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A Hybrid Subgradient Algorithm for Finding a Common Solution of Pseudomonotone Equilibrium Problems and Hierarchical Fixed Point Problems of Nonexpansive Mappings

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Abstract : In this paper, we propose a new strongly convergent algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings in a real Hilbert space. The strong convergence theorem of proposed algorithms is investigated without the Lipschitz condition for the bifunctions. Our results complement many known recent results in the literature.

Keywords : nonexpansive mappings; pseudomonotone equilibrium problems; fixed point problems; hybrid subgradient algorithm; Hilbert spaces.

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot\rangle$ and a norm $\|\cdot\|$ associated with this inner product, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. A mapping $T : C \to C$ is called *contraction*, if there exists a constant $\delta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

From (1.1), if $\delta = 1, T$ is called *nonexpansive*. Further, we consider the following fixed point problem for a nonexpansive mapping $T : C \to C$:

$$\text{Find } x \in C \text{ such that } Tx = x. \tag{1.2}$$

We denote the set of solutions of fixed point problem (1.2) by $\mathrm{Fix}(T)$. It is well known that if $\mathrm{Fix}(T) \neq \emptyset$, $\mathrm{Fix}(T)$ is closed and convex. Next, let f be a bifunction from $C \times C$ to $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. An equilibrium problem in the sense of Blum and Oettli [1] is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C. \tag{1.3}$$

Problem of the form (1.3) on one hand covers many important problems in optimization as well as in nonlinear analysis such as (generalized) variational inequality, nonlinear complementary problem, nonlinear optimization problem, just to name a few. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. On the other hand, it is rather convenient for reformulating many practical problems in economic, transportation and engineering (see [1, 2]) and the references quoted therein). We denote the set of solutions of the problem (1.3) by $\mathrm{Sol}(f, C)$.

The existence of solution and its characterizations can be found, for example, in [3], while the methods for solving problem (1.3) have been developed by many researchers [4,5]. On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems [6]. The problem $P(C, f, T)$ of finding a common point in the solution set of problem $EP(C, f)$ and the set of fixed points of a nonexpansive mapping T recently becomes an attractive subject, and various methods have been developed for solving this problem. Most of the existing algorithms for this problem are based on the proximal point method applying to equilibrium problem $EP(C, f)$ combining with a Mann's iteration to the problem of finding a fixed point of T.

In 2006, Takahashi and Takahashi [7] proposed an iterative scheme under the name *viscosity approximation methods* for finding a common element of set of solutions of (1.3) and the set of fixed points of non-expansive mapping T in a real Hilbert space $\mathcal{H}$. This method generated an iteration sequence $\{x^k\}$ starting from a given initial point $x^0 \in \mathcal{H}$ and computed x^{k+1} as

$$\begin{cases} \text{Find } u^k \in C \text{ such that } f(u^k, y) + \frac{1}{r_k}\langle y - u^k, u^k - x^k\rangle \geq 0, \text{ for all } y \in C, \\ \text{Compute } x^{k+1} = \alpha_k g(x^k) + (1 - \alpha_k)T(u^k), k \geq 0, \end{cases}$$

where g is a contraction of $\mathcal{H}$ into itself, the sequences of parameters $\{r_k\}$ and $\{\alpha_k\}$ were chosen appropriately. Under certain choice of $\{\alpha_k\}$ and $\{r_k\}$, the authors showed that two iterative sequences $\{x^k\}$ and $\{u^k\}$ converge strongly to $z = P_{\mathrm{Fix}(T,C)\cap \mathrm{Sol}(f,C)}(g(z))$, where P_C denotes the projection onto C.

Recently, Anh in [8] proposed to use the extragradient-type iteration instead of the proximal point iteration given in [9] for solving Problem $P(C,f,T)$. More precisely, given $z^k \in C$, the proximal point iteration given in [9] is replaced by the two following mathematical programs, which seems numerically easier than previous ones. More precisely, It is suggested in [8] the following algorithm:

$$\begin{cases} \text{For an initial point } x^0 \in C, \\ y^k = \mathrm{argmin}\left\{ f(x^k, y) + \frac{1}{2\lambda_k}\|y - x^k\|^2 : y \in C \right\}, \\ z^k = \mathrm{argmin}\left\{ f(y^k, z) + \frac{1}{2\lambda_k}\|z - x^k\|^2 : z \in C \right\}, k \geq 0. \end{cases} \quad (1.4)$$

It was proved that if f is pseudomonotone and satisfies the Lipschitz-type condition: there are Lipschitz constants $c_1 > 0$ and $c_2 > 0$ if

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \forall x,y,z \in C. \quad (1.5)$$

then the sequence $\{z^k\}$ strongly converges to a solution of Problem $P(C,f,T)$. Recently, Anh and Muu [10] emphasized that the Lipschitz-type condition (1.5), in general is not satisfied, and if yes, finding the constants c_1 and c_2 is not an easy task. Furthermore solving the strongly convex programs (1.4) is expensive excepts special cases when C has a simple structure. They suggested and studied a new algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings in a real Hilbert space. The proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions. More precisely, they introduced the following algorithm:

$$\begin{cases} \text{Pick any } x^0 \in C; \\ y^k \in \partial_{\varepsilon_k} f(x^k, \cdot)(x^k); \\ \gamma_k := \max\{\lambda_k, \|y^k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ w^k = P_C(x^k - \alpha_k y^k); \\ x^{k+1} = \delta_k x^k + (1-\delta_k) T w^k, \text{ for each } k = 0, 1, \ldots \end{cases}$$

where $\partial_\varepsilon f(x,\cdot)(x)$ stands for ε- subdifferential of the convex function $f(x,\cdot)$ at x, $\{\varepsilon_k\}, \{\lambda_k\}$, $\{\beta_k\}$ and $\{\delta_k\}$ were chosen appropriately. Under the certain conditions, $\{x^k\}$ converse strongly to a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings in a real Hilbert space.

