



Characterizations of Fuzzy Ideals of Semigroups by Soft Sets

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Abstract : Let A be a subsemigroup of a semigroup S and Δ be a nonempty subset of $[0, 1]$. The aim of this paper is to discuss characterizations of fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy left ideals, fuzzy right ideals and fuzzy ideals of a semigroup S by using soft sets over $A \times \Delta$, over A and over Δ . For a nonempty family $\{f_i \mid i \in I\}$ of fuzzy subsets of S , we show equivalent conditions for the fuzzy subset $\bigwedge_{i \in I} f_i$ which is a fuzzy subsemigroup, a fuzzy generalized bi-ideal, a fuzzy bi-ideal, a fuzzy left ideal, a fuzzy right ideal and a fuzzy ideal of S by using soft sets over $A \times \Delta$ and over Δ . Finally, regular semigroups are characterized by soft sets over $S \times [0, 1]$ and over S .

Keywords : fuzzy bi-ideals; fuzzy ideals; soft bi-ideals; soft ideals.

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1 Introduction

In the real physical world, some classes of objects, such as the class of beautiful women, the class of animals, the class of tall men and others, do not have precisely defined criteria of membership by using the usual mathematical sense. However, they play an important role in human thinking, communication of information and abstraction. The theory of fuzzy sets and soft sets are mathematical methods for

solving such classes and also describing complex systems. From the past several years, there has been a rapid growth worldwide in the interest of fuzzy set theory and soft set theory with their applications seen in various fields such as: mathematics, medical science, computer science, control engineer, tourism, robotics, management science, expert systems and others. In this paper, we focus on fuzzy semigroup theory related to soft set theory.

In 1965 [1], Zadeh introduced a concept of a fuzzy subset (or fuzzy set) which was a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function which assigns to each object a grade of membership ranging between zero and one [1]. The concept of fuzzy algebraic structures developed from the concept of a fuzzy set was firstly studied by Rosenfeld [2]. Rosenfeld introduced the concept of fuzzy groups and showed that many theories in groups can be extended in an elementary manner to develop the theory of fuzzy groups. Since then, the literature of various fuzzy algebraic concepts has been growing very rapidly. For examples, Kuroki [3–8] studied some properties of fuzzy subsemigroups, fuzzy ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, fuzzy quasi-ideals of a semigroup S and characterized a semigroup S in terms of such fuzzy subsets of S . In particular, he showed that any fuzzy subset f is a fuzzy ideal of S if and only if for each $\alpha \in [0, 1]$ the upper level subset $U(f : \alpha) = \{x \in S \mid f(x) \geq \alpha\}$ is either empty or an ideal of S . In this paper, the concept of the upper level subset is used to create one method related to soft set theory. Another equivalent condition for such fuzzy subsets is the condition in term of the product “ \circ ” of fuzzy subsets, for example, any fuzzy subset f is a fuzzy ideal of S if and only if $C_S \circ f \leq f$ and $f \circ C_S \leq f$ [9]. Wang, Mo and Liu [10] characterized fuzzy ideals as fuzzy points of semigroups. The relation between subsets of semigroups and subsets of the set of all fuzzy points of the semigroups was studied, and necessary and sufficient conditions for fuzzy left (right, two sided, interior)-ideals of semigroups via fuzzy points were discussed by Kim [11]. Kazanci and Yilmaz [12] introduced special types of soft sets and used them to study relationships between soft bi-ideals and (generalized) fuzzy bi-ideals in semigroups. The concept of soft sets, introduced by Molodtsov in 1999 [13], offering a general mathematical method for dealing with uncertain, have not clearly defined objects. At present, works on soft set theory are progressing rapidly. Motivated and inspired by the works above, we are interested in mathematical methods, with respect to soft set theory, for characterizing of fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy interior ideals, fuzzy left ideals, fuzzy right ideals and fuzzy ideals of a semigroup.

We organize this paper as follows. In Section 2, we give basic definitions for this paper. In Section 3, we give certain patterns of soft sets over $A \times \Delta$, over A and over Δ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$. After that, we use the certain patterns of soft sets to characterize fuzzy subsemigroups in Section 3, fuzzy (generalized) bi-ideals in Section 4, and fuzzy ideals in Section 5. In particular, for any nonempty family $\{f_i \mid i \in I\}$ of fuzzy subsets of S , we show equivalent conditions for the fuzzy subset $\bigwedge_{i \in I} f_i$ which is a fuzzy subsemigroup, a fuzzy generalized bi-ideal, a fuzzy bi-ideal, a fuzzy left

ideal, a fuzzy right ideal and a fuzzy ideal of S by using the certain patterns of soft sets over $A \times \Delta$ and over Δ . In Section 6, we characterize regular semigroups by using the certain patterns of soft sets related to their fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy left ideals, fuzzy right ideals and fuzzy ideals.

2 Preliminaries

In this section, we shall give some basic definitions which are use in this paper. Let S be a semigroup. A nonempty subset A of S is called a **subsemigroup** of S if $AA \subseteq A$. A nonempty subset A of S is called a **left ideal** [**right ideal**] of S if $SA \subseteq A$ [$AS \subseteq A$]. A nonempty subset of S is called an **ideal** of S if it is both a left ideal and a right ideal of S . A nonempty subset A of S is called a **generalized bi-ideal** of S if $ASA \subseteq A$. A subsemigroup A of S is called a **bi-ideal** of S if $ASA \subseteq A$. In any semigroup S , a bi-ideal of S is generalized bi-ideal of S but the converse does not hold. Next, S is called **regular** if for each $x \in S$ there exists $a \in S$ such that $x = xax$. In a regular semigroup, we see that definitions of a bi-ideal and a generalized bi-ideal coincide.

For $\alpha, \beta \in [0, 1]$ and $\Delta, \Omega \subseteq [0, 1]$, let $\alpha \wedge \beta := \inf\{\alpha, \beta\}$ and $\Delta \wedge \Omega := \{\alpha \wedge \beta \mid \alpha \in \Delta, \beta \in \Omega\}$. Then, $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\Delta \wedge \Omega = \Omega \wedge \Delta$.

Let S be a semigroup. Define a binary operation “ $*$ ” on $S \times [0, 1]$ as follows:

$$(x, \alpha) * (y, \beta) := (xy, \alpha \wedge \beta) \text{ for all } (x, \alpha), (y, \beta) \in S \times [0, 1].$$

Then, $(S \times [0, 1], *)$ is a semigroup.

Remark 2.1. For any subsemigroup A of a semigroup S and any nonempty subset Δ of $[0, 1]$, we see that $(A \times \Delta, *)$ is a subsemigroup of the semigroup $(S \times [0, 1], *)$. In what follows, let $A \times \Delta$ denote the semigroup under the binary operation “ $*$ ” unless otherwise specified.

