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Existence Results for Second Order Neutral Integrodifferential Equations of Sobolev Type in Banach Spaces

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Abstract : In this paper, we established the existence of mild solution for second order nonlinear impulsive integrodifferential equation of Sobolev type with nonlocal initial conditions. The results are obtained by using strongly continuous cosine families of operators and the fixed point approach. An example is provided to illustrate the theory.

Keywords : neutral impulsive integrodifferential equation; strongly continuous cosine families of operators; fixed point theorem.
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1 Introduction

Many fundamental laws of science can be formulated as differential equations. Differential equations are used to mathematical modelling the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Many of the modelled may be described by the same second-order differential equation. The wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water.

Second-order nonlinear differential and integrodifferential equations arise in problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena. So it is quite significant to study the existence and controllability problem for such systems in Banach spaces. In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. A useful study of abstract second order equations is the theory of strongly cosine families. The theory of strongly continuous cosine families of bounded linear operators has applications in many branches of analysis and in particular in finding the solutions of initial and boundary value problems for second order partial differential equations. Roughly speaking, every second order differential equation of the form $x'' = Ax$ which is well posed in a certain sense gives rise to a strongly continuous cosine family of bounded linear operators with infinitesimal generator A and conversely every strongly continuous cosine family of bounded linear operators with infinitesimal generator A may be associated with a well-posed second order differential equation $x'' = Ax$. The most fundamental and extensive work on cosine families is that of Travis and Webb [1, 2, 3]. This paper is devoted to the study of existence of mild solutions for initial value problems described as second order nonlinear impulsive neutral integrodifferential equation of Sobolev type with nonlocal conditions. Particularly, we are deal with problems that can be modeled as an abstract Cauchy problem on a Banach space X of the form

$$\frac{d}{dt}\big[(Bu(t))'+g(t,u(t))\big]=Au(t)+f(t,u(t))+\int_0^t h(t,s,u(s))ds,\ t\in I,\ t\neq t_k, \quad (1.1)$$

$$u(0)+\sum_{i=1}^{n} c_i u(t_i) = u_0, \qquad u'(0)+q(u)=v_0 \quad (1.2)$$

$$\Delta u(t_k) = I_k(u_{t_k}), \quad \Delta u'(t_k) = J_k(u_{t_k}), \quad k=1,2,\dots,m, \quad (1.3)$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{R}\}$ of bounded linear operators in the Banach space X. Here $\mathcal{I} = [0, b]$. B is the linear operator with domain contained in a Banach space X and range contained in a Banach space X and the nonlinear operators $f : I \times X \to X$, $h : I^2 \times X \to X$, $g : I \times X \to X$, $q : \mathcal{PC}(I, X) \to X$ and $I_k, J_k : X \to X$ are appropriate functions; $0 \le t_1 < t_2 < \cdots < t_m \le b$ are prefixed numbers and the symbol $\Delta u(t_k)$ represent the jump of the function u at t, which is defined by $\Delta u(t_k) = u(t^+) - u(t^-)$.

Neutral differential equations arise in many areas of applied mathematics. For this reason, these equations have received much attention in recent years. A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functional of the past history are involved as well as the present state of the system. The literature concerning first and second order ordinary neutral differential equations is very extensive. A good guide to the literature for neutral functional differential equations is the book by Hale and Verduyn Lunel [4] and the references therein. First order partial neutral differential and functional differential equations have been studied by several authors [5]. There is an extensive literature in which Sobolev type equations are investigated, in the abstract framework, see for instance [6, 7, 8, 9] and Radhakrishnan et al. [10].

On the other hand, the theory of impulsive differential equations [11] has become an important area of investigation in recent years, stimulated by their numerous applications to problems from physics, population dynamics, ecology,biological systems, biotechnology, optimal control and so forth. However, one may easily visualize that abrupt changes such as shock, harvesting and disasters may occur in nature. These phenomena are short time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. Ordinary differential equations first order and second order with impulses have been treated in several works. We refer the reader to [12, 13]. Partial differential equations with impulses have been studied by Liu [14], Rogovchenko [15, 16].

