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Common Fixed Points of an Iterative Method for Berinde Nonexpansive Mappings

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Abstract : A mapping T form a nonempty closed convex subset C of a uniformly Banach space into itself is called a Berinde nonexpansive mapping if there is $L \geq 0$ such that $\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\|$ for any $x, y \in C$. In this paper, we prove weak and strong convergence theorems of an iterative method for approximating common fixed points of two Berinde nonexpansive mappings under some suitable control conditions in a Banach space. Moreover, we apply our results to equilibrium problems and fixed point problems in a Hilbert space.

Keywords : S-iterations; Berinde nonexpansive mappings; equilibrium problems.
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1 Introduction

Let X be a Banach space and C a nonempty closed convex subset of X. A map T form C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. A map T is called *k-contraction* if there exists $k \in [0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for any $x, y \in C$. A point x in C is called a *fixed point* of T if $x = Tx$, the set of all fixed points of T is denoted by $F(T)$. If T_1 and T_2 are

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self-mappings on C, a point $x \in C$ is called a *common fixed point* of T_i $(i = 1, 2)$ if x is a fixied point of T_i for each $i \in \{1, 2\}$.

To find a solution of the common fixed point problems, several iterative approximation methods were introduced and studied. This problem can be applied in solving solutions of various problems in science and applied science, see [1–3] for instance. In many researches, the iterative approximation methods for finding a fixed point of nonlinear mappings have been studied extensively such as the following schemes:

The *Mann iteration process* [4] is defined by the sequence $\{x_n\}$

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. In case of $\alpha_n = 1$ for all $n \geq 1$, this iteration process reduces to the Picard iteration process.

The *Ishikawa iteration process* [5] is defined by the sequence $\{x_n\}$

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

The *S-iteration process* [6] is defined by the sequence $\{x_n\}$

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In 2007, Agqrwal, O'Regan and Sahu [6] proved that this iteration process is independent of Mann and Ishikawa iteration process and more converge faster than both of them.

In 2003, Berinde [7] introduced a new type of contractive mappings. A map T of a metric space X into itself is called *weak contraction* or (δ, L)-*contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$. He proved an existence and uniqueness theorem for this mapping. A few years later, Berinde [8, 9] proved convergence theorems for approximating fixed points by Picard iteration process.

In 2009, Chumpungam [10] studied $(1, L)$-contraction mappings, or simply Berinde nonexpansive mappings, and proved existence theorems of their fixed points. Moreover, she proved strong convergence theorems for some proposed iteration processes such as Mann, Noor, Ishikawa iterations etc. and also studied stability and rate of convergence of these iteration processes.

Recently, Kosol [3] proved weak and strong convergence theorems for a common fixed point of three Berinde nonexpansive mappings in unuformly convex Banach space by iteration process which is defined as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T_1 x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n, \\ x_{n+1} = (1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n, n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are some suitable sequences in $[0,1]$.

In this paper, we prove weak and strong convergence theorems for a common fixed point of two Berinde nonexpansive mappings which satisfying some condition in Banach space $\{x_n\}$ with iteration process defined by

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} = (1-\alpha_n)T_2 x_n + \alpha_n T_2 y_n, n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences in $[0,1]$. Note that the iteration process in (1.4) can not to reduce iteration process in (1.5) under the control conditions studied in [10]. Finally, we apply our result to the equilibrium problems in Hilbert spaces.

2 Preliminaries

In this section, the definitions and fundamental theorems that will be used in our work will be given. For a sequence $\{x_n\} \subset X$, the strong and weak convergence of $\{x_n\}$ to x is denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Definition 2.1. Let X be a Banach space and C a nonempty closed convex subset of X. A mapping T of C into itself is said to be

- *nonexpansive* if for any $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

- *quasi-nonexpansive* if $F(T) \neq \emptyset$ and for any $x \in C, p \in F(T)$,

$$\|Tx - p\| \leq \|x - p\|.$$

- *Berinde nonexpansive* or $(1, L)$-*contraction* if there exists $L \geq 0$ such that for any $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\|.$$

- *demicompact* if for any sequence $\{x_n\} \subset X$ such that $\|x_n - Tx_n\| \to 0$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ for some $x \in C$.