A function from a set X to $[0, 1]$ is called a **fuzzy set** (or a **fuzzy subset**) [1] of X . For fuzzy subsets f and g of X , define

$$f \leq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in X.$$

Let $Imf := \{f(x) \mid x \in X\}$ and $U(f : \alpha) := \{x \in X \mid f(x) \geq \alpha\}$ where $\alpha \in [0, 1]$. If $\alpha, \beta \in [0, 1]$ and $\alpha \leq \beta$, then $U(f : \beta) \subseteq U(f : \alpha)$ and hence the set $\{U(f : \alpha) \mid \alpha \in [0, 1]\}$ is a chain of subsets of X under the inclusion relation “ \subseteq ”. For a nonempty family $\{f_i \mid i \in I\}$ of fuzzy subsets of X , the fuzzy subset $\bigwedge_{i \in I} f_i$ of X is defined by

$$\left(\bigwedge_{i \in I} f_i\right)(x) := \inf\{f_i(x) \mid i \in I\} \text{ for all } x \in X.$$

Molodtsov [13] have introduced the basic concepts of the theory of soft sets which are a general mathematical tool for dealing with uncertainty, fuzzy and not

clearly defined objects.

Let U be an initial universe set and let E be a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and let X be a subset of E .

Definition 2.2. [13] If $F : X \rightarrow \mathcal{P}(U)$ is a mapping, then the pair (F, X) is called a **soft set** over U .

In other words, a soft set over U is a parameterized family of subsets of the universe U . For $x \in X$, $F(x)$ may be considered as the set of x -elements of the soft set (F, X) , or as the set of x -approximate elements of the soft set.

Definition 2.3. [14] Let (F, X) and (G, Y) be two soft sets over a set U . Then (F, X) is called a **soft subset** of (G, Y) if $X \subseteq Y$ and $F(x) \subseteq G(x)$ for all $x \in X$. $(F, X) = (G, Y)$ if (F, X) is a soft subset of (G, Y) and (G, Y) is a soft subset of (F, X) . We note here that $(F, X) = (G, Y)$ if and only if $X = Y$ and $F = G$.

Definition 2.4. [15] Let (F, X) and (G, X) be two soft sets over U . The **intersection** of (F, X) and (G, X) , denoted by $(F, X) \cap (G, X)$, is defined as the soft set (H, X) , where $H(x) = F(x) \cap G(x)$ for all $x \in X$.

Definition 2.5. [14] Let (F, X) and (G, Y) be two soft sets over a semigroup S . The **restricted product** of (F, X) and (G, Y) , denote by $(F, X) \hat{\circ} (G, Y)$, is defined as the soft set (H, Z) , where $Z = X \cap Y$ and $H(z) = F(z)G(z)$ for all $z \in Z$.

3 Fuzzy Subsemigroups of Semigroups

In this section, we discuss characterizations of fuzzy subsemigroups of a semigroup S by using soft semigroups over $A \times \Delta$, soft semigroups over A and soft sets over Δ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Let f be a fuzzy subset of a semigroup S , A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. For each $x \in A$, define

$$\begin{aligned} F_{A \times \Delta}^f(x) &= \{(a, \alpha) \in A \times \Delta \mid f(a) \geq f(x) \geq \alpha\}, \\ F_A^f(x) &= \{a \in A \mid f(a) \geq f(x)\}, \\ F_\Delta^f(x) &= \{\alpha \in \Delta \mid f(x) \geq \alpha\}, \\ G_{A \times \Delta}^f(x) &= \{(a, \alpha) \in A \times \Delta \mid f(x) \wedge f(a) \geq \alpha\}. \end{aligned}$$

Then $(F_{A \times \Delta}^f, A)$ and $(G_{A \times \Delta}^f, A)$ are soft sets over $A \times \Delta$, (F_A^f, A) is a soft set over A and (F_Δ^f, A) is a soft set over Δ . We see that $G_{A \times \Delta}^f(x) = F_{A \times \Delta}^f(x) \cup \{(a, \alpha) \in A \times \Delta \mid f(x) \geq f(a) \geq \alpha\}$, $F_A^f(x) \subseteq U(f : f(x))$, $F_\Delta^f(x) = \{\alpha \in \Delta \mid (a, \alpha) \in F^f(x) \text{ for some } a \in A\}$ and $F_{A \times \Delta}^f(x) = F_A^f(x) \times F_\Delta^f(x)$. If $A = S$, then $F_S^f(x) = U(f : f(x))$. The soft sets are main mathematical methods for study of fuzzy semigroup theory in this paper.

Definition 3.1. [3] A fuzzy subset f of a semigroup S is called a **fuzzy subsemigroup** of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

Definition 3.2. [14] A soft set (F, X) over a semigroup S is called a **soft semigroup** over S if $F(x)$ is either empty or a subsemigroup of S for all $x \in X$.

It is well known that if f and g are fuzzy subsemigroups of a semigroup S , then the fuzzy subset $f \wedge g$ is a fuzzy subsemigroup of S [9]. However, there exists a family $\{f_i \mid i \in I\}$ of fuzzy subsets, not fuzzy subsemigroups, of S such that $\bigwedge_{i \in I} f_i$ is a fuzzy subsemigroup of S . We shall see in Example 3.3.

Example 3.3. Let $S = \{a, b, c, d\}$ and define a binary operation “ \cdot ” on S as follows:

\cdot	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then, (S, \cdot) is a semigroup. Let f, g and h be fuzzy subsets of S such that

$$\begin{aligned} f(a) = f(b) = 0.6, & \quad f(c) = 0.7, & \quad f(d) = 0.5. \\ g(a) = 0.8, & \quad g(c) = 0.4, & \quad g(b) = g(d) = 0.9. \\ h(a) = 0.7, & \quad h(b) = h(c) = 0.8, & \quad h(d) = 0.3. \end{aligned}$$

Thus, f, g and h are not fuzzy subsemigroups of S , but $f \wedge g \wedge h$ is a fuzzy subsemigroup of S .

In Theorem 3.4, equivalent conditions for a fuzzy subsemigroup $\bigwedge_{i \in I} f_i$ of S are shown by using soft semigroups over $A \times \Delta$ and soft sets over Δ .

Theorem 3.4. Let $\{f_i \mid i \in I\}$ be a family of fuzzy subsets of a semigroup S . The following statements are equivalent.