The literature related to abstract second order functional differential equations is extensive. Balachandran and Park [17] prove the existence of second order nonlinear differential equations with nonlocal conditions in Banach spaces. We refer the reader to Hernandez et al. [18, 19] for second order impulsive differential equations. Many of the real life problems are modeled as matrix second-order systems. Necessary and sufficient conditions for controllability of matrix second-order linear systems have been established by Hughes and Skelton [20]. The solution of the second-order system, using sine and cosine matrices, is given by Hargreaves and Higham [21] and the controllability of second order Sobolev-type integrodifferential systems by Radhakrishnan and Anukokila [22]. Controllability of some second-order system has been studied by Park and Han [23]. Sobolev-type equation appears in a physical problems such as flow of fluid through fissured rocks [24], thermodynamics, propagation of long waves of small amplitude and shear in second order fluids and so on. Brill [25] and Showalter [26] established the existence of solutions of semilinear Sobolev type evolution equations in Banach space. For more details, we refer the reader to [27, 28].

From the above literatures, it should be noted that there are several contributions on the existence and controllability of the second order nonlinear impulsive neutral integrodifferential equation and existence and controllability of integrodifferential equations with and without randomness using one or more parameter families. Till now, there is no work reported on the existence results for second order Sobolev-type neutral impulsive integrodifferential system using fixed point approach. Motivated by this fact, in this paper, we make an attempt to fill this gap by studying existence of second order Sobolev-type neutral impulsive integrodifferential equation in Banach spaces.

2 Preliminaries

Let X be Banach spaces endowed with the norm $\|\cdot\|$. In what follows, we put $t = 0,\ t_{n+1} = b$ and we denote by $\mathcal{PC}$ the space formed by the functions $u : \mathcal{I} \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i, x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exist for all $i = 1, 2, ..., m$. It is clear that $\mathcal{PC}$, endowed with the norm $\|x\|_{\mathcal{PC}} := \sup_{t\in\mathcal{I}} \|x(t)\|$, is a Banach space. Similarly, $\mathcal{PC}'$ will be the space of the functions $x(\cdot) \in \mathcal{PC}$ such that $x(\cdot)$ is continuously differentiable on $I\backslash t_i, i = 1, 2, ..., n$ and the derivatives $u'_R(t) = \lim_{s\to 0} \dfrac{u(t+s) - u(t^+)}{s}$, $u'_L(t) = \lim_{s\to 0} \dfrac{u(t+s) - u(t^-)}{s}$ are continuous on $[0, b[$ and $]0, b]$, respectively. Next, for $x \in \mathcal{PC}'$, we represent, by $u'(t)$, the left derivative at $t \in\ ]0, b]$ and, by $u'(0)$, the right derivative at zero. It easy to see that $\mathcal{PC}'$, provided with the norm $\|u\|_{\mathcal{PC}'} := \|u\|_{\mathcal{PC}} + \|u'\|_{\mathcal{PC}}$ is Banach space.

The operator-valued function $\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is strongly continuous group of linear operators on the space $\mathcal{Q} \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times \mathcal{Q}$. From this, it follows that $AS(t) : \mathcal{Q} \to X$ is bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in \mathcal{Q}$. Furthermore, if $x : [0, \infty[\ \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function, which is a consequence of the fact that $\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)ds \\ \int_0^t C(t-s)x(s)ds \end{bmatrix}$ defines an $\mathcal{Q} \times X$-valued continuous function.

Definition 2.1 ([2]). A one parameter family $\{C(t), t \in \mathcal{R}\}$ of bounded linear operators in the Banach space X is called a *strongly continuous cosine family* if and only if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathcal{R}$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is continuous in t on $\mathcal{R}$, for each $x \in X$.

If $C(t), t \in \mathcal{R}$ is a strongly continuous cosine family in X, then $S(t), t \in \mathcal{R}$ is the associated one parameter sine family of operators in X is defined by

$$S(t)x := \int_0^t C(s)xds, \quad x \in X, t \in \mathcal{R}.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{R}\}$ is the operator $A : X \to X$, defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) := \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$.

Define $\mathcal{Q} := \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. We assume

(A1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{R}\}$ of bounded linear operators in the Banach space X.

To establish our main theorem, we need the following lemmas:

Lemma 2.2 ([2])**.** *Let (A1) hold. Then*

(i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq Me^{\omega|t|}$ and

$$\|S(t) - S(t^*)\| \leq M\Big|\int_t^{t^*} e^{\omega|s|}ds\Big|, \text{ for } t,\ t^* \in \mathcal{R};$$

(ii) $S(t)X \subset \mathcal{Q}$ and $S(t)\mathcal{Q} \subset D(A)$, for $t \in \mathcal{R}$;

(iii) $\frac{d}{dt}C(t)x = AS(t)x$, for $x \in \mathcal{Q}$ and $t \in \mathcal{R}$

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$, for $x \in D(A)$ and $t \in \mathcal{R}$.