A class of nonexpansive, quasi-nonexpansive, Berinde nonexpansive mappings are denoted by $\mathscr{C}_N, \mathscr{C}_{QN}, \mathscr{C}_{BN}$, respectively. It is clear that $\mathscr{C}_N \subset \mathscr{C}_{QN}$ and $\mathscr{C}_N \subset \mathscr{C}_{BN}$. The following examples show that $\mathscr{C}_{QN}$ and $\mathscr{C}_{BN}$ dose not belong to each other.

Example 2.2.

(a) Let $X = l_\infty, C = \{x \in l_\infty : \|x\|_\infty \leq 1\}$ and $T : C \to C$ a mapping defined by

$$Tx = (0, x_1^2, x_2^2, x_3^2, \ldots),$$

for $x = (x_1, x_2, x_3, \ldots) \in C$. In [11], the authors proved that the mapping T is quasi-nonexpansive but not nonexpansive. It is note that T is not Berinde nonexpansive. To see this, let $L \geq 0$ for arbitrary. Put $x = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$ and $y = (0, 0, \frac{1}{4}, \frac{1}{4}, \ldots)$. Then, we see that

$$\|Tx - Ty\| = \frac{7}{16} > \frac{1}{4} = \|x - y\| + L\|y - Tx\|.$$

Thus, T is not Berinde nonexpansive.

(b) Let $X = \mathbf{R}, C = [0, 1]$ and a map $T : C \to C$ defined by

$$Tx = \begin{cases} x^2, \text{ if } x \in [0, \frac{1}{2}), \\ 1, \text{ if } x \in [\frac{1}{2}, 1]. \end{cases}$$

In [10], the author showed that the mapping T is Berinde nonexpansive with $L = 4$ but not nonexpansive. We observe that the set of fixed points $F(T)$ is $\{0, 1\}$. By choosing $(\frac{1}{3}, 1) \in C \times F(T)$, we have $\left\|T(\frac{1}{3} - 1)\right\| = \frac{8}{9} > \left|\frac{1}{3} - 1\right|$, so the map T is not quasi-nonexpansive mapping.

The following condition forces a mapping to be quasi-nonexpansive. It is called *condition* $(*)$.

Definition 2.3 (condition $(*)$)**.** A map T of a set C into itself is called satisfying the condition $(*)$ if there exists $L \geq 0$ such that for any $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\| + L\|x - Tx\|.$$

Remark 2.4. *A map $T : C \to C$ satisfying condition $(*)$ with $F(T) \neq \emptyset$ is quasi-nonexpansive.*

Proof. Suppose that T satisfies the condition $(*)$. For $p \in F(T)$ and $x \in C$, there exists $L \geq 0$ such that

$$\|Tx - p\| = \|Tx - Tp\| \leq \|x - p\| + L\|p - Tp\| = \|x - p\|.$$

□

Lemma 2.5. [12] *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Definition 2.6. A Banach space X is said to satisfy *Opial's condition* if for any sequence $\{x_n\} \subset C$, $x_n \rightharpoonup x$ for some $x \in C$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for any $y \in C, y \neq x$.

Lemma 2.7. [3] *Let X be a Banach space such that Opial's condition holds, C a nonempty closed convex subset of X and T a mapping on C. If T satisfies the condition $(*)$, then $I - T$ is demiclosed at 0, i.e. for any $\{x_n\} \subset C$, $x_n \rightharpoonup x$ for some $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x \in F(T)$.*

3 Main Results

In this section, we prove weak and strong convergence of an interation (1.5). Throughout this section, we assume that X is a uniformly convex Banach space and C is a nonempty closed convex subset of X. To obtain our results, some useful lemmas are needed.