1. The fuzzy subset $\bigwedge_{i \in I} f_i$ is a fuzzy subsemigroup of S .
2. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $\bigcap_{i \in I} (G_{A \times \Delta}^{f_i}, A)$ is a soft semigroup over $A \times \Delta$.
3. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $(F_\Delta, A) := \bigcap_{i \in I} (F_\Delta^{f_i}, A)$ over Δ satisfies the condition that $F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xy)$ for all $x, y \in A$.

Proof. • $(1 \Rightarrow 2)$: Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A$ and $(a, \alpha), (b, \beta) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Then $(a, \alpha), (b, \beta) \in G_{A \times \Delta}^{f_i}(x)$ for all $i \in I$, so we have $f_i(a) \wedge f_i(x) \geq \alpha$ and $f_i(b) \wedge f_i(x) \geq \beta$ for all $i \in I$. Thus

$\inf\{f_i(a) \mid i \in I\} \geq \alpha$, $\inf\{f_i(b) \mid i \in I\} \geq \beta$ and $\inf\{f_i(x) \mid i \in I\} \geq \alpha \wedge \beta$. By the statement (1), we have for all $i \in I$

$$f_i(ab) \geq \inf\{f_i(ab) \mid i \in I\} \geq \inf\{f_i(a) \mid i \in I\} \wedge \inf\{f_i(b) \mid i \in I\} \geq \alpha \wedge \beta.$$

Hence, $f_i(ab) \wedge f_i(x) \geq \alpha \wedge \beta$ for all $i \in I$. Since A is a subsemigroup of S and $\alpha, \beta \in \Delta$, we have $(ab, \alpha \wedge \beta) \in A \times \Delta$. Therefore, $(ab, \alpha \wedge \beta) \in G_{A \times \Delta}^{f_i}(x)$ for all $i \in I$. Consequently, $(ab, \alpha \wedge \beta) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. It is shown that $\bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$ is a subsemigroup of $A \times \Delta$.

• (2 \Rightarrow 3): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x, y \in A$ and $\alpha \in F_\Delta(x) \wedge F_\Delta(y)$. There exist $\beta \in F_\Delta(x)$ and $\gamma \in F_\Delta(y)$ such that $\alpha = \beta \wedge \gamma$. Then, $\beta \in F_\Delta^{f_i}(x)$ and $\gamma \in F_\Delta^{f_i}(y)$ for all $i \in I$, which imply that $f_i(x) \geq \beta$ and $f_i(y) \geq \gamma$ for all $i \in I$. Thus, for all $i \in I$,

$$f_i(x) \wedge f_i(y) \geq \beta \wedge \gamma = \alpha.$$

Hence, $(x, \alpha), (y, \alpha) \in G_{A \times \Delta}^{f_i}(x)$ for all $i \in I$ and so $(x, \alpha), (y, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. By the statement (2), we have $(xy, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Therefore, $(xy, \alpha) \in G_{A \times \Delta}^{f_i}(x)$ for all $i \in I$, which implies that

$$f_i(xy) \geq f_i(x) \wedge f_i(y) \geq \beta \wedge \gamma = \alpha$$

for all $i \in I$. Consequently, we get $\alpha \in F_\Delta(xy)$. It is shown that $F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xy)$.

• (3 \Rightarrow 1): Let $x, y \in S$. Then, $f_i(x) \geq \inf\{f_i(x) \mid i \in I\}$ and $f_i(y) \geq \inf\{f_i(y) \mid i \in I\}$ for all $i \in I$. Choose a nonempty subset Δ of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$. Thus $\inf\{f_i(x) \mid i \in I\} \in F_\Delta^{f_i}(x)$ and $\inf\{f_i(y) \mid i \in I\} \in F_\Delta^{f_i}(y)$ for all $i \in I$, so we have $\inf\{f_i(x) \mid i \in I\} \in F_\Delta(x)$ and $\inf\{f_i(y) \mid i \in I\} \in F_\Delta(y)$. By the statement (3), we get

$$\inf\{f_i(x) \mid i \in I\} \wedge \inf\{f_i(y) \mid i \in I\} \in F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xy).$$

Hence, $\inf\{f_i(x) \mid i \in I\} \wedge \inf\{f_i(y) \mid i \in I\} \in F_\Delta^{f_i}(xy)$ for all $i \in I$, which imply that $f_i(xy) \geq \inf\{f_i(x) \mid i \in I\} \wedge \inf\{f_i(y) \mid i \in I\}$ for all $i \in I$. Therefore,

$$\begin{aligned} \left(\bigwedge_{i \in I} f_i\right)(xy) &= \inf\{f_i(xy) \mid i \in I\} \\ &\geq \inf\{f_i(x) \mid i \in I\} \wedge \inf\{f_i(y) \mid i \in I\} \\ &= \left(\bigwedge_{i \in I} f_i\right)(x) \wedge \left(\bigwedge_{i \in I} f_i\right)(y). \end{aligned} \quad \square$$

Note that: if Δ is a nonempty subset of $[0, 1]$ and $\bigcap_{i \in I} (G_{S \times \Delta}^{f_i}, S)$ is a soft semi-group over $S \times \Delta$, then in the proof of Theorem 3.4 (2 \Rightarrow 3), we get $\bigcap_{i \in I} (F_\Delta^{f_i}, S)$ over Δ satisfies the condition as the statement (3). Hence, by the proofs of Theorem 3.4 (2 \Rightarrow 3) and (3 \Rightarrow 1), we have the following remark.

Remark 3.5. If Δ is a nonempty subset of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$ and one of the following two statements holds.

1. The soft set $\bigcap_{i \in I} (G_{S \times \Delta}^{f_i}, S)$ is a soft semigroup over $S \times \Delta$.
2. The soft set $(F_\Delta, S) := \bigcap_{i \in I} (F_\Delta^{f_i}, S)$ over Δ satisfies the condition that

$$F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xy) \text{ for all } x, y \in S.$$

Then $\bigwedge_{i \in I} f_i$ is a fuzzy subsemigroup of S .

In Theorem 3.6, we discuss equivalent conditions for fuzzy subsemigroups of S by using soft semigroups over $A \times \Delta$, soft semigroups over A and soft sets over $[0, 1]$ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Theorem 3.6. Let f be a fuzzy subset of S . The following statements are equivalent.

1. The fuzzy subset f is a fuzzy semigroup of S .
2. The soft set (F_A^f, A) is a soft semigroup over A for every subsemigroup A of S .
3. The soft set (F_S^f, S) is a soft semigroup over S .
4. The soft set $(F_{A \times \Delta}^f, A)$ is a soft semigroup over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .
5. The soft set $(F_{S \times \Delta}^f, S)$ is a soft semigroup over $S \times \Delta$ where $\text{Im} f \subseteq \Delta \subseteq [0, 1]$.
6. The soft set $(G_{A \times \Delta}^f, A)$ is a soft semigroup over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .
7. The soft set $(G_{S \times \Delta}^f, S)$ is a soft semigroup over $S \times \Delta$ where $\text{Im} f \subseteq \Delta \subseteq [0, 1]$.
8. The soft set $(F_{[0,1]}^f, S)$ over $[0, 1]$ satisfies the condition that $F_{[0,1]}^f(x) \wedge F_{[0,1]}^f(y) \subseteq F_{[0,1]}^f(xy)$ for all $x, y \in S$.