Lemma 2.3 ([2])**.** *Let (A1) hold and $v : \mathcal{R} \to X$ be such that v is continuous and let $p(t) = \int_0^t S(t-s)v(s)ds$. Then p is twice continuously differentiable and, for $t \in \mathcal{R},\ p(t) \in D(A)$, $p'(t) = \int_0^t C(t-s)v(s)ds$ and $p''(t) = Ap(t) + v(t)$.*

In this article, we assume that there exists an operator E on $\mathcal{D}(E) = X$ given by the formula

$$E = \Big[I + \sum_{i=1}^{n} c_i B^{-1}C(t_i)B\Big]^{-1}$$

with the existence of E can be observed from the following fact (see [29]). Suppose that $\|B^{-1}C(t_i)B\| \leq \mathcal{C}e^{-\delta t_i} (i = 1, 2, \ldots, n)$ where δ is a positive constant and $\mathcal{C} \leq 1$. If $\sum_{i=1}^{p} |c_i|e^{-\delta t_i} < 1/\mathcal{C}$ then $\|\sum_{i=1}^{p} c_i B^{-1}C(t_i)B\| < 1$. So such an operator E exists on X.

First we study the following second order equation with nonlocal condition

$$(Bu(t))'' = Au(t) + f(t, u(t)), \qquad t \in (0, b], \tag{2.1}$$

$$u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0, \qquad u'(0) + q(u) = v_0. \tag{2.2}$$

Definition 2.4. A continuous solution u of the integral equation

$$\begin{aligned} u(t) = {} & B^{-1}C(t)BEu_0 + B^{-1}S(t)[v_0 - q(u)] - \sum_{i=1}^{n} c_i B^{-1}C(t)BE \\ & \times \Big\{\int_0^{t_i} B^{-1}S(t_i - s)f(s, u(s))ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\} \\ & + \int_0^t B^{-1}S(t-s)f(s, u(s))ds \end{aligned} \tag{2.3}$$

is said to be a *mild solution of problem* (2.1)-(2.2) on I.

If $u(\cdot)$ is a mild solution of the problem (2.1)-(2.2), then by the properties of a second order differential equation and Lemma 2.2, we have

$$\begin{aligned} u'(t) &= B^{-1}C(t)[v_0 - q(u)] + B^{-1}AS(t)BEu_0 + \int_0^t B^{-1}C(t-s)f(s,u(s))ds \\ &\quad - \sum_{i=1}^{n} c_i B^{-1}AS(t)BE\Big\{ \int_0^{t_i} B^{-1}S(t_i - s)f(s,u(s))ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\}. \end{aligned} \tag{2.4}$$

Remark 2.5. A mild solution of (2.1)-(2.2) satisfies the condition (2.2). For, from (2.3)

$$u(0) = Eu_0 - \sum_{i=1}^{n} c_i E\Big\{ \int_0^{t_i} S(t_i - s)B^{-1}f(s,u(s))ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\} \text{ and}$$

$$\begin{aligned} u(t_j) &= B^{-1}C(t_j)BEu_0 + B^{-1}S(t_j)[v_0 - q(u)] \\ &\quad - \sum_{i=1}^{n} c_i B^{-1}C(t_j)BE\Big\{ \int_0^{t_i} B^{-1}S(t_i - s)f(s,u(s))ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\} \\ &\quad + \int_0^{t_j} S(t_j - s)B^{-1}f(s,u(s))ds. \end{aligned}$$

Therefore

$$\begin{aligned} u(0) + \sum_{j=1}^{n} c_j u(t_j) = &\Big[I + \sum_{j=1}^{n} c_j B^{-1}C(t_j)B\Big]Eu_0 - \Big[I + \sum_{j=1}^{n} c_j B^{-1}C(t_j)B\Big] \\ &\times \sum_{i=1}^{n} c_i E\Big\{ \int_0^{t_i} S(t_i - s)B^{-1}f(s,u(s))ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\} \\ &+ \int_0^{t_j} S(t_j - s)B^{-1}f(s,u(s))ds + B^{-1}S(t_j)[v_0 - q(u)] \\ &= u_0. \end{aligned}$$

To prove the existence result, the following hypotheses are needed:

(H1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathfrak{R}\}$ of bounded linear operators in Banach space X. There exist constants $\mathcal{M}_1 \geq 1$ and $\mathcal{M}_2 \geq 0$ such that $\|S(t)\| \leq \mathcal{M}_1$ and $\|C(t)\| \leq \mathcal{M}_2$, for every $t \in [0, b]$. Furthermore we take $\mathcal{M}_3 = \sup_{0 \leq t \leq b} \|AS(t)\|$ and $\mathcal{R}_0 = \|B^{-1}\|$. Let $c = \sum_{i=1}^{p} |c_i|$.