Our main purpose in this paper is to present a method for finding hierarchically a common element in $Fix(T) \cap \mathrm{Sol}(f,C)$ with respect to a nonexpansive mapping S, namely

$$\text{Find } \tilde{x} \in \Omega := Fix(T) \cap \mathrm{Sol}(f,C) \text{ such that } \langle \tilde{x} - S(\tilde{x}), \tilde{x} - x \rangle \leq 0, \forall x \in \Omega, \quad (1.6)$$

i.e., $0 \in (I-S)\tilde{x} + N_\Omega \tilde{x}$, where $N_\Omega \tilde{x}$ is the normal cone of Ω at $\tilde{x} \in \Omega$. It is not hard to check that solving (1.6) is equivalent to the following fixed point problem

$$\text{Find } \tilde{x} \in C \text{ such that } \tilde{x} = P_\Omega \circ S(\tilde{x}). \tag{1.7}$$

It is worth mentioning that when $S = I$, the solution set S of (1.6) is nothing but Ω. From now on, we assume that

$$SOL := \{\tilde{x} \in C | \tilde{x} = P_\Omega \circ S(\tilde{x})\} \neq \emptyset.$$

Now, let us consider some special cases of the problem (1.6).

- If $f \equiv 0$, then the problem (1.6) is reduced to the problem of finding hierarchically a fixed-point of a nonexpansive mapping T with respect to a nonexpansive mapping S, namely

$$\text{Find } \tilde{x} \in Fix(T) \text{ such that } \langle \tilde{x} - S(\tilde{x}), \tilde{x} - x\rangle \leq 0 \, \forall x \in Fix(T). \tag{1.8}$$

  This problem was studied by Moudafi [11]. He extended the KM iteration in order to analyze an algorithm in a more broad setting. More precisely, it is proposed in [11] the following algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)T x_n), \text{ for } n \geq 0, \tag{1.9}$$

  where $x_0 \in C, \{\sigma_n\}$ and $\{\alpha_n\} \subset (0,1)$.

- By setting $S = I - \gamma\mathcal{F}$ in (1.8), where $\mathcal{F}$ is η-Lipschitzian and κ-strongly monotone with $\gamma \in \left(0, \frac{2\kappa}{\eta^2}\right]$, (1.8) reduces to

$$\text{find } \tilde{x} \in Fix(T) \text{ such that } \langle x - \tilde{x}, \mathcal{F}(\tilde{x})\rangle \geq 0 \, \forall x \in Fix(T),$$

  a variational inequality studied in Yamada [12].

- For a given a maximal monotone operator A, by setting $T = J_\lambda^A := (I + \lambda A)^{-1}$ and $S = I - \gamma\nabla\psi$ where ψ is a convex function such that $\nabla\psi$ is η-Lipschitzian (which is equivalent to the fact that $\nabla\psi$ is η^{-1} cocoercive) with $\gamma \in \left(0, \frac{2}{\eta^2}\right]$, and thanks to the fact that $Fix\big(J_\lambda^A\big) = (A)^{-1}(0)$, (1.8) reduces to the following mathematical program with generalized equation constraint:

$$\min_{0 \in A(x)} \psi(x),$$

  a problem considered in [13].

- By taking $A = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a lower semicontinuous convex function, the latter reduces to the following hierarchical minimization problem considered in Cabot [14] and Solodov [15]:

$$\min_{x \in \text{argmin}\varphi} \psi(x).$$

For related works, please refer to [16–24].

Now, we are in a position to propose new iterative scheme for finding a solution of (1.6).

Algorithm 1.1. *Let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfy the following conditions:* **Step 1:**

(i) *For each x, $f(x,x) = 0$ and $f(x,\cdot)$ is lower semicontinuous convex on C;*

(ii) *If $\{x^k\} \subseteq C$ is bounded and $\varepsilon_k \downarrow 0$ as $k \to \infty$, then the sequence $\{y^k\}$ with $y^k \in \partial_{\varepsilon_k} f(x^k,\cdot)(x^k)$ is bounded;*

(iii) *f is pseudomonotone on C with respect to every solution of $EP(C,f)$ and satisfies the following condition, called strict paramonotonicity property:*

$$x \in \mathrm{Sol}(C,f), y \in C, f(y,x) = 0 \Rightarrow y \in \mathrm{Sol}(C,f); \tag{1.10}$$

(iv) *For each $x \in C$, $f(\cdot,x)$ is weakly upper semicontinuous on the open set C.*

Step 2: *Suppose that the sequences $\{\lambda_k\}, \{\beta_k\}, \{\varepsilon_k\}, \{\delta_k\}$ and $\{\sigma_k\}$ of nonnegative numbers satisfy the following conditions:*

(i) $0 < \lambda_k < \bar{\lambda},\ 0 < a < \delta_k < b < 1,\ \delta_k \to \frac{1}{2},\ \ 0 < a' < \sigma_k < b' < 1,\ \sigma_k \to \frac{1}{2}$;

(ii) $\beta_k > 0, \sum_{k=0}^{\infty} \beta_k = +\infty$ *and* $\sum_{k=0}^{\infty} \beta_k^2 < +\infty$*;*

(iii) $\sum_{k=0}^{\infty} \beta_k \varepsilon_k < +\infty$.