Proof. $(1 \Leftrightarrow 6 \Leftrightarrow 7 \Leftrightarrow 8)$ Follow from Theorem 3.4 and Remark 3.5. For the proofs of $(2 \Rightarrow 3)$ and $(4 \Rightarrow 5)$, they are clear.

• $(1 \Rightarrow 2)$: Let A be a subsemigroup of S , $x \in A$ and $a, b \in F_A^f(x)$. Then $f(a) \geq f(x)$ and $f(b) \geq f(x)$. By the statement (1) and A is a subsemigroup of S , we have $ab \in A$ and

$$f(ab) \geq f(a) \wedge f(b) \geq f(x).$$

Thus $ab \in F_A^f(x)$. Therefore $F_A^f(x)$ is a subsemigroup of A .

• (2 \Rightarrow 4): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A$ and $(a, \alpha), (b, \beta) \in F_{A \times \Delta}^f(x)$. Then $f(a) \geq f(x) \geq \alpha$ and $f(b) \geq f(x) \geq \beta$. Thus $a, b \in F_A^f(x)$. By the statement (2), we have $ab \in F_A^f(x)$. Hence $f(ab) \geq f(x) \geq \alpha \wedge \beta$, so we have $(ab, \alpha \wedge \beta) \in F_{A \times \Delta}^f(x)$. Therefore $F_{A \times \Delta}^f(x)$ is a subsemigroup of $A \times \Delta$.

• (3 \Rightarrow 5): Its proof is similar to the proof of (2 \Rightarrow 4).

• (5 \Rightarrow 1): Let $x, y \in S$ and choose a nonempty subset Δ of $[0, 1]$ such that $Imf \subseteq \Delta$. There exists $z \in S$ such that $f(z) = f(x) \wedge f(y)$. Since $Imf \subseteq \Delta$, we have $(x, f(z)), (y, f(z)) \in F_{S \times \Delta}^f(z)$. By the statement (5), we have $(F_{S \times \Delta}^f, S)$ is a soft semigroup over $S \times \Delta$. Hence $(xy, f(z)) \in F_{S \times \Delta}^f(z)$. Therefore, $f(xy) \geq f(z) = f(x) \wedge f(y)$. \square

4 Fuzzy (Generalized) Bi-Ideals of Semigroups

In this section, we discuss characterizations of fuzzy (generalized) bi-ideals of a semigroup S by using soft (generalized) bi-ideals over $A \times \Delta$, soft (generalized) bi-ideals over A and soft sets over Δ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Definition 4.1. [5] A fuzzy subset f of a semigroup S is called a **fuzzy generalized bi-ideal** of S if $f(xzy) \geq f(x) \wedge f(y)$ for all $x, y, z \in S$.

Definition 4.2. A soft set (F, X) over S is called a **soft generalized bi-ideal** over S if $F(x)$ is either empty or a general bi-ideal of S for all $x \in X$.

Let $\{f_i \mid i \in I\}$ be a family of fuzzy subsets of a semigroup S . In Theorem 4.3, we discuss equivalent conditions for a fuzzy generalized bi-ideal $\bigwedge_{i \in I} f_i$ of S by using soft generalized bi-ideals over $A \times \Delta$ and soft sets over Δ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Theorem 4.3. Let $\{f_i \mid i \in I\}$ be a family of fuzzy subsets of a semigroup S . The following statements are equivalent.

1. The fuzzy subset $\bigwedge_{i \in I} f_i$ is a fuzzy generalized bi-ideal of S .
2. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $\bigcap_{i \in I} (G_{A \times \Delta}^{f_i}, A)$ is a soft generalized bi-ideal over $A \times \Delta$.
3. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $(F_\Delta, A) := \bigcap_{i \in I} (F_\Delta^{f_i}, A)$ over Δ satisfies the condition that $F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xzy)$ for all $x, y, z \in A$.

Proof. • (1 \Rightarrow 2): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A, (c, \gamma) \in A \times \Delta$ and $(a, \alpha), (b, \beta) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Then, $\inf\{f_i(a) \mid i \in I\} \geq \alpha$, $\inf\{f_i(b) \mid i \in I\} \geq \beta$, $f_i(x) \geq \alpha \wedge \beta \wedge \gamma$ for all $i \in I$ and $(acb, \alpha \wedge \beta \wedge \gamma) \in A \times \Delta$. By the statement (1), we get

$$f_i(acb) \geq \inf\{f_i(acb) \mid i \in I\} \geq \inf\{f_i(a) \mid i \in I\} \wedge \inf\{f_i(b) \mid i \in I\} \geq \alpha \wedge \beta \wedge \gamma.$$

Thus, $f_i(acb) \wedge f_i(x) \geq \alpha \wedge \beta \wedge \gamma$ for all $i \in I$. Hence, $(acb, \alpha \wedge \beta \wedge \gamma) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Therefore, $\bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$ is a generalized bi-ideal of $A \times \Delta$.

• (2 \Rightarrow 3): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x, y, z \in A$ and $\alpha \in F_\Delta(x) \wedge F_\Delta(y)$. There exist $\beta \in F_\Delta(x)$ and $\gamma \in F_\Delta(y)$ such that $\alpha = \beta \wedge \gamma$. Then, $f_i(x) \wedge f_i(y) \geq \beta \wedge \gamma = \alpha$ for all $i \in I$. Thus, $(x, \alpha), (y, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Clearly $(z, \alpha) \in A \times \Delta$. By the statement (2), we have $(xzy, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Therefore,

$$f_i(xzy) \geq f_i(x) \wedge f_i(xzy) \geq \beta \wedge \gamma = \alpha$$

for all $i \in I$. Hence, $\alpha \in F_\Delta(xzy)$. It is shown that $F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xzy)$.