(H2) The function $q : \mathcal{P}C(I, X) \to X$ is continuous and there exists a constant $\mathcal{M}_q \geq 0$ such that

$$\|q(u)\| \leq \mathcal{M}_q \|u\|, \quad \text{for } u \in X.$$

(H3) The function $f : I \times X \to X$ satisfies the following conditions:

(i) For each $t \in I$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in X$, the function $f(\cdot, X) : I \to X$ is strongly measurable.

(ii) There exist constants $\mathcal{L}_1 > 0,\ \mathcal{L}_f > 0$ such that

$$\|f(t, u(t))\| \leq \mathcal{L}_1 \|u\|, \text{ for } t \in I \text{ and } u \in X \qquad \text{and}$$

$$\|f(t, u_1(t)) - f(t, u_2(t))\| \leq \mathcal{L}_f \|u_1 - u_2\|, \text{ for } t \in I \text{ and } u_i \in X,\ i = 1, 2.$$

(H4) Let us take

$$\left.\begin{array}{r}\mathcal{R}_0\mathcal{M}_1[\|v_0\| + \mathcal{M}_q r] + \mathcal{R}_0\mathcal{M}_2\|BEu_0\| + b\mathcal{R}_0\mathcal{M}_1\mathcal{L}_1 r \\ + c\mathcal{R}_0^2\|BE\|\mathcal{M}_2\big\{b\mathcal{M}_1\mathcal{L}_1 r + \mathcal{M}_1[\|v_0\| + \mathcal{M}_q r]\big\}\end{array}\right\} \leq \xi$$

and

$$\left.\begin{array}{r}\mathcal{R}_0\mathcal{M}_2[\|v_0\| + \mathcal{M}_q r] + \mathcal{R}_0\|BEu_0\|\mathcal{M}_3 + b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_1 r \\ + c\mathcal{R}_0^2\|BE\|\mathcal{M}_3\big\{b\mathcal{M}_1\mathcal{L}_1 r + \mathcal{M}_1[\|v_0\| + \mathcal{M}_q r]\big\}\end{array}\right\} \leq \widehat{\xi}.$$

For our convenience,

$$\Lambda_1 = [b\mathcal{R}_0\mathcal{M}_1\mathcal{L}_f + bc\mathcal{R}_0^2\|BE\|\mathcal{M}_2\mathcal{M}_1\mathcal{L}_f] \text{ and}$$

$$\Lambda_2 = [b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_f + bc\mathcal{R}_0^2\|BE\|\mathcal{M}_2\mathcal{M}_3\mathcal{L}_f].$$

3 Existence Results

Theorem 3.1. *If assumptions* **(H1)-(H4)** *hold, then* (2.1)-(2.2) *has a mild solution on I.*

Proof. Let $\mathcal{F}$ be the subset of $\mathcal{C}(I, X)$ defined by

$$\mathcal{F} = \{u : u(t) \in \mathcal{C}(I, X), \|u(t)\| \leq r, \text{ for } \ t \in I\}.$$

We define a mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ by

$$\begin{aligned}(\Gamma u)(t) = {} & B^{-1}S(t)[v_0 - q(u)] + B^{-1}C(t)BEu_0 - \sum_{i=1}^{n} c_i B^{-1}C(t)BE \\ & \times \Big\{ \int_0^{t_i} B^{-1}S(t_i - s)f(s, u(s))ds + B^{-1}S(t_i)[v_0 - q(u)] \Big\} \\ & + \int_0^t S(t-s)B^{-1}f(s, u(s))ds.\end{aligned}$$

