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On Generalized Strong Vector Variational Inequality Problem with Fuzzy Mappings

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Abstract : In this paper, the generalized strong vector variational inequality problem with fuzzy mappings is introduced. The relationship between the solution set of it with the solution sets of the problems generalized vector complementarity problem with fuzzy mappings (GVCPFM) and generalized vector variational inequality problem with fuzzy mappings (GVVIPFM) is investigated. Also the equality of the solution sets of (GVCPFM) and (GVVIPFM) under mild assumptions is presented. Finally some existence results of a solution for the above problems by relaxing the upper semicontinuity and $0-$digonally convexity on the mappings, which are assumed in some papers, are established. The results of this paper can be viewed as an improvement of the corresponding results in the literature.

Keywords : KKM mappings; complementarity problems; variational inequalities; positive homogeneous mappings; fuzzy mappings.

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1 Introduction

The concept of fuzzy set has penetrated almost all branches of mathematics ever since the introduction of fuzzy sets by Zadeh [1]. Fuzzy sets have been applied to many fields including information sciences and control theory. In 1989, Chang and Zhu [2] introduced the concept of variational inequalities for fuzzy mappings in abstract spaces and investigated the existence problem for solutions of some classes of variational inequalities for fuzzy mappings. The variational inequality problem with its extensions and its numerous applications has been intensively studied in the last years, see, for examples, [3–5] and the references therein.

Recently Chang et al. [6] introduced and studied a new class of generalized vector variational-like inequalities and generalized vector variational inequalities in fuzzy environment by using the KKM-technique and Maximal Element Lemma. Several kinds of variational inequalities and complementarity problems for fuzzy mapping were studied by Chang et al. [7], Chang and Salahuddin [8], Anastassiou and Salahuddin [9]. Very recently, Kilicman et al. [10] introduced and studied a generalized vector complementarity problem with fuzzy mappings. They under suitable conditions, showed that generalized vector complementarity problem with fuzzy mappings is equivalent to generalized vector variational inequality problem with fuzzy mappings and they also derived some existence results for their problems.

The purpose of this paper is to introduce a new kind of generalized vector complementarity problems and to establish an existence result of a solution for it with mild assumptions. The results of this paper can be viewed as an improvement of the corresponding results in the literature, mainly the main results of the reference [10] by relaxing some conditions and using the KKM theory for proving the main theorem.

2 Preliminaries

Let X be a nonempty set. A fuzzy set A in X is characterized by its membership mapping $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in the fuzzy set A, for each $x \in X$. The collection of all fuzzy sets of X is denoted by $\mathfrak{F}(X)$ and a mapping F from D into $\mathfrak{F}(X)$ is called *fuzzy*. If $F : D \to \mathfrak{F}(X)$ is a fuzzy mapping, then $F(x)$, $x \in D$ (denoted by F_x , in the sequel) is a fuzzy set in $\mathfrak{F}(X)$ and $F_x(y)$, $y \in X$ is the degree of membership of y in F_x. Let $A \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$. The set

$$(A)_\alpha = \{x \in X : A(x) \geq \alpha\}$$

is called *α-cut set of A*.

Let E and Z be Hausdorff topological vector spaces. We denote by $L(E,Z)$ the space of all continuous linear operators from E into Z and $\langle l, x\rangle$, the evaluation of $l \in L(E,Z)$ at $x \in E$.

Let $T : K \to \mathfrak{F}(L(E,Z))$ be a fuzzy mapping and $\alpha : K \to [0,1]$ be a mapping. The mapping T induces a new multivalued mapping $\breve{T} : K \to 2^{L(E,Z)}$, where $2^{L(E,Z)}$ denotes the family of all nonempty subsets of $L(E,Z)$, which is defined by

$$\breve{T}(x) = (T_x)_{\alpha(x)}, \ \ \forall x \in K.$$

Let X and Y be two topological spaces and let $T : X \to 2^Y$ be a multivalued mapping. T is said to be *upper semicontinuous*, if for each $x \in X$ and each open set V in Y with $T(x) \subseteq V$, then there exists an open neighborhood U of x in X such that $T(u) \subseteq V$, for each $u \in U$.

Let K be a cone, that is, $tK \subseteq K$ for all $t \geq 0$, Y a topological vector space and $F : K \to Y$ be a mapping. F is said to be *positively homogenous* if

$$F(tx) = tF(x), \ \forall (t,x) \in [0,\infty) \times K.$$

Also a mapping $G : K \times K \to Y$ is said to be *positive homogeneous* if

$$G(tx,tx) = tG(x,x), \ \forall (t,x) \in [0,\infty) \times K,$$

(for more details, see [10]). Remark that if we define $\mathbf{K} = K \times K$ and $F(x) = G(x,x)$ for all $x \in \mathbf{K}$, then G is positively homogenous if and only if F is positively homogenous.

