac{d(gu_{n+1}, gx) + d(gv_{n+1}, gy) + d(gw_{n+1}, gz)}{3} \\
&\leq \psi^n\left(\frac{d(gu_1, gx) + d(gv_1, gy) + d(gw_1, gz)}{3}\right)
\end{aligned} \tag{3.17}$$

for each $n \geq 1$. Letting $n \to +\infty$ in (3.17) and using lemma 2.7. Which implies

$$\lim_{n\to\infty} d(gu_{n+1}, gx) = \lim_{n\to\infty} d(gv_{n+1}, gy) = \lim_{n\to\infty} d(gw_{n+1}, gz) = 0. \tag{3.18}$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(gu_{n+1}, gx^*) = \lim_{n\to\infty} d(gv_{n+1}, gy^*) = \lim_{n\to\infty} d(gw_{n+1}, gz^*) = 0. \tag{3.19}$$

Hence, from (3.18), (3.19), we get $gx^* = gx$, $gy^* = gy$ and $gz^* = gz$.
Since $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$, by commutativity of F and g, we have

$$\begin{aligned} g(g(x)) &= g(F(x, y, z)) = F(g(x), g(y), g(z)), \\ g(g(y)) &= g(F(y, x, y)) = F(g(y), g(x), g(y)), \\ g(g(z)) &= g(F(z, y, x)) = F(g(z), g(y), g(x)). \end{aligned} \tag{3.20}$$

Denote $a = g(x)$, $b = g(y)$ and $c = g(z)$. Then from (3.20),

$$g(a) = F(a, b, c),\ \ g(b) = F(b, a, b) \text{ and } g(c) = F(c, b, a). \tag{3.21}$$

Thus (a, b, c) is a tripled coincidence point of F and g. Then from (3.12) with $x^* = a$, $y^* = b$ and $z^* = c$. It follows $g(x) = g(a)$, $g(y) = g(b)$ and $g(z) = g(c)$, that is,

$$g(a) = a,\ \ g(b) = b \text{ and } g(c) = c. \tag{3.22}$$

From (3.21) and (3.22),

$$a = g(a) = F(a, b, c),\ \ b = g(b) = F(b, a, b) \text{ and } w = g(c) = F(c, b, a).$$

Therefore, (a, b, c) is a tripled common fixed point of F and g.

To prove the uniqueness, assume that (p, q, r) is another coupled common fixed point. Then by (3.12) we have $p = g(p) = g(a) = a$, $q = g(q) = g(b) = b$ and $r = g(r) = g(c) = c$. □

Acknowledgement(s) : This research was supported by Chiang Mai University and the author would like to express sincere appreciation to Prof. Suthep Suantai for very helpful suggestions and many kind comments.

References

[1] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435-1443.

[2] J.J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation, Order 22 (2005) 223-239.

[3] J.J. Nieto, R. Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation, Acta Math. Sin. (Engl.Ser.) 23 (12) (2007) 2205-2212.

[4] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008) 1-8.

[5] D. Guo, V. Shmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (1987) 623-632.

[6] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379-1393.

[7] S.M. Alsulami, A.Alotaibi, Tripled coincidence points for monotone operators in partially ordered metric spaces, Int. Math. Forum 7 (2012) 1811-1824.

[8] M. Mursaleen, S.A. Mohiuddine, R.P. Agawal, Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2012:228 (2012) doi:10.1186/1687-1812-2012-228 [Corrigendum to Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2013:127 (2013) doi:10.1186/1687-1812-2013-127].

[9] M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (2012) 1319-1334.

[10] B. Samet, C. Vetro, Coupled fixed point F-invariant set and fixed point of N-order, Ann. Funct. Anal. 1 (2010) 46-56.

[11] B. Samet, C. Vetro, P. Vetro, Fixed point theorem for α-ψ-contractive type mappings, Nonlear Anal. 75 (2012) 2154-2165.

[12] A. Amini-Harandi, Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem, Math. Comput. Modelling 57 (2013) 2343-2348.

[13] N. Hussain, A. Latif, M.H. Shah, Coupled and tripled coincidence point results without compatibility, Fixed Point Theory Appl. 2012:77 (2012) doi:10.1186/1687-1812-2012-77.

[14] V. Berinde, Coupled fixed point theorems for φ-contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 75 (6) (2012) 3218-3228.

[15] V. Berinde, Coupled coincidence point theorems for mixed monotone nonlinear operators, Comput. Math. Appl. 64 (6) (2012) 1770-1777.