Step 3: *Let T and S be two nonexpansive mappings of C into itself such that $SOL \neq \emptyset$. Now the iteration scheme for finding a common point in SOL can be written as follows:*

$$\begin{cases} x^0 \in C; \\ y^k \in \partial_{\varepsilon_k} f(x^k,\cdot)(x^k); \\ \gamma_k := \max\{\lambda_k, \|y^k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ w^k = P_C(x^k - \alpha_k y^k); \\ x^{k+1} = \delta_k x^k + (1-\delta_k)(\sigma_k S w^k + (1-\sigma_k) T w^k), \text{ for each } k = 0,1,\ldots \end{cases} \tag{1.11}$$

Remark 1.2. [10, Remark 2.1]

1. If f is pseudomonotone on C with respect to the solution set $\mathrm{Sol}(C,f)$ of the problem $EP(C,f)$, then under Step 1 (i) and (iv), the set $\mathrm{Sol}(C,f)$ is convex.

2. Step 1 (ii) is true if whenever Step 1 (i) is satisfied and the bifunction f is continuous on $C \times C$.

3. Step 1 (iii) is true if f is pseudomonotone on C and satisfies the paramonotone property :

$$x \in \mathrm{Sol}(C,f), y \in C, f(y,x) = f(y,x) = 0 \Rightarrow y \in \mathrm{Sol}(C,f). \tag{1.12}$$

4. Since $f(x, \cdot)$ is lower semicontinuous convex on C, applying Remark 2.5, we conclude that $\partial_{\varepsilon_k} f(x^k, \cdot)(x^k) \neq \emptyset$. Thus the Algorithm (1.1) is well defined.

In this paper, the strong convergence of proposed algorithms is investigated under certain assumptions. Our results complement many known recent results in the literature.

2 Preliminaries

Let C be a nonempty convex subset of a real Hilbert space $\mathcal{H}$. We write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x as $k \to \infty$, and $x^k \to x$ to indicate that the sequence $\{x^k\}$ converges strongly to x as $k \to \infty$. In a real Hilbert space $\mathcal{H}$, we have

$$\|\delta x + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|y\|^2 - \delta(1-\delta)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in \mathcal{H}$ and $\delta \in \mathbb{R}$. Since C is closed, convex, for any $x \in \mathcal{H}$, there exists a unique nearest point of C, denoted by $P_C(x)$ satisfying

$$\|x - P_C(x)\| \leq \|x - y\|, \forall y \in C.$$

P_C is called the metric projection of $\mathcal{H}$ to C. It is well known that P_C satisfies the following properties:

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \quad \forall x, y \in \mathcal{H}, \tag{2.2}$$

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \quad \forall x \in \mathcal{H}, y \in C, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C. \tag{2.4}$$

Lemma 2.1. [25] *If $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \sigma_k)a_n + \delta_k,$$

where $\{\sigma_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{k=1}^{\infty} \sigma_k = \infty$;

(ii) $\limsup_{n\to\infty} \dfrac{\delta_k}{\sigma_k} \leq 0$ *or* $\sum_{k=1}^{\infty} |\delta_k| < \infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.2. *Let $\{\alpha_k\}$ be a sequence of nonnegative real numbers such that*

$$\alpha_{k+1} \leq (1 - \eta_k)\alpha_k + \beta_k, \ k \geq 0,$$

where $\{\eta_k\} \subset (0, 1), \sum_{n=1}^{\infty} \eta_k = \infty$, $\lim_{k\to\infty} \eta_k = 0$ and $\sum_{k=0}^{\infty} \beta_k < \infty$. Then

(i) $\alpha_{k+1} \leq \alpha_k + \beta_k$, *for all* $k \geq 0$;

(ii) *the sequence* $\{\alpha_k\}$ *is convergent.*

Proof. (i) Since $\lim_{k\to\infty}\sigma_k = 0$. Then $\lim_{k\to\infty}(1-\sigma_k) = 1$. So, we get

$$\alpha_{k+1} \le \alpha_k + \beta_k, \text{ for all } k \ge 0.$$

(ii) In Lemma 2.1 setting $\eta_k = \sigma_k$ and $\beta_k = \delta_k$, we get the sequence $\{\alpha_k\}$ is convergent as desired. □

Lemma 2.3. [26] *Let* $\mathcal{H}$ *be a real Hilbert space,* $\{\delta_k\}$ *be a sequence of real numbers such that* $0 < \overline{L} < \delta_k < 1$ *for all* $k = 0, 1, \ldots$ *and let* $\{v^k\}, \{u^k\}$ *be sequences of* $\mathcal{H}$ *such that*

$$\limsup_{k\to\infty} \|v^k\| \le c, \limsup_{k\to\infty} \|u^k\| \le c,$$

and

$$\lim_{k\to\infty} \|\delta_k v^k + (1-\delta_k)u^k\| = c, \text{ for some } c > 0.$$

Then

$$\lim_{k\to\infty} \|v^k - u^k\| = 0.$$

The following idea of the ε-subdifferential of convex functions can be found in the work of Bronsted and Rockafellar [27] but the theory of ε-subdifferential calculus was given by Hiriart-Urruty [28].

Definition 2.4. Consider a proper convex function $\phi : C \to \bar{\mathbb{R}}$. For a given $\varepsilon > 0$, the ε-subdifferential of ϕ at $x_0 \in \operatorname{dom}\phi$ is given by

$$\partial_\varepsilon \phi(x_0) = \{x \in C : \phi(y) - \phi(x_0) \ge \langle x, y - x_0\rangle - \varepsilon, \text{ for all } y \in C\}.$$

Remark 2.5. *It is know that if the function* ϕ *is proper lower semicontinuous convex, then for every* $x \in dom\phi$*, the* ε*-subdifferential* $\partial_\varepsilon\phi(x)$ *is a nonempty closed convex set (see* [29]*).*

Lemma 2.6.