First we show that the operator Γ maps $\mathcal{F}$ into itself. Now

$$\begin{aligned}
&\|(\Gamma u)(t)\| \\
&\leq \|B^{-1}S(t)[v_0-q(u)]\|+\|B^{-1}C(t)BEu_0\|+\|\int_0^t S(t-s)B^{-1}f(s,u(s))ds\| \\
&\quad +\|\sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{\int_0^{t_i} B^{-1}S(t_i-s)f(s,u(s))ds+B^{-1}S(t_i)[v_0-q(u)]\Big\}\| \\
&\leq \mathcal{R}_0\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_2+b\mathcal{R}_0 M_1\mathcal{L}_f r \\
&\quad + bc\mathcal{R}_0^2\|BE\|\mathcal{M}_2\big\{\mathcal{M}_1\mathcal{L}_f r+\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r]\big\}
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Gamma u)'(t)\| \\
&\leq \|B^{-1}C(t)[v_0-q(u)]\|+\|B^{-1}AS(t)BEu_0\|+\|\int_0^t C(t-s)B^{-1}f(s,u(s)) \\
&\quad +\|\sum_{i=1}^{n} c_i B^{-1}AS(t)BE\Big\{\int_0^{t_i} B^{-1}S(t_i-s)f(s,u(s))ds+B^{-1}S(t_i)[v_0-q(u)]\Big\}\| \\
&\leq \mathcal{R}_0\mathcal{M}_2[\|v_0\|+\mathcal{M}_q r]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_3+b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_f r \\
&\quad + bc\mathcal{R}_0^2\|BE\|\mathcal{M}_3\big\{\mathcal{M}_1\mathcal{L}_f r+\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r]\big\}.
\end{aligned}$$

From the assumption (**H4**), $\|(\Gamma u)(t)\| \leq \xi$ and $\|(\Gamma u)'(t)\| \leq \widehat{\xi}$. Therefore Γ maps $\mathcal{F}$ into itself. Also, if $u_1, u_2 \in \mathcal{F}$, then

$$\begin{aligned}
&\|(\Gamma u_1)(t)-(\Gamma u_2)(t)\| \\
&\leq \|\sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{\int_0^{t_i} B^{-1}S(t_i-s)[f(s,u_1(s))-f(s,u_2(s))]ds\Big\}\| \\
&\quad +\|\int_0^t S(t-s)B^{-1}[f(s,u_1(s))-f(s,u_2(s))]ds\| \\
&\leq [b\mathcal{R}_0\mathcal{M}_1\mathcal{L}_f+bc\mathcal{R}_0^2\|BE\|\mathcal{M}_2\mathcal{M}_1\mathcal{L}_f]\|u_1-u_2\| \\
&\leq \Lambda_1\|u_1-u_2\|,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\|(\Gamma u_1)'(t)-(\Gamma u_2)'(t)\| \\
&\leq \|\sum_{i=1}^{n} c_i B^{-1}AS(t)BE\Big\{\int_0^{t_i} B^{-1}S(t_i-s)[f(s,u_1(s))-f(s,u_2(s))]ds\Big\}\| \\
&\quad +\|\int_0^t C(t-s)B^{-1}[f(s,u_1(s))-f(s,u_2(s))]ds\| \\
&\leq [b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_f+bcR_0^2\|BE\|\mathcal{M}_2\mathcal{M}_3\mathcal{L}_f]\|u_1-u_2\| \\
&\leq \Lambda_2\|u_1-u_2\|.
\end{aligned} \tag{3.2}$$

Since $\Lambda_1 < 1$ and $\Lambda_2 < 1$, (3.1) and (3.2) show that the operator Γ is contraction on $\mathcal{C}(I,X)$ and so, by Banach fixed point theorem, there exists a unique fixed point $u \in \mathcal{F}$ such that $(\Gamma u)(t) = u(t)$. This fixed point is then the solution of the problem (2.1)-(2.2). Thus Theorem 3.1 is proved. □

4 Neutral Integrodifferential Equation

Consider the following second order neutral integro-differential equation of Sobolev type

$$\frac{d}{dt}[(Bu(t))' + g(t,u(t))] = Au(t) + f(t,u(t)) + \int_0^t h(t,s,u(s))ds, \quad t \in (0,b], \tag{4.1}$$

$$u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0, \qquad u'(0) + q(u) = v_0. \tag{4.2}$$

To prove the existence result of the system (4.1)-(4.2). For this we impose following conditions:

(H5) The function $g : I \times X \to X$ satisfy the following conditions:

(i) For each $t \in I$, the function $g(t,\cdot) : X \to X$ is continuous and for each $x \in X$, the function $g(\cdot, X) : I \to X$ is strongly measurable.

(ii) There exist constants $\mathcal{L}_2 > 0$, $\mathcal{L}_g > 0$ and $\mathcal{L}_g^0 > 0$ such that

$$\begin{aligned} \|g(t,u(t))\| &\leq \mathcal{L}_2\|u\|, \quad \text{for } t \in I \text{ and } u \in X \\ \|g(t,u_1) - g(t,u_2)\| &\leq \mathcal{L}_g\|u_1 - u_2\|, \text{ for } t \in I \text{ and } u_i \in X,\ i = 1,2. \end{aligned}$$

and $\|g(0,u(0))\| \leq \mathcal{L}_g^0$.