• (3 \Rightarrow 1): Let $x, y, z \in S$ and choose a nonempty subset Δ of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$. Then, $\inf\{f_i(x) \mid i \in I\} \in F_\Delta^{f_i}(x)$ and $\inf\{f_i(y) \mid i \in I\} \in F_\Delta^{f_i}(y)$ for all $i \in I$. By the statement (3), we have

$$\inf\{f_i(x) \mid i \in I\} \in F_\Delta^{f_i}(x) \wedge \inf\{f_i(y) \mid i \in I\} \in F_\Delta^{f_i}(y) \in F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xzy).$$

Therefore,

$$\begin{aligned} \left(\bigwedge_{i \in I} f_i\right)(xzy) &= \inf\{f_i(xzy) \mid i \in I\} \\ &\geq \inf\{f_i(x) \mid i \in I\} \wedge \inf\{f_i(y) \mid i \in I\} \\ &= \left(\bigwedge_{i \in I} f_i\right)(x) \wedge \left(\bigwedge_{i \in I} f_i\right)(y). \quad \square \end{aligned}$$

By the proofs of Theorem 4.3 (2 \Rightarrow 3) and (3 \Rightarrow 1), we have the following remark.

Remark 4.4. *If Δ is a nonempty subset of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$ and one of the following two statements holds.*

1. *The soft set $\bigcap_{i \in I} (G_{S \times \Delta}^{f_i}, S)$ is a soft generalized bi-ideal over $S \times \Delta$.*
2. *The soft set $(F_\Delta, S) := \bigcap_{i \in I} (F_\Delta^{f_i}, S)$ over Δ satisfies the condition that $F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xzy)$ for all $x, y, z \in S$.*

Then $\bigwedge_{i \in I} f_i$ is a fuzzy generalized bi-ideal of S .

In Theorem 4.5, we discuss equivalent conditions for fuzzy generalized bi-ideals of S by using soft generalized bi-ideals over $A \times \Delta$, soft generalized bi-ideals over A and soft sets over $[0, 1]$ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Theorem 4.5. *Let f be a fuzzy subset of S . The following statements are equivalent.*

1. *The fuzzy subset f is a fuzzy generalized bi-ideal of S .*
2. *The soft set (F_A^f, A) is a soft generalized bi-ideal over A for every subsemigroup A of S .*
3. *The soft set (F_S^f, S) is a soft generalized bi-ideal over S .*
4. *The soft set $(F_{A \times \Delta}^f, A)$ is a soft generalized bi-ideal over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .*
5. *The soft set $(F_{S \times \Delta}^f, S)$ is a soft generalized bi-ideal over $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.*
6. *The soft set $(G_{A \times \Delta}^f, A)$ is a soft generalized bi-ideal over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .*
7. *The soft set $(G_{S \times \Delta}^f, S)$ is a soft generalized bi-ideal over $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.*
8. *The soft set $(F_{[0,1]}^f, S)$ over $[0, 1]$ satisfies the condition that*

$$F_{[0,1]}^f(x) \wedge F_{[0,1]}^f(y) \subseteq F_{[0,1]}^f(xzy) \text{ for all } x, y, z \in S.$$

Proof. $(1 \Leftrightarrow 6 \Leftrightarrow 7 \Leftrightarrow 8)$ Follow from Theorem 4.3 and Remark 4.4. For the proofs of $(2 \Rightarrow 3)$ and $(4 \Rightarrow 5)$, they are clear.

• $(1 \Rightarrow 2)$: Let A be a subsemigroup of S . Let $x, y \in A$ and $a, b \in F_A^f(x)$. Then, $f(a) \wedge f(b) \geq f(x)$ and $ayb \in A$. By the statement (1), we get $f(ayb) \geq f(x)$. Thus, $ayb \in F_A^f(x)$. Therefore, $F_A^f(x)$ is a generalized bi-ideal of A .

• $(2 \Rightarrow 4)$: Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A$, $(y, \gamma) \in A \times \Delta$ and $(a, \alpha), (b, \beta) \in F_{A \times \Delta}^f(x)$. Then

$$f(a) \wedge f(b) \geq f(x) \geq \alpha \wedge \beta \geq \alpha \wedge \beta \wedge \gamma.$$

Thus, $a, b \in F_A^f(x)$. By the statement (2), we have $ayb \in F_A^f(x)$. Hence, $f(ayb) \geq f(x) \geq \alpha \wedge \beta \wedge \gamma$. Therefore, $(ayb, \alpha \wedge \beta \wedge \gamma) \in F_{A \times \Delta}^f(x)$. It conclude that $F_{A \times \Delta}^f(x)$ is a generalized bi-ideal of $A \times \Delta$.

• $(3 \Rightarrow 5)$: Its proof is similar to the proof of $(2 \Rightarrow 4)$.

• $(5 \Rightarrow 1)$: Let $x, y, z \in S$ and choose a nonempty subset Δ of $[0, 1]$ such that $Imf \subseteq \Delta$. There exists $a \in S$ such that $f(a) = f(x) \wedge f(y)$. Since $Imf \subseteq \Delta$, we have $(x, f(a)), (y, f(a)) \in F_{S \times \Delta}^f(a)$. Clearly, $(z, f(a)) \in S \times \Delta$. By the statement (5), we have $(xzy, f(a)) \in F_{S \times \Delta}^f(a)$. Therefore, $f(xzy) \geq f(a) = f(x) \wedge f(y)$. \square

Definition 4.6. [3] A fuzzy subsemigroup f of a semigroup S is called a **fuzzy bi-ideal** of S if $f(xzy) \geq f(x) \wedge f(y)$ for all $x, y, z \in S$.

Definition 4.7. [14] A soft set (F, X) over S is called a **soft bi-ideal** over S if $F(x)$ is either empty or a bi-ideal of S for all $x \in X$.

By the results of Section 3 and this section, we conclude characterizations for fuzzy bi-ideals of a semigroup S by using soft bi-ideals over $A \times \Delta$, soft bi-ideals over A and soft sets over Δ where A is a subsemigroup of S and Δ is a nonempty subset of $[0, 1]$.

Theorem 4.8. Let $\{f_i \mid i \in I\}$ be a family of fuzzy subsets of a semigroup S . The following statements are equivalent.

1. The fuzzy subset $\bigwedge_{i \in I} f_i$ is a fuzzy bi-ideal of S .
2. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $\bigcap_{i \in I} (G_{A \times \Delta}^{f_i}, A)$ is a soft bi-ideal over $A \times \Delta$.
3. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $(F_\Delta, A) := \bigcap_{i \in I} (F_\Delta^{f_i}, A)$ over Δ satisfies the condition that for all $x, y, z \in A$,

$$F_\Delta(x) \wedge F_\Delta(y) \subseteq F_\Delta(xy) \cap F_\Delta(xzy).$$

Proof. It follows from Theorem 3.4 and Theorem 4.3. \square

Theorem 4.9. Let f be a fuzzy subset of S . The following statements are equivalent.