(i) *Let* A *be a maximal monotone operator, then* $\{t_k^{-1}\}$ *graph converges to* $N_{A^{-1}(0)}$ *as* $t_k \to 0$ *provided that* $A^{-1}(0) \neq \emptyset$*.*

(ii) *Let* $\{B_k\}$ *be a sequence of maximal monotone operators which graph converges to an operator* B*. If* A *is a Lipschitz maximal monotone operators, then* $\{A + B_k\}$ *graph converges to* $A + B$ *and* $A + B$ *is a maximal monotone.*

Remark 2.7. *It is well-known that since* T *is a nonexpansive mapping on* $C, I - T$ *is a maximal monotone operator on* C *which is* $\frac{1}{2}$*-co-coercive. In addition,* T *is a demiclosed on* C *in the sense that, if* $\{x_k\}$ *converges weakly to* x *in* C *and* $\{x_k - Tx_k\}$ *strongly converges to* 0*, then* x *is a fixed point of* T*.*

3 Main Results

Theorem 3.1. *Suppose that Step* 1. *and Step* 2. *of Algorithm 1.1 are satisfied. Then the sequences* $\{x^k\}$ *and* $\{w^k\}$ *generated by* (1.11) *converge strongly to the same point* $\bar{x} \in \Omega$ *which solves the problem* (1.6), *where* $\bar{x} = \lim_{k\to\infty} P_\Omega(x^k)$.

Proof. We divide the proof into five steps as follows.
Step 1. For every $x^* \in C$ and every k, we show that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2(1-\delta_k)\alpha_k(f(x^k, x^*) + \varepsilon_k) + 2(1-\delta_k)\beta_k^2, \quad (3.1)$$

and there exists the limit

$$c := \lim_{k\to\infty} \|x^k - x^*\|. \quad (3.2)$$

Let $\{x^k\}$ and $\{w^k\}$ be two sequences generated by 1.11 and $x^* \in C$. Then, for all $k \geq 1$, we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|\delta_k x^k + (1-\delta_k)(\sigma_k S w^k + (1-\sigma_k)T w^k) - x^*\|^2 \\
&\leq \delta_k \|x^k - x^*\|^2 + (1-\delta_k)\big\|\sigma_k(Sw^k - Sx^*) + (1-\sigma_k)(Tw^k - Tx^*)\big\|^2 \\
&\leq \delta_k \|x^k - x^*\|^2 + (1-\delta_k)\big[\sigma_k\|Sw^k - Sx^*\| + (1-\sigma_k)\|Tw^k - Tx^*\|\big]^2 \\
&\leq \delta_k \|x^k - x^*\|^2 + (1-\delta_k)\big[\sigma_k\|w^k - x^*\| + (1-\sigma_k)\|w^k - x^*\|\big]^2 \\
&= \delta_k \|x^k - x^*\|^2 + (1-\delta_k)\big\|w^k - x^*\big\|^2 \\
&= \delta_k \|x^k - x^*\|^2 + (1-\delta_k)\big(\|x^k - x^*\|^2 - \|w^k - x^*\|^2 \\
&\quad + 2\left\langle x^k - w^k, x^* - w^k \right\rangle\big) \\
&\leq \|x^k - x^*\|^2 + 2(1-\delta_k)\left\langle x^k - w^k, x^* - w^k \right\rangle. \quad (3.3)
\end{aligned}$$

Since $w^k = P_C(x^k - \alpha_k y^k)$ and $x^* \in C$, we have

$$\left\langle x^k - w^k, x^* - w^k \right\rangle \leq \alpha_k \left\langle y^k, x^* - w^k \right\rangle. \quad (3.4)$$

Combining this inequality with (3.3) yields

$$\begin{aligned}
\|x^{k+1}-x^*\|^2 &\leq \|x^k-x^*\|^2+2(1-\delta_k)\left\langle x^k-w^k, x^*-w^k\right\rangle \\
&\leq \|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-w^k\right\rangle \\
&= \|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-x^k\right\rangle \\
&\quad +2(1-\delta_k)\alpha_k\left\langle y^k, x^*-w^k\right\rangle \\
&\leq \|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-x^k\right\rangle \\
&\quad +2(1-\delta_k)\alpha_k\|y^k\|\|x^k-w^k\| \\
&= \|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-x^k\right\rangle \\
&\quad +2(1-\delta_k)\frac{\beta_k}{\max\{\lambda_k,\|y^k\|\}}\|y^k\|\|x^k-w^k\| \\
&\leq \|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-x^k\right\rangle \\
&\quad +2(1-\delta_k)\beta_k\|x^k-w^k\|. \qquad (3.5)
\end{aligned}$$

Using again $w^k=P_C(x^k-\alpha_k y^k)$ and $x^k\in C$, we have

$$\begin{aligned}
\|x^k-w^k\|^2 &\leq \alpha_k\left\langle y^k, x^*-w^k\right\rangle \\
&\leq \alpha_k\|y^k\|\|x^k-w^k\| \\
&= \frac{\beta_k}{\max\{\lambda_k,\|y^k\|\}}\|y^k\|\|x^k-w^k\| \\
&\leq \beta_k\|x^k-w^k\|,
\end{aligned}$$

which implies that

$$\|x^k-w^k\|\leq\beta_k. \qquad (3.6)$$

Consequently,

$$\|x^k-w^k\|\to 0 \,\text{as}\, k\to\infty. \qquad (3.7)$$

Therefore from (3.5) and (3.6), we get

$$\|x^{k+1}-x^*\|^2\leq\|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\left\langle y^k, x^*-x^k\right\rangle+2(1-\delta_k)\beta_k^2. \qquad (3.8)$$

Since $y^k\in\partial_{\varepsilon_k}f(x^k,.)(x^k), x^*\in C$ and $f(x,x)=0$ for all $x\in C$, we have

$$\left\langle y^k, x^*-x^k\right\rangle\leq f(x^k,x^*)-f(x^k,x^k)+\varepsilon_k\leq f(x^k,x^*)+\varepsilon_k. \qquad (3.9)$$