Lemma 3.1. *Let $T_i : C \to C, i = 1, 2,$ be quasi-nonexpansive mappings. Suppose that $F(T_1) \cap F(T_2) \neq \emptyset$, a sequence $\{x_n\}$ defined by (1.5) and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then,*

(1) $\|x_{n+1} - p\| \leq \|x_n - p\|$ *for any $n \geq 1$ and $p \in F(T_1) \cap F(T_2)$,*

(2) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. Let $p \in F(T_1) \cap F(T_2)$. By using (1.5), we obtain that

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T_1 x_n - p\| \\ &= \|(1-\beta_n)(x_n - p) + \beta_n (T_1 x_n - p)\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n \|T_1 x_n - p\| \\ &\leq \|x_n - p\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)T_2 x_n + \alpha_n T_2 y_n - p\| \\ &= \|(1-\alpha_n)(T_2 x_n - p) + \alpha_n (T_2 y_n - p)\| \\ &\leq (1-\alpha_n) \|T_2 x_n - p\| + \alpha_n \|T_2 y_n - p\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|y_n - p\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

By using (1) and $\{\|x_n - p\|\}$ bounded below, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 3.2. *Let $T_i : C \to C, i = 1, 2,$ be quasi-nonexpansive mappings such that $F(T_1) \cap F(T_2) \neq \emptyset$ and let a sequence $\{x_n\}$ be generated by (1.5) where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1,$

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$

Then $\{\|x_n - T_1x_n\|\}, \{\|T_2x_n - T_2y_n\|\}, \{\|y_n - x_n\|\}, \{\|x_{n+1} - T_2x_n\|\}$ *converse to* 0.

Proof. Let $p \in F(T_1) \cap F(T_2)$. It is easily to see by Lemma 3.1 that a sequence $\{x_n - p\}$ is bounded, i.e. there exists a real number $M > 0$ such that $\|x_n - p\| \leq M$ for all $n \geq 1$. Replacing y_n and using quasi-nonexpansiveness of T_1, we get

$$\|y_n - p\| \leq (1 - \beta_n) \|x_n - p\| + \beta_n \|T_1x_n - p\| \leq M,$$

for any $n \geq 1$. Again with quasi-nonexpansiveness of T_i, we finally obtain that $\{x_n - p\}, \{y_n - p\}, \{T_ix_n - p\}, \{T_iy_n - p\},\ i = 1, 2,$ are subsets of $B_M(0)$. By lemma 2.5, there is a continuous strictly convex function g from $[0,\infty)$ into $[0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}\|y_n - p\|^2 &= \|(1 - \beta_n)x_n + \beta_nT_1x_n - p\|^2\\ &= \|(1 - \beta_n)(x_n - p) + \beta_n(T_1x_n - p)\|^2\\ &\leq (1 - \beta_n) \|x_n - p\|^2 + \beta_n \|T_1x_n - p\|^2 - (1 - \beta_n)\beta_n g(\|x_n - T_1x_n\|),\end{aligned}$$

and then,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)T_2x_n + \alpha_nT_2y_n - p\|^2\\ &= \|(1 - \alpha_n)(T_2x_n - p) + \alpha_n(T_2y_n - p)\|^2\\ &\leq (1-\alpha_n) \|T_2x_n - p\|^2 + \alpha_n \|T_2y_n - p\|^2 - (1 - \alpha_n)\alpha_n g(\|T_2x_n - T_2y_n\|)\\ &\leq (1-\alpha_n) \|x_n - p\|^2 + \alpha_n \|y_n - p\|^2 - (1 - \alpha_n)\alpha_n g(\|T_2x_n - T_2y_n\|)\\ &\leq (1-\alpha_n) \|x_n - p\|^2 + \alpha_n(1 - \beta_n) \|x_n - p\|^2 + \alpha_n\beta_n \|T_1x_n - p\|^2\\ &\quad - \alpha_n(1 - \beta_n)\beta_n g(\|x_n - T_1x_n\|) - (1 - \alpha_n)\alpha_n g(\|T_2x_n - T_2y_n\|).\end{aligned}$$

So, we have

$$\alpha_n(1 - \beta_n)\beta_n g(\|x_n - T_1x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 .$$