(H6) The function $h : I^2 \times X \to X$ satisfy the following conditions:

(i) For each $t, s \in I$, the function $h(t,s,\cdot) : X \to X$ is continuous and for each $x \in X$, the function $h(t,s,\cdot) : I \times I \to X$ is strongly measurable.

(ii) There exist constants $\mathcal{L}_3 > 0$, $\mathcal{L}_h > 0$ such that

$$\|h(t,s,u)\| \leq \mathcal{L}_3\|u\|, \text{ for } \ t,s \in I \text{ and } u \in X, \quad \textit{and}$$

$$\|h(t,s,u_1) - h(t,s,u_2)\| \leq \mathcal{L}_h\|u_1 - u_2\|, \text{ for } t,s \in I \text{ and } u_i \in X,\ i = 1,2.$$

(H7) For convenience

$$\left.\begin{aligned} &\mathcal{R}_0\mathcal{M}_1[\|v_0\| + \mathcal{M}_q r_1] + R_0\|BEu_0\|\mathcal{M}_2 + \mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_2 r_1 \\ &+ b\mathcal{R}_0\mathcal{M}_1[\mathcal{L}_1 r_1 + b\mathcal{L}_3 r_1] + c\mathcal{R}_0^2\|BE\|\mathcal{M}_2\Big[b\mathcal{M}_2\mathcal{L}_2 r_1 + \mathcal{M}_1\mathcal{L}_g^0 \\ &\qquad + \mathcal{M}_1 b[\mathcal{L}_1 r_1 + b\mathcal{L}_3 r_1] + \mathcal{M}_1[\|v_0\| + \mathcal{M}_q r_1]\Big] \end{aligned}\right\} \leq \eta$$

and

$$\left.\begin{array}{l}\mathcal{R}_0\mathcal{M}_2[\|v_0\|+\mathcal{M}_q r_1]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_3+\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2 r_1\\ +b\mathcal{R}_0\mathcal{M}_2[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+c\mathcal{R}_0^2\|BE\|\mathcal{M}_3\Big[b\mathcal{M}_2\mathcal{L}_2 r_1+\mathcal{M}_1\mathcal{L}_g^0\\ \qquad +\mathcal{M}_1 b[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_1]\Big]\end{array}\right\}\leq\widehat{\eta}.$$

Definition 4.1. A continuous solution u of the integral equation

$$\begin{aligned}u(t)&=B^{-1}C(t)BEu_0+B^{-1}S(t)[v_0-q(u)]+B^{-1}S(t)g(0,u(0))\\&\quad-\int_0^t B^{-1}C(t-s)g(s,u(s))ds-\sum_{i=1}^n c_iB^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0))\\&\quad-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds+B^{-1}S(t_i)[v_0-q(u)]\\&\quad+\int_0^{t_i}B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\Big\}\\&\quad+\int_0^t S(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\end{aligned}\tag{4.3}$$

is said to be a *mild solution of problem* (4.1)-(4.2) on I.

If $u(\cdot)$ is a mild solution of the problem (4.1)-(4.2), then by the properties of a second order differential equation and Lemma 2.2, we have

$$\begin{aligned}u'(t)&=B^{-1}AS(t)BEu_0+B^{-1}C(t)[v_0-q(u)]+B^{-1}C(t)g(0,u(0))\\&\quad-\int_0^t B^{-1}AS(t-s)g(s,u(s))ds-\sum_{i=1}^n c_iB^{-1}AS(t)BE\Big\{B^{-1}S(t_i)g(0,u(0))\\&\quad-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds+B^{-1}S(t_i)[v_0-q(u)]\\&\quad+\int_0^{t_i}B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\Big\}\\&\quad+\int_0^t C(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds.\end{aligned}\tag{4.4}$$