Combining (3.8) and (3.9), we obtain that

$$\|x^{k+1}-x^*\|^2\leq\|x^k-x^*\|^2+2(1-\delta_k)\alpha_k(f(x^k,x^*)+\varepsilon_k)+2(1-\delta_k)\beta_k^2. \quad (3.10)$$

On the other hand, since $x^*\in\mathrm{Sol}(C,f)$, that is, $f(x^*,x)\geq 0$ for all $x\in C$, by pseudomonotonicity of f with respect to x^*, we have $f(x^k,x^*)\leq 0$ for all $x\in C$. Replacing x by $x^k\in C$, we get $f(x^k,x^*)\leq 0$. Then, from (3.10), it follows that

$$\|x^{k+1}-x^*\|^2\leq\|x^k-x^*\|^2+2(1-\delta_k)\alpha_k\varepsilon_k+2(1-\delta_k)\beta_k^2. \qquad (3.11)$$

Using Lemma 2.2 and (3.11), we get the existence of

$$c := \lim_{k\to\infty} \|x^k - x^*\|. \tag{3.12}$$

Step 2. For every $x^* \in C$, we show that

$$\lim_{k\to\infty} \sup f(x^*, x^k) = 0. \tag{3.13}$$

Since f is pseudomonotone on C and $f(x^*, x^k) \geq 0$, we have $-f(x^k, x^*) \geq 0$. Then by Step 1., for every k, we have

$$\begin{aligned} 2(1-\delta_k)\alpha_k[-f(x^k, x^*)] &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2(1-\delta_k)\alpha_k\varepsilon_k \\ &\quad + 2(1-\delta_k)\beta_k^2 \\ &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\alpha_k\varepsilon_k + 2\beta_k^2. \end{aligned} \tag{3.14}$$

Summing up the above inequalities for every k, we obtain that

$$0 \leq 2\sum_{k=0}^{\infty} \alpha_k[-f(x^k, x^*)] \leq \|x^0 - x^*\|^2 + 2\sum_{k=0}^{\infty} \alpha_k\varepsilon_k + 2\sum_{k=0}^{\infty} \beta_k^2 < +\infty. \tag{3.15}$$

It follows from the boundedness of the sequences $\{y^k\}$ and $\{\lambda_k\}$ that we can assume that

$$\max\{\lambda_k, \|y^k\|\} \leq M, \tag{3.16}$$

for a constant $M \geq 0$. Thus,

$$\alpha_k = \frac{\beta_k}{\gamma_k} = \frac{\beta_k}{\max\{\lambda_k, \|y^k\|\}} \geq \frac{\beta_k}{M}, \tag{3.17}$$

which together with $0 < a < \delta_k < b < 1$ and (3.15), implies

$$0 \leq \frac{2(1-a)}{M} \sum_{k=0}^{\infty} \beta_k[-f(x^k, x^*)] \leq 2\sum_{k=0}^{\infty} (1-\delta_k)\alpha_k[-f(x^k, x^*)] < +\infty. \tag{3.18}$$

Thus

$$\sum_{k=0}^{\infty} \beta_k[-f(x^k, x^*)] < +\infty. \tag{3.19}$$

Then, by $\sum_{k=0}^{\infty} \beta_k = \infty$ and $-f(x^k, x^*) \geq 0$, we can deduce that $\limsup_{k\to\infty} f(x^k, x^*) = 0$ as desired.

Step 3. For any $x^* \in \Omega$, suppose that x^{k_j} is the subsequence of x^k such that

$$\limsup_{k\to\infty} f(x^k, x^*) = \lim_{j\to\infty} (x^{k_j}, x^*), \tag{3.20}$$

and, without loss of generality, we may assume that $x^{k_j} \rightharpoonup, \bar{x}$ as $j \to \infty$ for some $\bar{x} \in C$. We show that $\bar{x}$ solves $EP(C, f)$. To this end, since $f(\cdot, x^*)$ is weakly upper semicontinuous, we have

$$f(\bar{x}, x^*) \geq \limsup_{j\to\infty} f(x^{k_j}, x^*) = \lim_{j\to\infty} f(x^{k_j}, x^*) = \limsup_{k\to\infty} f(x^k, x^*) = 0. \tag{3.21}$$

On the other hand, since f is pseudomonotone with respect to x^* and $f(x^*, \bar{x}) \geq 0$, we have

$$f(x^*, \bar{x}) \leq 0. \tag{3.22}$$

From 3.21 and 3.22, we can conclude that $f(\bar{x}, x^*) = 0$. By Step 1. of Algorithm 1.1, we can deduce that $\bar{x}$ is a solution of $EP(f, C)$ as well.

Step 4. We prove that any weakly cluster point of the sequence $\{x^k\}$ is a fixed point of T. In particular, $\bar{x} \in \text{Fix}(T)$. Let $\tilde{x} \in \text{Fix}T$ and $T_{\sigma_k} := \sigma_k S + (1 - \sigma_k)T$. By (1.9), we can write

$$\begin{aligned} \|x^{k+1} - \tilde{x}\| &\leq (1 - \delta_k)\|x^k - \tilde{x}\| + \delta_k \|T_{\sigma_k}(x^k) - T\tilde{x}\| \\ &\leq \|x^k - \tilde{x}\| + \delta_k \|T_{\sigma_k}(\tilde{x}) - T\tilde{x}\| \\ &= \|x^k - \tilde{x}\| + \delta_k \sigma_k \|S(\tilde{x}) - T\tilde{x}\|. \end{aligned}$$