1. The fuzzy subset f is a fuzzy bi-ideal of S .
2. The soft set (F_A^f, A) is a soft bi-ideal over A for every subsemigroup A of S .
3. The soft set (F_S^f, S) is a soft bi-ideal over S .
4. The soft set $(F_{A \times \Delta}^f, A)$ is a soft bi-ideal over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .
5. The soft set $(F_{S \times \Delta}^f, S)$ is a soft bi-ideal over $S \times \Delta$ where $Im f \subseteq \Delta \subseteq [0, 1]$.
6. The soft set $(G_{A \times \Delta}^f, A)$ is a soft bi-ideal over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .
7. The soft set $(G_{S \times \Delta}^f, S)$ is a soft bi-ideal over $S \times \Delta$ where $Im f \subseteq \Delta \subseteq [0, 1]$.
8. The soft set $(F_{[0,1]}^f, S)$ over $[0, 1]$ satisfies the condition that for all $x, y, z \in S$,

$$F_{[0,1]}^f(x) \wedge F_{[0,1]}^f(y) \subseteq F_{[0,1]}^f(xy) \cap F_{[0,1]}^f(xzy).$$

Proof. It follows from Theorem 3.6 and Theorem 4.5. \square

5 Fuzzy Ideals of Semigroups

In this section, we discuss characterizations of fuzzy right ideals, fuzzy left ideals and fuzzy ideals of a semigroup S by using soft sets. In proofs of results of this section, we will prove the right cases. For other cases, we can prove by analogous way.

Definition 5.1. [3] A fuzzy subset f of a semigroup S is called a **fuzzy right ideal** [resp., **left ideal**] of S if $f(xy) \geq f(x)$ [resp., $f(xy) \geq f(y)$] for all $x, y \in S$. A fuzzy subset f of S is called a **fuzzy ideal** of S if f is both a fuzzy left ideal and a fuzzy right ideal of S .

Definition 5.2. [14] A soft set (F, X) over a semigroup S is called a **soft right ideal** [resp., **soft left ideal**] over S if $F(x)$ is either empty or a right ideal [resp., left ideal] of S for all $x \in X$. A soft set (F, X) over S is called a **soft ideal** if (F, X) is both a soft left ideal and a soft ideal over S .

Theorem 5.3. Let $\{f_i \mid i \in I\}$ be a family of fuzzy subsets of a semigroup S . The following statements are equivalent.

1. The fuzzy subset $\bigwedge_{i \in I} f_i$ is a fuzzy right ideal [resp., fuzzy left ideal, fuzzy ideal] of S .
2. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $\bigcap_{i \in I} (G_{A \times \Delta}^{f_i}, A)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $A \times \Delta$.
3. For every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S , the soft set $(F_\Delta, A) := \bigcap_{i \in I} (F_\Delta^{f_i}, A)$ over Δ satisfies the condition that for all $x, y \in A$,

$$F_\Delta(x) \subseteq F_\Delta(xy) \text{ [resp., } F_\Delta(y) \subseteq F_\Delta(xy), F_\Delta(x) \cup F_\Delta(y) \subseteq F_\Delta(xy)].$$

Proof. • (1 \Rightarrow 2): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A$, $(a, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$ and $(b, \beta) \in A \times \Delta$. Then, $(ab, \alpha \wedge \beta) \in A \times \Delta$, $\inf\{f_i(a) \mid i \in I\} \geq \alpha$ and $\inf\{f_i(x) \mid i \in I\} \geq \alpha$. By the statement (1), we get

$$\begin{aligned} f_i(ab) \wedge f_i(x) &\geq \inf\{f_i(ab) \mid i \in I\} \wedge \inf\{f_i(x) \mid i \in I\} \\ &\geq \inf\{f_i(a) \mid i \in I\} \wedge \inf\{f_i(x) \mid i \in I\} \\ &\geq \alpha \wedge \beta \end{aligned}$$

for all $i \in I$. Hence, $(ab, \alpha \wedge \beta) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$. Therefore, $\bigcap_{i \in I} G_{A \times \Delta}^{f_i}(x)$ is a right ideal of $A \times \Delta$.

• (2 \Rightarrow 3): Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x, y \in A$ and $\alpha \in F_\Delta(x)$. Then, $f_i(x) \geq \alpha$ for all $i \in I$. Thus, $(x, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}$. Clearly, $(y, \alpha) \in A \times \Delta$. By the statement (2), we have $(xy, \alpha) \in \bigcap_{i \in I} G_{A \times \Delta}^{f_i}$. Hence,

$$f_i(xy) \geq f_i(x) \wedge f_i(xy) \geq \alpha$$

for all $i \in I$. Therefore, $\alpha \in F_\Delta(xy)$. Consequently, we get $F_\Delta(x) \subseteq F_\Delta(xy)$.

• (3 \Rightarrow 1): Let $x, y \in S$ and choose a nonempty subset Δ of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$. Then, $\inf\{f_i(x) \mid i \in I\} \in F_\Delta^{f_i}(x)$ for all $i \in I$. By the statement (3), we have

$$\inf\{f_i(x) \mid i \in I\} \in F_\Delta(x) \subseteq F_\Delta(xy).$$

Therefore, $(\bigwedge_{i \in I} f_i)(xy) \geq (\bigwedge_{i \in I} f_i)(x)$. \square

By the proofs of Theorem 5.3 (2 \Rightarrow 3) and (3 \Rightarrow 1), we get the following remark.

Remark 5.4. *If Δ is a nonempty subset of $[0, 1]$ such that $\bigwedge_{i \in I} \text{Im} f_i \subseteq \Delta$ and one of the following two statements holds.*

1. *The soft set $\bigcap_{i \in I} (G_{S \times \Delta}^{f_i}, S)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $S \times \Delta$.*
2. *The soft set $(F_\Delta, S) := \bigcap_{i \in I} (F_\Delta^{f_i}, S)$ over Δ satisfies the condition that for all $x \in S$,*

$$F_\Delta(x) \subseteq F_\Delta(xy) [\text{resp.}, F_\Delta(y) \subseteq F_\Delta(xy), F_\Delta(x) \cup F_\Delta(y) \subseteq F_\Delta(xy)].$$

Then $\bigwedge_{i \in I} f_i$ is a fuzzy right ideal [resp., fuzzy left ideal, fuzzy ideal] of S .