[16] W. Sintunavarat, P. Kumam, Y.J. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, Fixed Point Theory Appl. 2012:170 (2012) doi:10.1186/1687-1812-2012-170.

[17] W. Sintunavarat, S. Radenović, Z. Golubović, P. Kuman, Coupled fixed point theorems for F-invariant set, Appl. Math. Inf. Sci. 7 (1) (2013) 247-255.

[18] R. Batra, S. Vashistha, Coupled coincidence point theorems for nonlinear contractions under (F, g)-invariant set in cone metric spaces, J. Nonlinear Sci. Appl. 6 (2013) 86-96.

[19] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonliear Anal. 70 (2009) 4341-4349.

[20] E. Karapinar, A. Roldan, J. Martinez-Moreno, C. Roldan, Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces, Abstr. Appl. Anal. 2013:406026 (2013) doi:10.1155/2013/406026.

[21] N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983-992.

[22] S.A. Mohiuddine, A. Alotaibi, Some results on a tripled fixed point for nonlinear contractions in partially ordered G-metric spaces, Fixed Point Theory Appl. 2012:179 (2012) doi:10.1186/1687-1812-2012-179.

[23] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889-4897.

[24] V. Berinde, M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (10) 5929-5936.

[25] M. Abbas, H. Aydi, E. Karapinar, Tripled fixed points of multi-valued nonlinear contraction mappings in partially ordered metric spaces, Abstr. Appl. Anal. 2011 (2011). doi:10.1155/2011/812690.

[26] M. Abbas, B. Ali, W. Sintunavarat, P. Kumam, Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces, Fixed Point Theory Appl. 2012:187 (2012).

[27] H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence points for weak ϕ-contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2012:44 (2012) doi:10.1186/1687-1812-2012-44.

[28] H. Aydi, E. Karapinar, C. Vetro, Meir-Keeler type contractions for tripled fixed points, Acta Math. Sci. 32 (2012) 2119-2130.

[29] H. Aydi, E. Karapinar, New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces, Math. Probl. Eng. 2012:409872 (2012) doi:10.1155/2012/409872.

[30] H. Aydi, M. Abbas, W. Sintunavarat, P. Kumam, Tripled fixed point of W-compatible mappings in abstract metric spaces, Fixed Point Theory Appl. 2012:134 (2012) doi:0.1186/1687-1812-2012-134.

[31] H. Aydi, M. Abbas, Tripled coincidence and fixed point results in partial metric spaces, Appl. Gen. Topol. 13 (2012) 193-206.

[32] H. Aydi, E. Karapinar, Tripled Fixed Points in Ordered Metric Spaces, Bull. Math. Anal. Appl. 4 (2012) 197-207.

[33] H. Aydi, E. Karapinar, W. Shatanawi, Tripled coincidence point results for generalized contractions in ordered generalized metric spaces, Fixed Point Theory Appl. 2012:101. (2012) doi:0.1186/1687-1812-2012-101.

[34] M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (2012) 7339-7346.

[35] B.S. Choudhury, E. Karapinar, A. Kundu, Tripled Coincidence Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces, Int. J. Math. Math. Sci. 2012:329298 (2012) doi:10.1155/2012/329298.

[36] P. Charoensawan, Tripled fixed points theorems for ϕ-contractive mixed monotone operators on Partially ordered metric spaces, Appl. Math. Sci. 6 (2012) 5229-5239.

[37] P. Charoensawan, C. Klanarong, Coupled coincidence point theorems for φ-contractive under (f,g)-invariant set in complete metric space, Int. J. Math. Anal. 7 (33-36) (2013) 1685-1701.

[38] P. Charoensawan, Tripled fixed points theorems for a ϕ-contractive mapping in a complete metric spaces without the mixed g-monotone property, Fixed Point Theory Appl. 2013:252 (2013) doi:10.1186/1687-1812-2013-252.

[39] A. Roldan, J. Martinez-Moreno, C. Roldan, Multidimensional fixed point theorems in partially ordered complete metric spaces, J. Math. Anal. Appl. 396 (2012) 536-545.

[40] A. Roldan, J. Martinez-Moreno, C. Roldan, E. Karapinar, Multidimensional fixed point theorems in partially ordered complete partial metric spaces under (ψ,φ)-contractivity conditions, Abstr. Appl. Anal. 2013:634371 (2013) doi:10.1155/2013/634371.

[41] P. Kaushik, S. Kumar, P. Kuman, Coupled coincidence point theorems for α-ψ-contractive type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2013:325. (2013) doi:0.1186/1687-1812-2013-325.

(Received 10 March 2014)
(Accepted 17 February 2015)