Remark 4.2. A mild solution of the neutral integrodifferential (4.1)-(4.2) satisfies the condition (4.2). For, from (4.3)

$$\begin{aligned}&u(0)\\&=Eu_0-\sum_{i=1}^n c_iE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds\\&\quad+\int_0^{t_i}B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]\Big\}\end{aligned}$$

and

$$
\begin{aligned}
&u(t_j)\\
&= B^{-1}S(t_j)[v_0 = q(u)] + B^{-1}C(t_j)BEu_0 + B^{-1}S(t_j)g(0,u(0))\\
&\quad - \int_0^{t_j} B^{-1}C(t_j-s)g(s,u(s))ds\\
&\quad - \sum_{i=1}^{n} c_i B^{-1}C(t_j)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds\\
&\quad + \int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s)) + \int_0^{s} h(s,\tau,u(\tau))d\tau\Big]ds + B^{-1}S(t_i)[v_0 - q(u)]\Big\}\\
&\quad + \int_0^{t_j} S(t_j-s)B^{-1}\Big[f(s,u(s)) + \int_0^{s} h(s,\tau,u(\tau))d\tau\Big]ds.
\end{aligned}
$$

Therefore $u(0) + \sum_{j=1}^{n} c_j u(t_j) = u_0$.

Theorem 4.3. *If assumptions* **(H1)-(H3)** *and* **(H5)-(H7)** *hold, then* (4.1)-(4.2) *has a mild solution on* I.

Proof. Let $\mathcal{F}_1$ be the subset of $\mathcal{C}(I,X)$ defined by $\mathcal{F}_1 = \{u : u(t) \in C(I,X), \|u(t)\| \leq r_1,\ \text{for } t \in I\}$. We define a mapping $\Psi : \mathcal{F}_1 \to \mathcal{F}_1$ by

$$
\begin{aligned}
&(\Psi u)(t)\\
&= B^{-1}S(t)v_0 + B^{-1}C(t)BEu_0 + B^{-1}S(t)g(0,u(0)) - \int_0^{t} B^{-1}C(t-s)g(s,u(s))ds\\
&\quad - \sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds\\
&\quad + \int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s)) + \int_0^{s} h(s,\tau,u(\tau))d\tau\Big]ds + B^{-1}S(t_i)v_0\Big\}\\
&\quad + \int_0^{t} S(t-s)B^{-1}\Big[f(s,u(s)) + \int_0^{s} h(s,\tau,u(\tau))d\tau\Big]ds.
\end{aligned}
$$

Now we show that the operator Ψ maps $\mathcal{F}_1$ into itself.

$$
\begin{aligned}
&\|(\Psi u)(t)\|\\
&\leq \|B^{-1}S(t)[v_0 - q(u)]\| + \|B^{-1}C(t)BEu_0\| + \|B^{-1}S(t)g(0,u(0))\|\\
&\quad + \|\int_0^{t} B^{-1}C(t-s)g(s,u(s))ds\|\\
&\quad + \Big\|\sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds
\end{aligned}
$$

$$+\int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]\Big\}\Big\|$$
$$+\|\int_0^t S(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\|$$
$$\le \mathcal{R}_0\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_1]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_2+\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_2 r_1$$
$$+b\mathcal{R}_0\mathcal{M}_1[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+cR_0^2\|BE\|\mathcal{M}_2\Big[b\mathcal{M}_2\mathcal{L}_2 r_1+\mathcal{M}_1\mathcal{L}_g^0$$
$$+\mathcal{M}_1 b[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_1]\Big]$$

and

$$\|(\Psi u)'(t)\|$$
$$\le \|B^{-1}C(t)[v_0-q(u)]\|+\|B^{-1}AS(t)BEu_0\|+\|B^{-1}C(t)g(0,u(0))\|$$
$$+\|\int_0^t B^{-1}AS(t-s)g(s,u(s))ds\|$$
$$+\Big\|\sum_{i=1}^n c_iB^{-1}AS(t)BE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds$$
$$+\int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]\Big\}\Big\|$$
$$+\|\int_0^t C(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\|$$
$$\le \mathcal{R}_0\mathcal{M}_2[\|v_0\|+\mathcal{M}_q r_1]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_3+\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2 r_1$$
$$+b\mathcal{R}_0\mathcal{M}_2[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+c\mathcal{R}_0^2\|BE\|\mathcal{M}_3\Big[b\mathcal{M}_2\mathcal{L}_2 r_1+\mathcal{M}_1\mathcal{L}_g^0$$
$$+\mathcal{M}_1 b[\mathcal{L}_1 r_1+b\mathcal{L}_3 r_1]+\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_1]\Big].$$