As an example we can characterize all positively homogenous mappings $F : [0,\infty) \to Y$, Y is a linear space over the real numbers, as follows: $F : [0,\infty) \to Y$ is positively homogenous if and only if

$$F(x) = xF(1), \forall x \in [0,\infty).$$

Let X be a nonempty set and Y a topological space. A multivalued mapping $G : X \to 2^Y$ is called *transfer closed-valued* if and only if

$$\bigcap_{x \in X} G(x) = \bigcap_{x \in X} clG(x),$$

where $clG(x)$ denotes the closure of $G(x)$.

It is clear the multivalued mapping G is transfer closed-valued when $G(x)$ is closed for each each x in X. While the example $G : X = (0,\infty) \to 2^{\mathbb{R}}$ defined by $G(x) = (-x,x]$, shows that the converse may drop.

Let X and Y be two vector spaces, K a convex subset of X and $C : X \to 2^Y$ be a multivalued mapping. A mapping $h : K \times K \to Y$ is said to be vector $0-$diagonally convex in the second variable, if for any subset $\Omega = \{x_1, x_2, ..., x_n\}$ of K, and $x = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$, the following relation holds

$$\sum_{i=1}^{n} t_i h(x, x_i) \in C(x),$$

(for more details, see [11, 12]).

A mapping $f : K \to Z$ is $C(x)-$*convex* if for any $x_1, x_2 \in K$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

that is

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x),$$

(for me details, see [6]). It is easy to check that if $h : K \times K \to Y$ is $C(x)-$convex in the second variable and $h(x,x) = 0$ for each $x \in K$ then h is vector $0-$diagonally convex in the second variable. While the converse may fail, because, for instance, if we take $K = X = \mathbb{R}, C(x) = [0, \infty)$ and $h(x,y) = -1$ for all $(x,y) \in K \times K$.

A multivalued mapping $\Gamma : K_0 \to 2^K$ is said to be the *KKM mapping* if

$$coA \subseteq \bigcup_{x \in A} \Gamma(x), \quad \forall A \in \langle K_0 \rangle,$$

where coA and $\langle K_0 \rangle$, respectively, stand for the convex hull of A and the set of all nonempty finite subsets of K_0.

The following result plays a crucial in the article.

Theorem 2.1. [13] *Let K be a nonempty subset of a topological vector space E and $\Gamma : K \to 2^E$ be a KKM mapping with closed values in K. Assume that there exist a nonempty compact convex subset B of K such that $D = \bigcap_{x \in B} \Gamma(x)$ is compact. Then*

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset.$$

3 Main Results

In this section, we show that, under sufficient conditions, the solution sets of the problems generalized vector complementarity problem with fuzzy mappings (GVCPFM) and generalized vector variational inequality problem with fuzzy mappings(GVVIPFM) are equal. Moreover, an existence result for (GSVVIPFM) is established. The results of this part improves and repairs the corresponding results given in this area, specially [10].

The following problems have been introduced and studied by Kiliman et al. [10]. They called them, respectively, generalized vector complementarity problem with fuzzy mappings (GVCPFM) and generalized vector variational inequality problem with fuzzy mappings (GVVIPFM) which consist of

(a) Finding $x \in K$ and $t \in (T_x)_{\alpha(x)}$ such that

$$\langle t, f(x,x) \rangle + H(x) = 0 \text{ and } \langle t, f(y,y) \rangle + H(y) \in C(x), \forall y \in K.$$

(b) Finding $x \in K$ and $t \in (T_x)_{\alpha(x)}$ such that

$$\langle t, f(y,x) \rangle + H(y) - H(x) \in C(x) \;\; \forall y \in K,$$

where K is a nonempty subset of topological vector space X, $T : K \to \mathfrak{F}(L(X,Y))$ is a fuzzy mapping, $\alpha : K \to [0,1]$ is a mapping and $f : K \times K \to Y$, $H : K \to Y$ are mappings and finally $C : K \to 2^Y$ is a multivalued mapping.

We call the following property *Condition* ($\triangle$), see [10].

Condition ($\triangle$) **:** $f(y,x) = f(y,y) - f(x,x), \ \forall x,y \in K.$

Theorem 3.1. [10] *If H is positive homogeneous and f is positive homogeneous of order 1 and condition ($\triangle$) is satisfied, then problems (GVCPFM) and (GVVIPFM) are equivalent (that is their solution sets are equal).*

The next result characterizes the class of all mappings which satisfy the condition ($\triangle$).