Taking $n \to \infty$, we obtain that $g(\|x_n - T_1x_n\|) \to 0$. In fact, $\|x_n - T_1x_n\| \to 0$ by continuity of g and $g(0) = 0$. Similarly, we also obtain $\|T_2x_n - T_2y_n\| \to 0$. Thus,

$$\|y_n - x_n\| \leq \beta_n \|x_n - T_1x_n\| \to 0,$$

and

$$\|x_{n+1} - T_2x_n\| \leq \alpha_n \|T_2x_n - T_2y_n\| \to 0.$$

□

Theorem 3.3. *Let $T_i : C \to C, i = 1, 2$, be Berinde nonexpansive mappings which are satisfying the condition $(*)$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by* (1.5) *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying conditions $(i), (ii)$ in Lemma* 3.2. *If T_1 is demicompact, then a sequence $\{x_n\}$ converges strongly to some element of $F(T_1) \cap F(T_2)$.*

Proof. Suppose that T_1 is demicompact. Since $\|x_n - T_1x_n\| \to 0$, there exists a subsequence $\{x_{n_k}\}$ of x_n converges strongly to some element p in C. We claim that $p \in F(T_1) \cap F(T_2)$. Indeed, for each $k \geq 1$,

$$\begin{aligned} \|T_1x_{n_k} - T_1p\| &\leq \|x_{n_k} - p\| + L\,\|p - T_1x_{n_k}\| \text{ for some } L \geq 0 \\ &\leq \|x_{n_k} - p\| + L\,\|p - x_{n_k}\| + L\,\|x_{n_k} - T_1x_{n_k}\| \to 0. \end{aligned}$$

Then,

$$\|p - T_1p\| \leq \|p - x_{n_k}\| + \|x_{n_k} - T_1x_{n_k}\| + \|T_1x_{n_k} - T_1p\| \to 0.$$

Hence, $p \in F(T_1)$. Next, we show that $p \in F(T_2)$. Since

$$\|p - T_2x_{n_k}\| \leq \left\|p - x_{n_{k+1}}\right\| + \left\|x_{n_{k+1}} - T_2x_{n_k}\right\| \to 0,$$

we have

$$\|p - T_2y_{n_k}\| \leq \|p - T_2x_{n_k}\| + \|T_2x_{n_k} - T_2y_{n_k}\| \to 0.$$

Then,

$$\begin{aligned} \|T_2p - T_2x_{n_k}\| &\leq \|T_2p - T_2y_{n_k}\| + \|T_2y_{n_k} - T_2x_{n_k}\| \\ &\leq \|p - y_{n_k}\| + L'\,\|p - T_2y_{n_k}\| + \|T_2y_{n_k} - T_2x_{n_k}\| \text{ for some } L' \geq 0 \\ &\leq \|p - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| + L'\,\|p - T_2y_{n_k}\| + \|T_2y_{n_k} - T_2x_{n_k}\| \to 0. \end{aligned}$$

Hence,

$$\|p - T_2p\| \leq \left\|p - x_{n_{k+1}}\right\| + \left\|x_{n_{k+1}} - T_2x_{n_k}\right\| + \|T_2x_{n_k} - T_2p\| \to 0.$$

Thus, $p \in F(T_1) \cap F(T_2)$. By existence of limit of $\|x_n - p\|$ in Lemma 3.1, we conclude that a sequence $\{x_n\}$ converges strongly to $p \in F(T_1) \cap F(T_2)$. □

Definition 3.4. A mapping $T : C \to C$ is said to be *weakly continuous* if for each $x_0 \in C$, $Tx \rightharpoonup Tx_0$ as $x \to x_0$.

Theorem 3.5. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed convex subset of X and let $T_i : C \to C, i = 1, 2$, be Berinde nonexpansive mappings which satisfy the condition $(*)$ and T_2 is weakly continuous. Assume that $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by* (1.5) *where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ with the following restrictions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, (*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(*iii*) $\lim_{n\to\infty} \alpha_n = \alpha$ *for some* $\alpha \in (0,1)$.

