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Strong Convergence for ANI Mappings

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Abstract : In this paper, we establish a strong convergence theorem of the modified Noor iteration process for an ANI mapping such that its image is contained in a compact subset of Banach spaces.

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1 Introduction

Let C be a nonempty closed convex subset of a Banach space E, and let $T : C \to C$ be a mapping. Then
(i) T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(ii) T is *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\}$, $k_n \geq 1$ with $\lim\limits_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in C$ and $n \geq 1$;
(iii) T is *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$ and $n \geq 1$;
(iv) T is *asymptotically nonexpansive in the intermediate sence* (in brief, ANI) [2]

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provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian. Every asymptotically nonexpansive mapping is ANI but ANI mapping is not necessarily Lipschitzian.

Iterative methods for the approximation of fixed points of non-Lipschitzian mapping have been studied by Agarwal et al. [3], Bruck et al. [2], Chidume et al. [4], Kim and Kim [5] and many others.

In 1998, Takahashi and Kim [6] gave a strong convergence theorem of the Ishikawa iteration process for a nonexpansive mapping defined on a noncompact domain in a strictly convex Banach space. Two years later, Tsukiyama and Takahashi [7] generalized the Takahashi and Kim's result to a nonexpansive mapping under less restrictions on the parameters.

In 2014, Kim [8] generalized the result due to Takahashi and Kim [6] to an ANI-self mapping on the modified Ishikawa iteration process as the following result: Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to C$ be an ANI mapping such that $T(C)$ is contained in a compact subset of C and for $x_1 \in C$, and the sequence $\{x_n\}$ defined by

$$\begin{aligned} y_n &= \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$. If $\alpha_n \in [a, b]$ and $\limsup_{n\to\infty} \beta_n = b < 1$ or $\liminf_{n\to\infty} \alpha_n > 0$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$, then $\{x_n\}$ converges strongly to a fixed point of T.

In this paper, we generalize the result due to Kim [8] by consider on the modified Noor iteration process as the following. For $x_1 \in C$, let the sequence $\{x_n\}$ defined by

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

If $\gamma_n = 0$ for all $n \geq 1$, then the iteration process (1.2) is reduced to (1.1). If $\gamma_n = \beta_n = 0$ for all $n \geq 1$, then the iteration process (1.2) is reduced to the modified Mann iteration process [9].

We prove that the iteration $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T under the appropiate conditions of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$. Finally, we give some examples which satisfy all assumptions of T.

2 Preliminaries

We denote by $F(T)$, the set of all fixed point of T. We define the modulus of convexity for a convex subset of a Banach space (see also [10]). Let C be a nonempty bounded convex subset of a Banach space E with $d(C) > 0$, where $d(C) > 0$ is the diameter of C. Then we define $\delta(C, \epsilon)$ with $0 \leq \epsilon \leq 1$ as follows:

$$\delta(C, \epsilon) = \frac{1}{r} \inf \left\{ \max\left(\|x-y\|, \|y-z\|\right) - \left\| z - \frac{x+y}{2} \right\| : x, y, z \in C, \|x-y\| \geq r\epsilon \right\}$$

where $r = d(C)$.

Lemma 2.1. [11] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$\sum_{n=1}^{\infty} b_n < \infty \quad \text{and} \quad a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. [7] *Let C be a nonempty compact convex subset of a Banach space E with $r = d(C) > 0$. Let $x, y, z \in C$ and suppose $\|x - y\| \geq \epsilon r$ for some ϵ with $0 \leq \epsilon \leq 1$. Then, for all λ with $0 \leq \lambda \leq 1$,*

$$\|\lambda(x-z) + (1-\lambda)(y-z)\| \leq \max(\|x-z\|, \|y-z\|) - 2\lambda(1-\lambda)r\delta(C, \epsilon).$$

Lemma 2.3. [7] *Let C be a nonempty compact convex subset of a strictly convex Banach space E with $r = d(C) > 0$. If $\lim_{n\to\infty} \delta(C, \epsilon_n) = 0$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

3 Main Results

We give some results which will be used in our main result.