Proposition 3.2. *A mapping $f : K \times K \to Y$ satisfies condition ($\triangle$) if and only if f equals to zero (i.e., $f(x,y) = 0$, for all $(x,y) \in K \times K$.)*

Proof. It is obvious that the zero function satisfies in the condition ($\triangle$). Conversely, let the mapping f fulfil condition ($\triangle$) and x be an arbitrary element of K. By taking $y = x$ in the condition ($\triangle$) we get $f(x,x) = 0$ and so $f(y,y) = 0$. Hence $f(y,x) = 0$ for all $x,y \in K$. This completes the proof. □

Hence, by applying Proposition 3.2, we can rewrite Theorem 3.1 by omitting the positively homogeneous on the mappings as follows:

Theorem 3.3. *Let K be a subset of a real vector space X with $2K \subset K$ and $0 \in K$. Assume that $f : K \times K \to Y$ satisfies the condition ($\triangle$) and $H : K \to Y$ a mapping with $H(2x) = 2H(x)$, for all $x \in K$. Then the problems (GVCPFM) and (GVVIPFM) are equivalent; that is their solution sets are equal.*

Proof. By using Proposition 3.2, the problems (GVCPFM) and (GVVIPFM) reduce to the following problems:

(GVCPFM) Find $x \in K$ and $t \in (T_x)_{\alpha(x)}$ such that

$$H(x) = 0 \ \text{ and } \ H(y) \in C(x), \forall y \in K,$$

(GVVIPFM) Find $x \in K$ and $t \in (T_x)_{\alpha(x)}$ such that

$$H(y) - H(x) \in C(x), \ \ \forall y \in K.$$

Hence the solution set of (GVCPFM) is a subset of the solution set of (GVVIPFM). Conversely, let $x \in K$ be a solution of (GVVIPFM). Then

$$H(y) - H(x) \in C(x) \ \ \forall y \in K.$$

Then by taking $y = 2x$ and $y = 0$ in the last equation we deduce that x is a solution of (GVCPFM). This completes the proof. □

Remark that if H is positively homogeneous then $H(2x) = 2H(x)$ for all $x \in K$. Although, the example, $X = Y = K = \mathbb{R}$ and $H(x) = x$ for x is a rational number and $H(x) = 0$ when x is an irrational number, shows that the converse is not true.

The following existence result given by Kilicman et al. [10].

Theorem 3.4. ([10, Theorem 3.2]) *Assume that*

(a) *for all finite subset A of K, the multi-valued mapping $G_A : coA \to 2^K$ defined by*

$$G_A(y) = \{x \in K : \langle t, f(y,x)\rangle + H(y) - H(x) \in C(x)\},$$

for all $y \in K$ and $t \in (T_x)_{\alpha(x)}$, is a transfer closed-valued mapping;

(b) (i) *there exists a mapping $h : K \times K \to Y$ such that h is $0-$diagonally convex in the second argument;*

(ii) $\langle t, f(y,x)\rangle + H(y) - H(x) - h(x,y) \in C(x)$, *for all $x, y \in K$ and $t \in (T_x)_{\alpha(x)}$;*

(c) *let the mapping $T : K \to 2^{L(X,Y)}$ be upper semi-continuous, compact valued and f, H be hemicontinuous;*

(d) *there exist a nonempty compact subset B and a non-empty compact convex subset D of K such that for each $x \in K \backslash B$, there exists $y \in D$ such that*

$$\langle t, f(y,x)\rangle + H(y).H(x) \in C(x), \forall (x,y) \in K \times K \, and \, t \in (T_x)_{\alpha(x)}.$$

Then, the generalized vector variational inequality problem with fuzzy mappings (GVVIPFM) is solvable. Moreover, the solution set is compact.

We are interested in providing a new version of Theorem 3.4 by relaxing conditions (b) and (c) of it.

In order to do it, we first introduce the following problem and then we establish an existence result for it. Our method for proving our main theorem is different from the proof of Theorem 3.4 presented in [10].

Generalized strong vector variational inequality problem with fuzzy mappings (GSVVIPFM), consists of finding $x \in K$ such that

$$\langle t, f(y,x)\rangle + H(y) - H(x) \in C(x), \quad \forall (y,t) \in K \times (T_x)_{\alpha(x)}.$$

It is obvious that the solution set of (GSVVIPFM) is a subset of the solution set (GVVIPFM).

Now, we are ready to present our main theorem.

Theorem 3.5. *Assume that the following conditions hold:*

(i) *The multivalued mapping*

$$z \to \{x \in K : \langle t, f(z,x)\rangle + H(z) - H(x) \in C(x), \forall t \in (T_x)_{\alpha(x)}\}$$

is transfer closed-valued and $f(z,z) = 0$, *for all* $z \in K$;

(ii) *For each* $z \in K$ *the set* $Y \backslash C(z)$ *is convex;*

(iii) *For each* $z \in K$, *the mappin* $(t, y) \to \langle t, f(y,z)\rangle + H(y) - H(z)$ *is convex;*

(iv) *There exist convex and convex and compact subsets* D *and* M *of* K *such that*

$$\forall x \in K \backslash M, \exists y \in D; f(y,x)\rangle + H(y) - H(x) \notin C(x).$$

Then the solution set of the problem (GSVVIPFM) is nonempty and compact.