Then $\{x_n\}$ *converges weakly to some element* x *in* $F(T_1) \cap F(T_2)$.

Proof. By using lemma 3.2, we have $\{\|x_n - T_1x_n\|\}$,$\{\|T_2x_n - T_2y_n\|\}$,$\{\|y_n - x_n\|\}$, $\{\|x_{n+1} - T_2x_n\|\}$ converge to 0. Since $\{x_n\}$ is bounded, there is a weakly convergence subsequence $\{x_{n_k}\} \subset \{x_n\}$. Without loss of generality, we may assume that $x_n \rightharpoonup x \in C$. Also, $y_n \rightharpoonup x$ since $\|y_n - x_n\| \to 0$. By using Lemma 2.7 with $\{x_n\}$ and $\|x_n - T_1x_n\| \to 0$, we get $x \in F(T_1)$. From $x_{n+1} = (1-\alpha_n)T_2x_n + \alpha_nT_2y_n$ and T_2 is weakly continuous, we obtain

$$x_{n+1} = (1-\alpha_n)T_2x_n + \alpha_nT_2y_n \rightharpoonup (1-\alpha)T_2x + \alpha T_2x = T_2x,$$

which implies that $x \in F(T_2)$. Therefore, $x_n \rightharpoonup x \in F(T_1) \cap F(T_2)$. □

4 Applications

Let C be a nonempty subset of a real Hilbert space X and F a bifinction of $C \times C$ into $\mathbf{R}$. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find an element $x \in C$ such that

$$F(x, y) \ge 0 \text{ for all } y \in C. \tag{4.1}$$

The set of all solutions in (4.1) is denoted by $EP(F)$. If an element x belongs to $EP(F)$, we called x an equilibrium point of F. In many researches, the equilibrium problem has been studied extensively, see [13–15] for instance. In this paper, we apply our results to find a common fixed point of Berinde nonexpansive mapping T and equilibrium point of bifunction F.

For solving the equilibrium problem for a bifunction F, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) for all $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemmas are crucial for our main results.

Lemma 4.1. [13] *Let* C *be a closed convex subset of a real Hilbert space* X*, let* F *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying* $(A1) - (A4)$*, and let* $r > 0$ *and* $x \in X$*. Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,$$

for all $y \in C$.

Lemma 4.2. [14] *Let C be a closed convex subset of a real Hilbert space X and let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying $(A1)-(A4)$. For all $r > 0$ and $x \in X$, define a mapping $T_r : X \to C$ by*

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Then the following holds:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, i.e. for all $x, y \in X$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

By using two above lemmas and our main result (Theorem 3.3), we obtain the following result.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert X. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying $(A1)-(A4)$ and S a Berinde nonexpansive mapping satisfying condition $(*)$. Suppose $F(S) \cap EP(F) \neq \emptyset$. Let $r > 0$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \\ F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \\ y_n = (1-\beta_n)x_n + \beta_n u_n, \\ x_{n+1} = (1-\alpha_n)Sx_n + \alpha_n S y_n, n \geq 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If T_r is demicompact, then a sequence $\{x_n\}$ converges strongly to $x \in F(S) \cap EP(F)$.

Proof. By Lemma 4.2, we know that T_r is a firmly nonexpansive-type mapping, so it is nonexpansive, hence it is Berinde nonexpansive. Thus, we apply Theorem 3.3 to obtain the result. □

Example 4.4. Let $X = \mathbf{R}$ and $C = [0,1]$. A map $S : [0,1] \to [0,1]$ is defined by

$$Sx = \begin{cases} x^2, \text{ if } x \in [0, \frac{1}{2}), \\ 0.3, \text{ if } x \in [\frac{1}{2}, 1]. \end{cases}$$

i	x_n	y_n
1	1	0.8125
2	0.3	0.23076923076923075
3	0.07530177514792899	0.057148668639053254
4	0.004639905763337993	0.0035036023110919545
5	$1.7416060453382352e-05$	$1.3119335012580785e-05$
6	$2.4368179328698657e-10$	$1.8332024816085228e-10$
7	$4.748489192088233e-20$	$3.5693879906744314e-20$
8	$1.7971260750321737e-39$	$1.3501724397910112e-39$
9	$2.5676873002008594e-78$	$1.9283963023024897e-78$
10	$5.231504810775423e-156$	$3.9279737117450024e-156$
11	$2.168308777326e-311$	$1.627720806056e-311$
12	0.0	0.0
13	0.0	0.0
14	0.0	0.0