Lemma 3.1. *Let C be a nonempty compact convex subset of a Banach space E, and let $T : C \to C$ be an ANI mapping. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Suppose that the sequence $\{x_n\}$ is defined by (1.2). Then $\lim_{n\to\infty} \|x_n - w\|$ exists for any $w \in F(T)$.

Proof. By Schauder's fixed point theorem [12], we have $F(T) \neq \emptyset$. Let $w \in F(T)$.

Since

$$\begin{aligned}\|z_n - w\| &\le \gamma_n\|T^n x_n - w\| + (1-\gamma_n)\|x_n - w\|\\ &\le \gamma_n(\|x_n - w\| + c_n) + (1-\gamma_n)\|x_n - w\|\\ &\le \|x_n - w\| + c_n,\\ \|y_n - w\| &\le \beta_n\|T^n z_n - w\| + (1-\beta_n)\|x_n - w\|\\ &\le \beta_n(\|z_n - w\| + c_n) + (1-\beta_n)\|x_n - w\|\\ &\le \beta_n(\|x_n - w\| + 2c_n) + (1-\beta_n)\|x_n - w\|\\ &\le \|x_n - w\| + 2c_n,\end{aligned}$$

we obtain

$$\begin{aligned}\|x_{n+1} - w\| &\le \alpha_n\|T^n y_n - w\| + (1-\alpha_n)\|x_n - w\|\\ &\le \alpha_n(\|y_n - w\| + c_n) + (1-\alpha_n)\|x_n - w\|\\ &\le \alpha_n(\|x_n - w\| + 3c_n) + (1-\alpha_n)\|x_n - w\|\\ &\le \|x_n - w\| + 3c_n.\end{aligned}$$

By Lemma 2.1, we get $\lim_{n\to\infty}\|x_n - w\|$ exists. □

Lemma 3.2. *Let C be a nonempty compact convex subset of a strictly convex Banach space E with $r = d(C) > 0$. Let $T : C \to C$ be an ANI mapping. Put*

$$c_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Suppose $x_1 \in C$, and the sequence $\{x_n\}$ defined by (1.2).

(i) If $\beta_n \in [a,b]$ for some $a,b \in (0,1), 0 < \liminf_{n\to\infty}\alpha_n$ and $\limsup_{n\to\infty}\gamma_n < 1$, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

(ii) If $\alpha_n, \in [a,b]$ for some $a,b \in (0,1)$ and $\limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

(iii) If $\gamma_n \in [a,b]$ for some $a,b \in (0,1)$, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. By Schauder's fixed point theorem [12], we have $F(T) \neq \emptyset$. Let $w \in F(T)$.
(i) If $\alpha_n, \beta_n \in [a,b]$ for some $a,b \in (0,1), 0 < \liminf_{n\to\infty}\alpha_n$ and $\limsup_{n\to\infty}\gamma_n < 1$. We will show that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Let $\epsilon_n = \frac{\|T^n z_n - x_n\|}{r}$, then we have $0 \le \epsilon_n \le 1$ because $\|T^n z_n - x_n\| \le r$. Since $\|T^n z_n - x_n\| = r\epsilon_n$ and by Lemma 2.2, we have

$$\begin{aligned}\|y_n - w\| &= \|\beta_n T^n(z_n - w) + (1-\beta_n)(x_n - w)\|\\ &\le \max\{\|T^n z_n - w\|, \|x_n - w\|\} - 2\beta_n(1-\beta_n)r\delta(C,\epsilon_n)\\ &\le \max\{\|x_n - w\| + 2c_n, \|x_n - w\|\} - 2\beta_n(1-\beta_n)r\delta(C,\epsilon_n)\\ &= \|x_n - w\| + 2c_n - 2\beta_n(1-\beta_n)r\delta(C,\epsilon_n).\end{aligned}$$

And so

$$2\beta_n(1-\beta_n)r\delta(C,\epsilon_n) \le \|x_n - w\| - \|y_n - w\| + 2c_n. \tag{3.1}$$

Since

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n(T^n y_n - w) + (1-\alpha_n)(x_n - w)\| \\ &\le \alpha_n\|T^n y_n - w\| + (1-\alpha_n)\|x_n - w\| \\ &\le \alpha_n[\|y_n - w\| + c_n] + (1-\alpha_n)\|x_n - w\|, \end{aligned}$$

we have

$$\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \le \|y_n - w\| - \|x_n - w\| + 2c_n. \tag{3.2}$$