Proof. Define $\Gamma : K \to 2^K$ by

$$\Gamma(y) = \{x \in K : \langle t, f(y,x)\rangle + H(y) - H(x) \in C(x), \forall t \in (T_x)_{\alpha(x)}\}\}.$$

It is obvious that $y \in \Gamma(y)$, for each $y \in K$. Also Γ is a KKM mapping. Because, otherwise there exist a finite subset $A = \{y_1, ..., y_n\}$ of K and $z = \sum_{i=1}^n \lambda_i y_i \in coA \backslash \cup_{i=1}^n \Gamma(y_i)$. Then

$$\langle t, f(y_i, z)\rangle + H(y_i) - H(z) \in Y \backslash C(z), \ \ \forall (t,i) \in (T(z))_{a(z)} \times \{1, 2, ..., n\}.$$

By multiplying the last equation by λ_i and summing them we get

$$\sum_{i=1}^n \lambda_i \langle t_i, f(y_i, z)\rangle + \sum_{i=1}^n \lambda_i H(y_i) - H(z) \in Y \backslash C(z), \tag{3.1}$$

(note that $Y \backslash C(z)$ is convex). Also it follows from (iii) that

$$\sum_{i=1}^n \lambda_i \langle t_i, f(y_i, z)\rangle - \langle \sum_{i=1}^n \lambda_i t_i, f((\sum_{i=1}^n \lambda_i, y_i) = z, z)\rangle + \sum_{i=1}^n \lambda_i H(y_i) - H(z) \in C(z). \tag{3.2}$$

Hence, since $f(z,z) = 0$, the relation (3.2) reduces to

$$\sum_{i=1}^n \lambda_i \langle t_i f(y_i, z)\rangle + \sum_{i=1}^n \lambda_i H(y_i) - H(z) \in C(z),$$

which is contradicted by (3.1). Consequently, the mapping Γ is a KKM mapping. It is clear from (iv) that $\cap_{x \in D} \Gamma(x) \subseteq M$ and so $\cap_{x \in D} \Gamma(x)$ is relatively compact. Then the multivalued mapping $x \to cl\Gamma(x)$ (the closure of $\Gamma(x)$) satisfies all assumptions of Theorem 2.1 and then $\cap_{x \in K} cl\Gamma(x)$ is nonempty. This means that there exists $z \in K$ such that $z \in \cap_{x \in K} cl\Gamma(x)$. Consequently, it follows from (i) that

$$z \in \cap_{x \in K} cl\Gamma(x) = \cap_{x \in K} \Gamma(x).$$

This means, for each $y \in K$ and $t \in T(z)_{a(z)}$, that

$$f(y,z)\rangle + H(y) - H(z) \in C(z).$$

Hence, z is a solution of the problem (GSVVIPFM). Further, since the solution set of the problem (GSVVIPFM) equals to the set $\cap_{x \in K} \Gamma(x) = \cap_{x \in K} cl\Gamma(x)$ which is a closed subset of the compact set M by the condition (iv), So the solution set of problem (GSVVIFM) is compact. This completes the proof. $\square$

Remark 3.6. We have the following remarks:

1. Since the solution set of (GSVVIFM) is a subset of (GVVIFM), then any existence result of (GSVVIFM) is an existence result of (GVVIFM). Hence Theorem 3.5 is a new version of Theorem 3.4 by relaxing conditions (b) and (c) of it. Moreover, it provides an accurate proof of an existence result of (GVVIFM).
2. Condition (iii) of Theorem 3.5 trivially holds when the mappings $z \to H(z)$ and $(t, x) \to \langle t, f(z, x)\rangle$ are convex, for each $z \in K$. Also one can omit condition (iv) when K is compact.
3. If for each $z \in K$, the set

$$\{x \in K : \langle t, f(z, x)\rangle + H(z) - H(x) \in C(x), \forall t \in (T_x)_{\alpha(x)}\}$$

is convex, then the solution set of (GSVVIFM) is a convex set.

Combining Theorem 3.3 and Theorem 3.5, we get the following theorem.

Theorem 3.7. *If all the assumptions of Theorem* 3.3 *and Theorem* 3.5 *are satisfied, then the generalized vector complementarity problem with fuzzy mappings (GVCPFM) is solvable. Moreover, the solution set is compact.*

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