Theorem 5.5. *Let f be a fuzzy subset of S . The following statements are equivalent.*

1. *The fuzzy subset f is a fuzzy right ideal [resp., fuzzy left ideal, fuzzy ideal] of S .*
2. *The soft set (F_A^f, A) is a soft right ideal [resp., soft left ideal, soft ideal] over A for every subsemigroup A of S .*
3. *The soft set (F_S^f, S) is a soft right ideal [resp., soft left ideal, soft ideal] over S .*
4. *The soft set $(F_{A \times \Delta}^f, A)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .*
5. *The soft set $(F_{S \times \Delta}^f, S)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $S \times \Delta$ where $\text{Im} f \subseteq \Delta \subseteq [0, 1]$.*

- 6. The soft set $(G_{A \times \Delta}^f, A)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $A \times \Delta$ for every nonempty subset Δ of $[0, 1]$ and every subsemigroup A of S .
- 7. The soft set $(G_{S \times \Delta}^f, S)$ is a soft right ideal [resp., soft left ideal, soft ideal] over $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.
- 8. The soft set $(F_{[0,1]}^f, S)$ over $[0, 1]$ satisfies the condition that ,

$$F_{[0,1]}^f(x) \subseteq F_{[0,1]}^f(xy) \text{ [resp., } F_{[0,1]}^f(y) \subseteq F_{[0,1]}^f(xy), \\ F_{[0,1]}^f(x) \cup F_{[0,1]}^f(y) \subseteq F_{[0,1]}^f(xy)] \text{ for all } x, y, z \in S.$$

Proof. $(1 \Leftrightarrow 6 \Leftrightarrow 7 \Leftrightarrow 8)$ Follow from Theorem 5.3 and Remark 5.4. For the proofs of $(2 \Rightarrow 3)$ and $(4 \Rightarrow 5)$, they are clear.

- $(1 \Rightarrow 2)$: Let A be a subsemigroup of S . Let $x, y \in A$ and $a \in F_A^f(x)$. Then, $f(a) \geq f(x)$ and $ay \in A$. By the statement (1), we get $f(ay) \geq f(a) \geq f(x)$. Thus, $ay \in F_A^f(x)$. Therefore, $F_A^f(x)$ is right ideal of A .

- $(2 \Rightarrow 4)$: Let A be a subsemigroup of S and Δ be a nonempty subset of $[0, 1]$. Let $x \in A$, $(a, \alpha) \in F_{A \times \Delta}^f(x)$ and $(y, \beta) \in A \times \Delta$. Then,

$$f(a) \geq f(x) \geq \alpha \geq \alpha \wedge \beta.$$

Thus, $a \in F_A^f(x)$. By the statement (2), we have $ay \in F_A^f(x)$. Hence, $f(ay) \geq f(x) \geq \alpha \wedge \beta$. Therefore, $(ay, \alpha \wedge \beta) \in F_{A \times \Delta}^f(x)$. It is concluded that $F_{A \times \Delta}^f(x)$ is right ideal of $A \times \Delta$.

- $(3 \Rightarrow 5)$: Its proof is similar to the proof of $(2 \Rightarrow 4)$.
- $(5 \Rightarrow 1)$: Let $x, y \in S$ and choose a nonempty subset Δ of $[0, 1]$ such that $Imf \subseteq \Delta$. Then $(x, f(x)) \in F_{S \times \Delta}^f(x)$. Clearly $(y, f(x)) \in S \times \Delta$. By the statement (5), we have $(xy, f(x)) \in F_{S \times \Delta}^f(x)$. Therefore $f(xy) \geq f(x)$. □

6 Characterizations of Regular Semigroups

In this section, we characterize regular semigroups by soft right ideals, soft left ideals, soft ideals, soft generalized bi ideals and soft bi ideals with respect to fuzzy subsets.

Let f and g be fuzzy subsets of a semigroup S . The fuzzy subsets C_S and $f \circ g$ of S are defined by for all $x \in S$, $C_S(x) = 1$ and

$$(f \circ g)(x) = \begin{cases} \sup\{f(y) \wedge g(z) \mid (y, z) \in T_x\}, & \text{if } T_x \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}$$

where $T_x = \{(y, z) \in S \times S \mid x = yz\}$. It is well known that “ \circ ” is a binary operation on set of all fuzzy subsets of S and equivalent conditions for fuzzy left ideals, fuzzy right ideals, fuzzy ideals and fuzzy bi-ideals of S have studied in the terms of product “ \circ ” seen in Lemma 6.1.

Lemma 6.1. [9] *Let f a fuzzy subset of a semigroup S . The following properties hold.*

1. f is a fuzzy left ideal of S if and only if $C_S \circ f \leq f$.
2. f is a fuzzy right ideal of S if and only if $f \circ C_S \leq f$.
3. f is a fuzzy ideal of S if and only if $f \circ C_S \leq f$ and $C_S \circ f \leq f$.
4. f is a fuzzy bi-ideal of S if and only if $f \circ C_S \circ f \leq f$ and $f \circ f \leq f$.

Theorem 6.2. [9] *Let S be a semigroup. The following statements are equivalent.*

1. S is regular.
2. $f \wedge g = f \circ g$ for every fuzzy right ideal f and every fuzzy left ideal g of S .
3. $f \wedge g = f \circ g \circ f$ for every fuzzy bi-ideal f and every fuzzy ideal g of S .

Theorem 6.3. [9] *Let S be a semigroup. The following statements are equivalent.*

1. S is regular.
2. $R \cap L = RL$ for every right ideal R and every left ideal L of S .
3. $G \cap I = GIG$ for every generalized bi-ideal G and every ideal I of S .

Lemma 6.4. *Let f and g be fuzzy subsets of a semigroup S . The following statements are equivalent.*

1. $(F_S^f, S) \cap (F_S^g, S) = (F_S^f, S) \hat{\circ} (F_S^g, S)$.
2. $(F_{S \times [0,1]}^f, S) \cap (F_{S \times [0,1]}^g, S) = (F_{S \times [0,1]}^f, S) \hat{\circ} (F_{S \times [0,1]}^g, S)$.

Proof. • (1 \Rightarrow 2): Let $x \in S$ and $(a, \alpha) \in F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$. Then, $a \in F_S^f(x) \cap F_S^g(x)$ and $f(x) \wedge g(x) \geq \alpha$. By the statement (1), we have $a \in F_S^f(x) \cdot F_S^g(x)$. There exist $b \in F_S^f(x)$ and $c \in F_S^g(x)$ such that $a = bc$. Thus, $(b, \alpha) \in F_{S \times [0,1]}^f(x)$ and $(c, \alpha) \in F_{S \times [0,1]}^g(x)$. Hence,

$$(a, \alpha) = (b, \alpha) * (c, \alpha) \in F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x).$$

Therefore, $F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x) \subseteq F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x)$.