Since $0 < \liminf_{n\to\infty} \alpha_n$, there is a positive integer n_0 and a positive number k such that $\alpha_n \ge k > 0$ for all $n \ge n_0$ and by (3.2), we obtain

$$\|x_n - w\| - \|y_n - w\| \le \frac{\|x_n - w\| - \|x_{n+1} - w\|}{k} + 2c_n. \tag{3.3}$$

From (3.1) and (3.3) it follow that

$$2\beta_n(1-\beta_n)r\delta(C,\epsilon_n) \le \frac{\|x_n - w\| - \|x_{n+1} - w\|}{k} + 2c_n. \tag{3.4}$$

Since $\beta_n \in [a,b]$,

$$2a(1-b)r\delta(C,\epsilon_n) \le \frac{\|x_n - w\| - \|x_{n+1} - w\|}{k} + 2c_n. \tag{3.5}$$

And so

$$2r\sum_{n=1}^{\infty} a(1-b)\delta(C, \frac{\|T^n z_n - x_n\|}{r}) < \infty.$$

Thus $\lim_{n\to\infty} \delta(C, \frac{\|T^n z_n - x_n\|}{r}) = 0$. By Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{3.6}$$

Since

$$\|y_n - x_n\| = \beta_n\|T^n z_n - x_n\| \le b\|T^n z_n - x_n\| \to 0 \text{ as } n \to \infty \tag{3.7}$$

and

$$\begin{aligned} \|T^n x_n - x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\| \\ &\le \|x_n - z_n\| + c_n + \|T^n z_n - x_n\| \\ &= \gamma_n\|T^n x_n - x_n\| + c_n + \|T^n z_n - x_n\|, \end{aligned}$$

we obtain

$$(1-\gamma_n)\|T^n x_n - x_n\| \leq \|T^n z_n - x_n\| + c_n.$$

Since $\limsup_{n\to\infty}\gamma_n < 1$, we have

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{3.8}$$

It follows that

$$\begin{aligned}\|z_n - y_n\| &= \|\gamma_n T^n x_n + (1-\gamma_n)x_n - y_n\| \\ &\leq \gamma_n\|T^n x_n - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.9}$$

Since

$$\begin{aligned}\|x_{n+1} - x_n\| &= \alpha_n\|T^n y_n - x_n\| \\ &\leq \|T^n y_n - T^n z_n\| + \|T^n z_n - x_n\| \\ &\leq \|y_n - z_n\| + c_n + \|T^n z_n - x_n\|\end{aligned}$$

and by using (3.6) and (3.9), we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Since

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &+ \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|x_n - x_{n+1}\| + c_{n+1} \\ &+ \|T^{n+1}x_n - Tx_n\| \\ &= 2\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + c_{n+1} \\ &+ \|T(T^n x_n - x_n)\|\end{aligned} \tag{3.11}$$

and by the uniform continuity of T, (3.8) and (3.10), we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.12}$$

(ii) Let $\alpha_n \in [a,b]$ for some $a,b\in(0,1)$ and $\limsup_{n\to\infty}(\beta_n+\gamma_n) < 1$. We will show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Let $\epsilon_n = \frac{\|T^n y_n - x_n\|}{r}$, then we have $0\leq\epsilon_n\leq 1$. Since $\|T^n y_n - x_n\| = r\epsilon_n$ and by Lemma 2.2, we have

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n T^n y_n + (1-\alpha_n)x_n - w\| \\ &= \|\alpha_n(T^n y_n - w) + (1-\alpha_n)(x_n - w)\| \\ &\leq \max\{\|T^n y_n - w\|, \|x_n - w\|\} - 2\alpha_n(1-\alpha_n)r\delta(C,\epsilon_n) \\ &\leq \max\{\|x_n - w\| + 3c_n, \|x_n - w\|\} - 2\alpha_n(1-\alpha_n)r\delta(C,\epsilon_n) \\ &= \|x_n - w\| + 3c_n - 2\alpha_n(1-\alpha_n)r\delta(C,\epsilon_n).\end{aligned}$$