Table 1: Numerical experiment of the iteration process 4.2

Then, we observe that S is Berinde nonexpansive satisfying condition $(*)$ and $F(S) = \{0\}$. Let $F : C \times C \to \mathbf{R}$ be a bifunction defined by

$$F(x, y) = y^2 + xy - 2x^2,$$

for any $x, y \in C$. Let $r > 0$ and a sequence $\{x_n\}$ generated by iteration process (4.2). From [16] [Example 4.1], we can compute that $T_r x = \frac{x}{3r+1}$ for all $x \in C$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, 1]$ given by

$$\alpha_n = \frac{n}{2n+1}, \beta_n = \frac{n^2}{3n^2+1}, n \geq 1.$$

Setting $r = 1$ and $x_1 = 1 \in C$. Then the iteration process (4.2) becomes:

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \dfrac{n^2}{3n^2+1})x_n + \dfrac{n^2}{3n^2+1} \cdot \dfrac{x_n}{4}, \\ x_{n+1} = (1 - \dfrac{n}{2n+1})Sx_n + \dfrac{n}{2n+1}Sy_n, n \geq 1. \end{cases} \tag{4.3}$$

From the Table 1, we see that the sequence $\{x_n\}$ converges to a point $0 \in F(S) \cap F(T_r) = F(S) \cap EP(F)$ which are garunteed by Theorem 4.3.

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References

[1] B. Gunduz, S. Akbulutl, Common fixed points of a finite family of i-asymptotically nonexpansive mappings by s-iteration process in banach spaces, Thai J. Math. 15 (3) (2017) 673-687.

[2] P. Jailoka, S. Suantai, Split common fixed point and null point problems for demicontractive operators in Hillbert spaces, Optim. Methods Softw. (2017) doi:10.1080/10556788.2017.1359265.

[3] S. Kosol, Weak and strong convergence theorems of some iterative methods for common fixed point of berinde nonexpansive mappings, Thai J. Math. 15 (3) (2017) 629-639.

[4] W.R. Mann, Mean value methods in iteration, Proceedings of the American Mathematical Society 4 (1953) 506-510.

[5] S. Ishikawa, Fixed points by a new iteration method, Proceedings of the American Mathematical Society 44 (1974) 147-150.

[6] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed point of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61-79.

[7] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9 (1) (2004).

[8] V. Berinde, Approximating fixed points of weak ϕ-contractions using the Picard iteration, Fixed Point Theory 4 (2) (2003) 131-142.

[9] V. Berinde, On the approximation of fxed points of weak contractive mappings, Carpathian J. Math. 19 (2003) 7-22.

[10] D. Chumpungam, Strong convergence theorems and rate of convergence of some iterative methods for common fixed points of weak contractions in Banach spaces, Masters thesis, Chiang Mai University 2009.

[11] R. Agarwal, D. ORegan, D. Sahu, Fixed Point Theory for Lipschitziantype Mappings with Applications, Topological Fixed Point Theory and Its Applications, Springer New York, 2009.

[12] Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (4) (2004) 707-717.

[13] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[14] P. Combettes, S. Hirstoaga, Equilibrium programming in hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 51-61.

[15] U. Kamraksa, C. Jaiboon, Strong convegence theorems for general equilibrium problems and fixed point problems in banach spaces, Thai J. Math. 13 (2) (2015) 481-495.

[16] U. Singthong, S. Suantai, Equilibrium problems and fixed point problems for nonspreading-type mappings in hilbert space, Int. J. Nonlinear Anal. Appl. 2 (2) (2011) 51-61.

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