In order to see that $F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x) \subseteq F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$, let $(a, \alpha) \in F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x)$. There exist $(b, \beta) \in F_{S \times [0,1]}^f(x)$ and $(c, \gamma) \in F_{S \times [0,1]}^g(x)$ such that $(a, \alpha) = (b, \beta) * (c, \gamma)$. Then, $b \in F_S^f(x)$, $c \in F_S^g(x)$ and $f(x) \wedge g(x) \geq \alpha$. By the statement (1), we have

$$a = bc \in F_S^f(x) \cdot F_S^g(x) = F_S^f(x) \cap F_S^g(x).$$

Thus, $(a, \alpha) \in F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$. Hence,

$$F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x) \subseteq F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x).$$

It is concluded that $F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x) = F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$.

- (2 \Rightarrow 1): Let $x \in S$ and $a \in F_S^f(x) \cap F_S^g(x)$. By the statement (2), we get

$$(a, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x) = F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x).$$

Then, $(a, f(x) \wedge g(x)) = (b, f(x)) * (c, g(x))$ for some $(b, f(x)) \in F_{S \times [0,1]}^f(x)$, $(c, g(x)) \in F_{S \times [0,1]}^g(x)$. Thus, $a = bc \in F_S^f(x) \cdot F_S^g(x)$. Hence,

$$F_S^f(x) \cap F_S^g(x) \subseteq F_S^f(x) \cdot F_S^g(x).$$

In order to see that $F_S^f(x) \cdot F_S^g(x) \subseteq F_S^f(x) \cap F_S^g(x)$, let $a \in F_S^f(x) \cdot F_S^g(x)$. There exist $b \in F_S^f(x)$ and $c \in F_S^g(x)$ such that $a = bc$. By the statement (2), we have

$$(a, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x) = F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x).$$

Thus, $a \in F_S^f(x) \cap F_S^g(x)$. Hence, $F_S^f(x) \cdot F_S^g(x) \subseteq F_S^f(x) \cap F_S^g(x)$. It is concluded that $F_S^f(x) \cdot F_S^g(x) = F_S^f(x) \cap F_S^g(x)$. \square

Lemma 6.5. *Any semigroup S is regular if and only if $S \times \Delta$ is regular where $\emptyset \neq \Delta \subseteq [0, 1]$.*

In Theorem 6.6, we show equivalent conditions for regular semigroups by using soft left ideals, soft right ideals, soft ideals, soft bi ideals and soft generalized bi-ideals. Let f be a fuzzy subset of a semigroup S , we denote (F^f, S) is $(F_{S \times [0,1]}^f, S)$ or (F_S^f, S) . For every statement in Theorem 6.6, if choose $(F^f, S) = (F_{S \times [0,1]}^f, S)$, we must choose $(F^g, S) = (F_{S \times [0,1]}^g, S)$ where g is a fuzzy subset of S . On the other hand, if choose $(F^f, S) = (F_S^f, S)$, we must choose $(F^g, S) = (F_S^g, S)$. In the proof of Theorem 6.6, we choose $(F^f, S) = (F_{S \times [0,1]}^f, S)$ and $(F^g, S) = (F_{S \times [0,1]}^g, S)$. The other choice, we can use Lemma 6.4 for the proof.

Theorem 6.6. *Let S be a semigroup. The following statements are equivalent.*

1. S is regular.
2. $(F^f, S) \cap (F^g, S) = (F^f, S) \hat{\circ} (F^g, S)$ for every fuzzy left ideal f and every fuzzy right ideal g of S .
3. $(F^f, S) \cap (F^g, S) = (F^f, S) \hat{\circ} (F^g, S) \hat{\circ} (F^f, S)$ for every fuzzy bi-ideal f and every fuzzy ideal g of S .

4. $(F^f, S) \cap (F^g, S) = (F^f, S) \hat{\circ} (F^g, S) \hat{\circ} (F^f, S)$ for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S .

Proof. • (1 \Rightarrow 2): Use Theorem 5.5(1 \Rightarrow 5), Theorem 6.3(1 \Rightarrow 2) and Lemma 6.5.

• (1 \Rightarrow 4): Use Theorem 4.5(1 \Rightarrow 5), Theorem 5.5(1 \Rightarrow 5), Theorem 6.3(1 \Rightarrow 3) and Lemma 6.5.

• (4 \Rightarrow 3): It is clear.

• (2 \Rightarrow 1): Let f and g be a fuzzy left ideal and a fuzzy right ideal of S , respectively. By applying Lemma 6.1(1, 2), we get $f \circ g \leq f \wedge g$.

In order to see that $f \wedge g \leq f \circ g$, let $x \in S$. Then, $(x, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$. By the statement (2), we have $(x, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x)$. Thus, $(x, f(x) \wedge g(x)) = (y, f(x)) * (z, g(x))$ for some $(y, f(x)) \in F_{S \times [0,1]}^f(x)$, $(z, g(x)) \in F_{S \times [0,1]}^g(x)$. Hence,

$$(f \circ g)(x) = \sup\{f(a) \wedge g(b) \mid (a, b) \in T_x\} \geq f(y) \wedge g(z) \geq f(x) \wedge g(x).$$

Therefore, $f \wedge g \leq f \circ g$. It is concluded that $f \wedge g = f \circ g$. By Theorem 6.2(2 \Rightarrow 1), we get S is regular.

• (3 \Rightarrow 1): Let f and g be a fuzzy bi-ideal and a fuzzy ideal of S , respectively. By Applying Lemma 6.1(3, 4), we have $f \circ g \circ f \leq f \wedge g$. In order to see that $f \wedge g \leq f \circ g \circ f$, let $x \in S$. Then, $(x, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) \cap F_{S \times [0,1]}^g(x)$. By the statement (3), we have $(x, f(x) \wedge g(x)) \in F_{S \times [0,1]}^f(x) * F_{S \times [0,1]}^g(x) * F_{S \times [0,1]}^f(x)$. There exist $(x_1, f(x)), (x_2, f(x)) \in F_{S \times [0,1]}^f(x)$ and $(x_3, g(x)) \in F_{S \times [0,1]}^g(x)$ such that

$$(x, f(x) \wedge g(x)) = (x_1, f(x)) * (x_3, g(x)) * (x_2, f(x)).$$

Thus,

$$\begin{aligned} (f \circ (g \circ f))(x) &= \sup\{f(a) \wedge (g \circ f)(b) \mid (a, b) \in T_x\} \\ &\geq f(x_1) \wedge (g \circ f)(x_3 x_2) \\ &= f(x_1) \wedge \sup\{f(a) \wedge (g \circ f)(b) \mid (a, b) \in T_{x_3 x_2}\} \\ &\geq f(x_1) \wedge f(x_2) \wedge g(x_3) \\ &\geq f(x) \wedge g(x). \end{aligned}$$

Therefore, $f \wedge g \leq f \circ g \circ f$. It is concluded that $f \wedge g = f \circ g \circ f$. By Theorem 6.2(3 \Rightarrow 1), we get S is regular. \square

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