And so

$$2\alpha_n(1-\alpha_n)r\delta(C,\epsilon_n) \le \|x_n - w\| - \|x_{n+1} - w\| + 3c_n.$$

Since $2r\sum_{n=1}^{\infty} a(1-b)\delta(C, \frac{\|T^n y_n - x_n\|}{r}) < \infty$, we have

$$\lim_{n\to\infty} \delta(C, \frac{\|T^n y_n - x_n\|}{r}) = 0.$$

By Lemma 2.3, we get

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{3.13}$$

Since

$$\begin{aligned}
\|T^n x_n - x_n\| &\le \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
&\le \|x_n - y_n\| + c_n + \|T^n y_n - x_n\| \\
&= \beta_n\|T^n z_n - x_n\| + c_n + \|T^n y_n - x_n\| \\
&= \beta_n[\|T^n z_n - T^n x_n\| + \|T^n x_n - x_n\|] + c_n + \|T^n y_n - x_n\| \\
&\le \beta_n[\|z_n - x_n\| + c_n + \|T^n x_n - x_n\|] + c_n + \|T^n y_n - x_n\| \\
&= \beta_n[\gamma_n\|T^n x_n - x_n\| + c_n + \|T^n x_n - x_n\|] + c_n + \|T^n y_n - x_n\|,
\end{aligned}$$

we have

$$(1-\beta_n-\gamma_n)\|T^n x_n - x_n\| \le 2c_n + \|T^n y_n - x_n\|.$$

From (3.13), $\sum_{n=1}^{\infty} c_n < \infty$ and $\limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$, it follows that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

Since

$$\|x_{n+1} - x_n\| = \|(1-\alpha_n)x_n + \alpha_n T^n y_n - x_n\| = \alpha_n\|T^n y_n - x_n\| \le b\|T^n y_n - x_n\|,$$

and by (3.13), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Since

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\
&\quad + \|T^{n+1}x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|x_n - x_{n+1}\| + c_{n+1} \\
&\quad + \|T^{n+1}x_n - Tx_n\| \\
&= 2\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + c_{n+1} \\
&\quad + \|T(T^n x_n - x_n)\|
\end{aligned} \tag{3.15}$$

by the uniform continuity of T, $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$ and (3.14), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.16}$$

(iii) Let $\gamma_n \in [a,b]$ for some a,b with $0 < a \le b < 1$. We will show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Let $\epsilon_n = \frac{\|T^n x_n - x_n\|}{r}$, then we have $0 \le \epsilon_n \le 1$. Since

$$\begin{aligned}\|z_n - w\| &= \|\gamma_n(T^n x_n - w) + (1-\gamma_n)(x_n - w)\|\\ &\le \max\{\|T^n x_n - w\|, \|x_n - w\|\} - 2\gamma_n(1-\gamma_n)2\delta(C,\epsilon_n)\\ &\le \max\{\|x_n - w\| + c_n, \|x_n - w\|\} - 2\gamma_n(1-\gamma_n)2\delta(C,\epsilon_n)\\ &= \|x_n - w\| + c_n - 2\gamma_n(1-\gamma_n)2\delta(C,\epsilon_n). \qquad (3.17)\end{aligned}$$

And so

$$2\gamma_n(1-\gamma_n)2\delta(C,\epsilon_n) \le \|x_n - w\| - \|z_n - w\| + c_n.$$

It implies that

$$2r\sum_{n=1}^{\infty} a(1-b)\delta(C, \frac{\|T^n x_n - x_n\|}{r}) < \infty.$$

Hence

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \qquad (3.18)$$

Since

$$\|z_n - x_n\| = \|\gamma_n T^n x_n + (1-\gamma_n)x_n - x_n\| = \gamma_n \|T^n x_n - x_n\| \le \|T^n x_n - x_n\|$$

and by using (3.18), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \qquad (3.19)$$

Since

$$\begin{aligned}\|T^n z_n - x_n\| &= \|T^n z_n - T^n x_n + T^n x_n - x_n\|\\ &\le \|z_n - x_n\| + c_n + \|T^n x_n - x_n\|,\end{aligned}$$

by (3.18) and (3.19), we have

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \qquad (3.20)$$

Since $\|y_n - x_n\| = \|(1-\beta_n)x_n + \beta_n T^n z_n - x_n\| = \beta_n \|T^n z_n - x_n\|$ and (3.20), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \qquad (3.21)$$