By Lemma 2.2 and taking into account the fact that $\sum_{k=0}^{\infty} \delta_k \sigma_k < +\infty$, we have that the limit

$$l(\tilde{x}) = \lim_{k \to +\infty} \|x^k - \tilde{x}\| \tag{3.23}$$

exists and is finite. Also the sequence $\{x^k\}$ is bounded. Then, by setting $\bar{x}^{k+1} = (1 - \delta_k)x^k + \delta_k T x^k$ and $G = I - T$ we obtain

$$\|x^{k+1} - \bar{x}^{k+1}\| = \delta_k \|T_{\sigma_k}(x^k) - Tx^k\| = \delta_k \sigma_k \|Tx^k - Sx^k\|.$$

On the other hand, using the fact that G is $\frac{1}{2}$-co-coercive, we obtain

$$\begin{aligned} \|\bar{x}^{k+1} - \tilde{x}\|^2 &= \|x^{k+1} - \tilde{x} - \delta_k G x^k\|^2 \\ &= \|x^k - \tilde{x}\|^2 - 2\langle x^k - \tilde{x}, Gx^k - G\tilde{x}\rangle + \delta_k^2 \|Gx^k\|^2 \\ &\leq \|x^k - \tilde{x}\|^2 - \delta_k(1 - \delta_k)\|Gx^k\|^2. \end{aligned}$$

Since $\{x^k\}$ is bounded, there is an $M_1 > 0$ such that $\|Tx^k - Sx^k\| \leq M_1 \, \forall k \in \mathbb{N}$ and we derive

$$\begin{aligned} \delta_k(1 - \delta_k)\|Gx^k\|^2 &\leq \|x^k - \tilde{x}\|^2 - \|\bar{x}^{k+1} - \tilde{x}\|^2 \\ &= \|x^k - \tilde{x}\|^2 - \|\bar{x}^{k+1} - x^{k+1} + x^{k+1} - \tilde{x}\|^2 \\ &\leq \|x^k - \tilde{x}\|^2 - \|x^{k+1} - \tilde{x}\|^2 - 2\langle \bar{x}^{k+1} - x^{k+1}, x^{k+1} - \tilde{x}\rangle \\ &\leq \|x^k - \tilde{x}\|^2 - \|x^{k+1} - \tilde{x}\|^2 + 2M_1 \delta_k \sigma_k. \end{aligned}$$

This leads to

$$\sum_{k=0}^{\infty} \delta_k(1 - \delta_k)\|Gx^k\|^2 \leq \|x_0 - \tilde{x}\|^2 + 2M_1 \sum_{k=0}^{\infty} \delta_k \sigma_k < +\infty.$$

As $\sum_{k=0}^{\infty} \delta_k(1-\delta_k) = +\infty$, we infer that $\liminf_{k \to +\infty} \|Gx^k\| = \liminf_{k \to +\infty} \|x^k - Tx^k\| = 0$. However, for all k

$$Tx^{k+1} - x^{k+1} = Tx^{k+1} - T_{\sigma_k}(x^k) + (1 - \delta_k)(T_{\sigma_k}(x^k) - x^k)$$

and therefore

$$\begin{aligned}\|x^{k+1}-Tx^{k+1}\| &= \|Tx^{k+1}-Tx^k+Tx^k-T_{\sigma_k}(x^k)+(1-\delta_k)(T_{\sigma_k}(x^k)-x^k)\| \\ &\leq \|Tx^{k+1}-Tx^k\|+\|Tx^k-T_{\sigma_k}(x^k)\| \\ &\quad +\|(1-\delta_k)(T_{\sigma_k}(x^k)-x^k)\| \\ &\leq \|x^{k+1}-x^k\|+\|Tx^k-T_{\sigma_k}(x^k)\|+(1-\delta_k)\|(T_{\sigma_k}(x^k)-x^k)\| \\ &\leq \|Tx^k-T_{\sigma_k}(x^k)\|+\|T_{\sigma_k}(x^k)-x^k\| \\ &\leq \|x^k-Tx^k\|+2M_1\sigma_k.\end{aligned}$$

Consequently as $\sum_{k=0}^{\infty}\sigma_k < +\infty$, Lemma 2.2 ensures that the sequence $\{\|x^k-Tx^k\|\}$ converges and thus

$$\lim_{k\to\infty}\|x^k-Tx^k\|=0. \tag{3.24}$$

As $\{x^k\}$ is bounded, there has a weak cluster point $\bar{x}$ which amounts to saying that there exists a subsequence $\{x^{k_m}\}$ that weakly converges $\bar{x}$. This combined with the fact that Tx^k is demiclosed yields $\bar{x}\in \operatorname{Fix} T$. Moreover, since

$$\|x^{k+1}-x^k\|=\delta_k\|(T_{\sigma_k}(x^k)-x^k)\|\leq \delta_k\sigma_k\|Sx^k-Tx^k\|+\delta_k\|x^k-Tx^k\|,$$

the sequence $\{x^k\}$ is asymptotically regular, in other words

$$\lim_{k\to+\infty}\|x^{k+1}-x^k\|=0. \tag{3.25}$$

It remains to show that $\bar{x}$ solves problem (1.6). Again by (1.9), we have

$$x^{k+1}-x^k=(1-\delta_k)\big(\sigma_k(Sx^k-x^k)+(1-\sigma_k)(Tx^k-x^k)\big),$$

that is

$$\frac{1}{(1-\delta_k)\sigma_k}(x^k-x^{k+1})=\Big((I-S)+\frac{1-\sigma_k}{\sigma_k}(I-T)\Big)x^k. \tag{3.26}$$

Lemma 2.6(i) assures that the operator sequence $\{\frac{1-\sigma_k}{\sigma_k}(I-T)\}$ graph converges to N_Ω which in the light of Lemma 2.6(ii) allows us to deduce that the operator $(I-S)+\frac{1-\sigma_k}{\sigma_k}(I-T)$ graph converges to $(I-S)+N_\Omega$.