Since $\|y_n - z_n\| = \|y_n - x_n + x_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\|$, by using (3.20) and (3.21), we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \qquad (3.22)$$

Since

$$\|T^n y_n - x_n\| = \|T^n y_n - T^n z_n + T^n z_n - x_n\| \leq \|y_n - z_n\| + c_n + \|T^n z_n - x_n\|,$$

by using (3.20) and (3.22), we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{3.23}$$

Since $\|T^n y_n - y_n\| = \|T^n y_n - x_n + x_n - y_n\| \leq \|T^n y_n - x_n\| + \|x_n - y_n\|$, by using (3.21) and (3.23), we obtain

$$\lim_{n\to\infty} \|T^n y_n - y_n\| = 0. \tag{3.24}$$

Since

$$\begin{aligned}\|T^n z_n - y_n\| &\leq \|T^n z_n - T^n x_n\| + \|T^n x_n - T^n y_n\| + \|T^n y_n - y_n\| \\ &\leq \|z_n - x_n\| + c_n + \|x_n - y_n\| + c_n + \|T^n y_n - y_n\| \\ &\leq \|z_n - x_n\| + \|x_n - y_n\| + 2c_n + \|T^n y_n - y_n\|,\end{aligned}$$

by using (3.19), (3.21) and (3.24), we obtain

$$\lim_{n\to\infty} \|T^n z_n - y_n\| = 0. \tag{3.25}$$

Since

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|(1-\alpha_{n-1})x_{n-1} + \alpha_{n-1}T^{n-1}y_{n-1} - x_{n-1}\| \\ &\leq \alpha_{n-1}\|T^{n-1}y_{n-1} - x_{n-1}\|\end{aligned}$$

and by (3.23), we get

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{3.26}$$

From

$$\begin{aligned}\|T^{n-1}x_n - x_n\| &\leq \|T^{n-1}x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\leq \|x_n - x_{n-1}\| + c_{n-1} + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &= 2\|x_n - x_{n-1}\| + c_{n-1} + \|T^{n-1}x_{n-1} - x_{n-1}\|\end{aligned}$$

and by (3.18) and (3.26), we obtain

$$\lim_{n\to\infty} \|T^{n-1}x_n - x_n\| = 0. \tag{3.27}$$

Since

$$\begin{aligned}\|x_n - Tx_n\| \quad &\leq \quad \|x_n - y_n\| + \|y_n - T^n z_n\| + \|T^n z_n - T^n y_n\| + \|T^n y_n - T^n x_n\| \\ &\quad + \quad \|T^n x_n - Tx_n\| \\ &\leq \quad \|x_n - y_n\| + \|y_n - T^n z_n\| + \|z_n - y_n\| + c_n + \|y_n - x_n\| + c_n \\ &\quad + \quad \|T^n x_n - Tx_n\| \\ &= \quad 2\|x_n - y_n\| + \|y_n - T^n z_n\| + \|z_n - y_n\| + 2c_n + \|T^n x_n - Tx_n\|\end{aligned}$$

and by the uniform continuity of T, (3.21), (3.22), (3.25) and (3.27), we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let $T : C \to C$ be an ANI mapping, and let $T(C)$ be contained in a compact subset of C. Put*

$$c_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Suppose $x_1 \in C$ and the sequence $\{x_n\}$ defined by (1.2). If

(i) $\beta_n \in [a, b]$ for some $a, b \in (0, 1), 0 < \liminf_{n\to\infty} \alpha_n$ and $\limsup_{n\to\infty} \gamma_n < 1$ or

(ii) $\alpha_n, \in [a, b]$ for some $a, b \in (0, 1)$ and $\limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ or

(iii) $\gamma_n \in [a, b]$ for some $a, b \in (0, 1)$,

then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $T(C)$ be contained in a compact subset of C, by Mazur's theorem [13] implies that $A := \overline{co}(\{x_1\} \cup T(C))$ is a compact subset of C containing $\{x_n\}$ which is invariant under T. Without loss of generality, we may assume that C is compact and $\{x_n\}$ is well defined. By Schauder's fixed point theorem [12], we have $F(T) \neq \emptyset$. If $d(C) = 0$, then done. So, we assume $d(C) > 0$. From Lemma 3.2 , we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.28}$$

Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and a point $w \in C$ such that $x_{n_k} \to w$. Thus we obtain $w \in F(T)$ by the continuity of T and (3.28). Hence, we obtain $\lim_{n\to\infty} \|x_n - w\| = 0$ by Lemma 3.2. □

For $\gamma_n \equiv 0$ in Theorem 3.3, we obtain the Kim's result as the following.

Corollary 3.4. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let $T : C \to C$ be an ANI mapping, and let $T(C)$ be contained in a compact subset of C. Put*

$$c_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Suppose $x_1 \in C$ and the sequence $\{x_n\}$ defined by (1.1) satisfies

(i) $\alpha_n \in [a, b]$ for some a, b with $0 < a \le b < 1$ and $\limsup_{n\to\infty} \beta_n < 1$ or

(ii) $\beta_n \in [a, b]$ for some a, b with $0 < a \le b < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

If T in Theorem 3.3 is an asymptotically nonexpansive mapping, then we have the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1, \sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $T(C)$ be contained in a compact subset of C. Suppose $x_1 \in C$ and the sequence $\{x_n\}$ defined by (1.2) satisfies*

(i) $\beta_n \in [a, b]$ for some $a, b \in (0, 1), 0 < \liminf_{n\to\infty} \alpha_n$ and $\limsup_{n\to\infty} \gamma_n < 1$ or

(ii) $\alpha_n, \in [a, b]$ for some $a, b \in (0, 1)$ and $\limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ or

(iii) $\gamma_n \in [a, b]$ for some $a, b \in (0, 1)$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

4 Examples

We give some mappings which is ANI but is not Lipschitzian.

Example 4.1. Let $E := \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers and $C := [0, 4]$. Define $T : C \to C$ by

$$Tx = \begin{cases} 2, & x \in [0, 2]; \\ \sqrt{4-x}, & x \in [2, 4]. \end{cases} \tag{4.1}$$

We see that $T^n x = 2$ for all $x \in C$ and $n \geq 2$ and $F(T) = \{2\}$. Clearly, T is uniformly continuous and ANI on C. Next, we will show that T is not a Lipschitzian mapping. Suppose not, i.e., there exists $L > 0$ such that

$$|Tx - Ty| \leq L|x - y|$$

for all $x, y \in C$. If we choose $y = 4$ and $x = 4 - \frac{1}{(L+1)^2} > 3$, then

$$\sqrt{4-x} \leq L(4-x) \Leftrightarrow \frac{1}{L} \leq \sqrt{4-x} \Leftrightarrow \frac{1}{L^2} \leq 4 - x = \frac{1}{(L+1)^2} \Leftrightarrow L + 1 \leq L.$$

This is a contradiction.

Example 4.2. Let $E := \mathbb{R}$ and $C := [-2\pi, 2\pi]$ and let $|h| < 1$. Let $T : C \to C$ be defined by $Tx = hx \cos nx$ for each $x \in C$ and for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Clearly, $F(T) = \{0\}$. Since

$$\begin{aligned} Tx &= hx \cos nx \\ T^2 x &= h(hx \cos nx) \cos n(hx \cos nx) = h^2 x \cos nx \cos n(hx \cos nx) \ldots, \end{aligned}$$

we obtain $T^n x \to 0$ uniformly on C as $n \to \infty$. Thus T is an ANI mapping. Next, we will show that T is not a Lipschitzian mapping. Suppose that there exists $h > 0$ such that

$$|Tx - Ty| \leq h|x - y|$$

for all $x, y \in C$. If we choose $x = \frac{2\pi}{n}$ and $y = \frac{\pi}{n}$, then

$$|Tx - Ty| = \left| h(\frac{2\pi}{n})\cos(\frac{2\pi}{n}) - h(\frac{\pi}{n})\cos(\frac{\pi}{n}) \right| = |2\pi h + \pi h| = 3\pi h,$$

and since

$$h|x - y| = h\left|\frac{2\pi}{n} - \frac{\pi}{n}\right| = \frac{\pi h}{n}.$$

Hence T is not a Lipschitz function.

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