Now, by replacing k by k_j and passing to the limit in (3.26), as $j\to\infty$ and by taking into account the fact that $\frac{1}{(1-\delta_k)\sigma_k}\|x^{k+1}-x^k\|\to 0$ and that the graph of $(I-S)+N_\Omega$ is weakly-strongly closed, we finally obtain $0\in(I-S)\bar{x}+N_\Omega\bar{x}$, in other words $\bar{x}$ solves problem (1.6).

Step 5. Finally, we prove that

$$\lim_{k\to\infty}x^k=\lim_{k\to\infty}w^k=\lim_{k\to\infty}P_\Omega(x^k)=\overline{x}. \tag{3.27}$$

It follows from (3.11) that, for all $x^*\in\Omega$,

$$\left\|x^{k+1}-x^*\right\|^2\leq\left\|x^k-x^*\right\|^2+\mu_k, \tag{3.28}$$

where $\mu_k = 2(1-\delta^k)\alpha_k\varepsilon_k + 2(1-\delta^k)\beta_k^2 > 0$ for all $k \geq 0$ and $\Sigma_{k=0}^{\infty}\mu_k < +\infty$. Now, using property (2.4) of the metric projection, we have

$$\begin{aligned}\left\|x^{k+1} - P_\Omega(x^{k+1})\right\|^2 &= \left\|\delta_k x^k + (1-\delta_k)\big(\sigma_k Sw^k + (1-\sigma_k)Tw^k\big) - P_\Omega(x^{k+1})\right\|^2 \\ &\leq \delta_k\left\|x^k - P_\Omega(x^k)\right\|^2 \\ &\quad + (1-\delta)\left\|(\sigma_k Sw^k + (1-\sigma_k)Tw^k) - P_\Omega(x^k)\right\|^2 \\ &\leq \delta_k\left\|x^k - P_\Omega(x^k)\right\|^2 \\ &\quad + (1-\delta_k)\left\|(\sigma_k Sw^k + (1-\sigma_k)Tw^k) - x^k\right\|^2 \\ &\quad - (1-\delta_k)\left\|x^k - P_\Omega(x^k)\right\|^2 \\ &= (2\delta_k - 1)\left\|x^k - P_\Omega(x^k)\right\|^2 \\ &\quad + (1-\delta_k)\left\|\sigma_k(Sw^k - Tw^k) + (Tw^k - x^k)\right\|^2 \\ &= (2\delta_k - 1)\left\|x^k - P_\Omega(x^k)\right\|^2 + (1-\delta_k)\left\|w^k - x^k\right\|^2. \quad (3.29)\end{aligned}$$

Since $\delta_k \to \frac{1}{2}$ and (3.7), $\|w^k - x^k\| \to 0$ as $k \to \infty$, it follows from (3.29) that

$$\|x^{k+1} - P_\Omega(x^{k+1})\| \to 0 \text{ as } k \to \infty. \quad (3.30)$$

For the simplicity of notation, let $z^k := P_\Omega(x^k)$ for each $k \geq 1$. Then, for all $m > k$, since Ω is convex, we have $\frac{1}{2}(z^m + z^k) \in \Omega$, and therefore

$$\begin{aligned}\left\|z^m - z^k\right\|^2 &= 2\left\|x^m - z^m\right\|^2 + 2\left\|x^m - z^k\right\|^2 - 4\left\|x^m - \frac{1}{2}(z^m + z^k)\right\|^2 \\ &\leq 2\left\|x^m - z^m\right\|^2 + 2\left\|x^m - z^k\right\|^2 - 4\left\|x^m - z^m\right\|^2 \\ &= 2\left\|x^m - z^k\right\|^2 - 2\left\|x^m - z^m\right\|^2. \quad (3.31)\end{aligned}$$

Replacing x^* with z^k in (3.28), we can obtain the following:

$$\begin{aligned}\left\|x^m - z^k\right\|^2 &\leq \left\|x^{m-1} - z^k\right\|^2 + \mu_{m-1} \\ &\leq \left\|x^{m-2} - z^k\right\|^2 + \mu_{m-1} + \mu_{m-2} \\ &\leq \cdots \\ &\leq \left\|x^k - z^k\right\|^2 + \sum_{j=k}^{m-1}\mu_j. \quad (3.32)\end{aligned}$$

Combining this inequality with (3.31), we have

$$\left\|z^m - z^k\right\|^2 \leq 2\left\|x^k - z^k\right\|^2 + 2\sum_{j=k}^{m-1}\mu_j - 2\left\|x^m - z^m\right\|^2, \quad (3.33)$$

which gives that

$$\lim_{m\to\infty, \to k\to\infty}\left\|z^m - z^k\right\| = 0, \quad (3.34)$$

which implies that $\{z^k\}$ is a Cauchy sequence. Hence, $\{z^k\}$ strongly converges to some point $\overline{z} \in \Omega$. However, since $z^{k_i} := P_\Omega(x^{k_i})$, letting $i \to \infty$, we obtain in the limit that

$$\overline{z} = \lim_{i\to\infty} P_\Omega(x^{k_i} = P_\Omega(\overline{x}) = \overline{x} \in \Omega. \tag{3.35}$$

Therefore, $z^k := P_\Omega(x^k) \to \overline{z} = \overline{x} \in \Omega$. Then, from (3.30), we can conclude that $x^k \to \overline{x}$. Finally, since $\lim_{k\to\infty} \|x^k - w^k\| = 0$, we have $\lim_{k\to\infty} w^k = \overline{x}$. □

If $S \equiv I$, the identity mapping, then we obtain the new algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a nonexpansive mappings in a real Hilbert space.

Corollary 3.2. *Suppose that Step* 1. *in Algorithm 1.1 is satisfied. Let T be a nonexpansive mapping of C into itself such that $\Omega := \mathrm{Fix}(T) \cap \mathrm{Sol}(C, f) \neq \emptyset$. Suppose that the sequences $\{\lambda_k\}, \{\beta_k\}, \{\varepsilon_k\}$, $\{\delta_k\}$ and $\{\sigma_k\}$ of nonnegative numbers satisfy the following conditions:*

1. $0 < \lambda_k < \bar{\lambda},\ 0 < a < \delta_k < b < 1,\ \delta_k \to \frac{1}{2},\ 0 < a' < \sigma_k < b' < 1,\ \sigma_k \to \frac{1}{2}$;
2. $\beta_k > 0, \sum_{k=0}^{\infty} \beta_k = +\infty$ *and* $\sum_{k=0}^{\infty} \beta_k^2 < +\infty$;
3. $\sum_{k=0}^{\infty} \beta_k \varepsilon_k < +\infty$.

Then, the sequences $\{x^k\}$ and $\{w^k\}$ are generated by

$$\begin{cases} x^0 \in C; \\ y^k \in \partial_{\varepsilon_k} f(x^k, \cdot)(x^k); \\ \gamma_k := \max\{\lambda_k, \|y^k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ w^k = P_C(x^k - \alpha_k y^k); \\ x^{k+1} = \delta_k x^k + (1-\delta_k)(\sigma_k w^k + (1-\sigma_k)Tw^k), \text{ for each } k = 0, 1, \ldots \end{cases} \tag{3.36}$$

converge strongly to the same point $\bar{x} \in \Omega$ and $\bar{x} = \lim_{k\to\infty} P_\Omega(x^k)$.

By setting $S = I - \gamma\mathcal{F}$ in (1.8), where $\mathcal{F}$ is η-Lipschitzian and κ-strongly monotone with $\gamma \in \left(0, \frac{2\kappa}{\eta^2}\right]$, the problem (1.6) reduces to the following variational inequality studied in Yamada [12]:

$$\text{find } \tilde{x} \in \Omega := \mathrm{Fix}(T) \cap \mathrm{Sol}(C, f) \text{ such that } \langle x - \tilde{x}, \mathcal{F}(\tilde{x})\rangle \geq 0,\ \ \forall x \in \mathrm{Fix}(T). \tag{3.37}$$

The solution set of this problem is denoted by Γ_1.

Corollary 3.3. *Suppose that Step* 1. *in Algorithm 1.1 is satisfied. Let T be a nonexpansive mapping of C into itself and $\mathcal{F}$ be η-Lipschitzian and κ-strongly monotone with $\gamma \in \left(0, \frac{2\kappa}{\eta^2}\right]$ such that $\Gamma_1 \neq \emptyset$. Suppose that the sequences $\{\lambda_k\}, \{\beta_k\}$, $\{\varepsilon_k\}$, $\{\delta_k\}$ and $\{\sigma_k\}$ of nonnegative numbers satisfy the following conditions*

1. $0 < \lambda_k < \bar{\lambda},\ 0 < a < \delta_k < b < 1,\ \delta_k \to \frac{1}{2},\ \ 0 < a' < \sigma_k < b' < 1,\ \sigma_k \to \frac{1}{2}$;

2. $\beta_k > 0, \sum_{k=0}^{\infty} \beta_k = +\infty$ *and* $\sum_{k=0}^{\infty} \beta_k^2 < +\infty$;

3. $\sum_{k=0}^{\infty} \beta_k \varepsilon_k < +\infty$.

Then, the sequences $\{x^k\}$ *and* $\{w^k\}$ *are generated by*

$$\begin{cases} x^0 \in C; \\ y^k \in \partial_{\varepsilon_k} f(x^k, \cdot)(x^k); \\ \gamma_k := \max\{\lambda_k, \|y^k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ w^k = P_C(x^k - \alpha_k y^k); \\ x^{k+1} = \delta_k x^k + (1-\delta_k)(\sigma_k(I - \gamma\mathcal{F})w^k + (1-\sigma_k)Tw^k), \text{ for each } k = 0, 1, \ldots \end{cases} \tag{3.38}$$

converge strongly to the same point $\bar{x} \in \Gamma_1$ *and* $\bar{x} = \lim_{k\to\infty} P_{\Gamma_1}(x^k)$.

If $f \equiv 0$, then the problem (1.6) is reduced to the problem of finding hierarchically a fixed-point of a nonexpansive mapping T with respect to a nonexpansive mapping S, namely

$$\text{Find } \tilde{x} \in \text{Fix}(T) \text{ such that } \langle \tilde{x} - S(\tilde{x}), \tilde{x} - x \rangle \leq 0, \ \forall x \in \text{Fix}(T). \tag{3.39}$$

The solution set of (3.39) is denoted by Γ_2. Applying Theorem 3.1, we have the following strong convergence theorem.

Corollary 3.4. *Let* C *be a nonempty convex subset of a real Hilbert space* H. *Let* T *and* S *be two nonexpansive mappings of* C *into itself such that* $\Gamma_2 \neq \emptyset$. *Suppose that the sequences* $\{\delta_k\}$ *and* $\{\sigma_k\}$ *of nonnegative numbers satisfy the following conditions* $0 < a < \delta_k < b < 1, \ \delta_k \to \frac{1}{2}, \ \ 0 < a' < \sigma_k < b' < 1, \ \sigma_k \to \frac{1}{2}$. *Then, the sequences* $\{x^k\}$ *generated by*

$$\begin{cases} x^0 \in C; \\ x^{k+1} = \delta_k x^k + (1-\delta_k)(\sigma_k S x^k + (1-\sigma_k)Tx^k), \text{ for each } k = 0, 1, \ldots \end{cases} \tag{3.40}$$

converge strongly to a point $\bar{x} \in \Gamma_2$.

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