From the assumption (**H7**), $\|(\Psi u)(t)\|\le\eta$ and $\|(\Psi u)'(t)\|\le\widehat{\eta}$. Therefore Ψ maps $\mathfrak{F}_1$ into itself. Moreover, if $u_1,u_2\in\mathfrak{F}_1$, then

$$\|(\Psi u_1)(t)-(\Psi u_2)(t)\|$$
$$\le \|B^{-1}S(t)[g(0,u_1(0))-g(0,u_2(0))]\|+\|\int_0^t B^{-1}C(t-s)[g(s,u_1(s))-g(s,u_2(s))]ds$$
$$+\Big\|\sum_{i=1}^n c_iB^{-1}C(t)BE\Big\{B^{-1}S(t_i)[g(0,u_1(0))-g(0,u_2(0))]$$
$$-\int_0^{t_i}B^{-1}C(t_i-s)[g(s,u_1(s))-g(s,u_2(s))]ds$$
$$+\int_0^{t_i}B^{-1}S(t_i-s)\Big[\Big(f(s,u_1(s))-f(s,u_2(s))\Big)+\int_0^s\Big(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\Big\}\Big\|$$
$$+\|\int_0^t S(t-s)B^{-1}\Big[\Big(f(s,u_1(s))-f(s,u_2(s))\Big)+\int_0^s\Big(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\|$$
$$\le\Big[\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g+b\mathcal{M}_1\{\mathcal{L}_f+b\mathcal{L}_h\}$$

$$+ c\mathcal{R}_0^2\|BE\|\mathcal{M}_2[\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{M}_2\mathcal{L}_g + b\mathcal{M}_1\{\mathcal{L}_f + b\mathcal{L}_h\}]\Big]\|u_1 - u_2\|$$
$$\leq \Theta_1\|u_1 - u_2\|, \tag{4.5}$$

where $\Theta_1 = \Big[\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g + b\mathcal{M}_1\{\mathcal{L}_f + b\mathcal{L}_h\} + c\mathcal{R}_0^2\|BE\|\mathcal{M}_2[\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{M}_2\mathcal{L}_g + b\mathcal{M}_1\{\mathcal{L}_f + b\mathcal{L}_h\}]\Big].$

Also

$$\|(\Psi u_1)'(t) - (\Psi u_2)'(t)\|$$
$$\leq \|B^{-1}C(t)[g(0,u_1(0)) - g(0,u_2(0))]\| + \|\int_0^t B^{-1}AS(t-s)[g(s,u_1(s)) - g(s,u_2(s))]ds\|$$
$$+ \Big\|\sum_{i=1}^n c_i B^{-1}AS(t)BE\Big\{B^{-1}S(t_i)[g(0,u_1(0)) - g(0,u_2(0))]$$
$$- \int_0^{t_i} B^{-1}C(t_i - s)[g(s,u_1(s)) - g(s,u_2(s))]ds$$
$$+ \int_0^{t_i} B^{-1}S(t_i - s)\Big[\Big(f(s,u_1(s)) - f(s,u_2(s))\Big) + \int_0^s\Big(h(s,\tau,u_1(\tau)) - h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\Big\}\Big\|$$
$$+ \|\int_0^t C(t-s)B^{-1}\Big[\Big(f(s,u_1(s)) - f(s,u_2(s))\Big) + \int_0^s\Big(h(s,\tau,u_1(\tau)) - h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\|$$
$$\leq \Big[\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_g + b\mathcal{M}_2\{\mathcal{L}_f + b\mathcal{L}_h\}$$
$$+ c\mathcal{R}_0^2\|BE\|\mathcal{M}_3[\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{M}_2\mathcal{L}_g + b\mathcal{M}_1\{\mathcal{L}_f + b\mathcal{L}_h\}]\Big]\|u_1 - u_2\|$$
$$\leq \Theta_2\|u_1 - u_2\|, \tag{4.6}$$

where $\Theta_2 = \Big[\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_g + b\mathcal{M}_2\{\mathcal{L}_f + b\mathcal{L}_h\} + c\mathcal{R}_0^2\|BE\|\mathcal{M}_3[\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{M}_2\mathcal{L}_g + b\mathcal{M}_1\{\mathcal{L}_f + b\mathcal{L}_h\}]\Big].$

Since $\Theta_1 < 1$ and $\Theta_2 < 1$, (4.5) and (4.6) show that the operator Ψ is contraction on $\mathcal{C}(I, X)$ and hence by Banach fixed point theorem there exists a unique fixed point $u \in \mathfrak{F}_1$ such that $(\Psi u)(t) = u(t)$. This fixed point is then the solution of the problem (4.1)-(4.2). □

5 Impulsive Integrodifferential Equation

To prove the existence result of the system (1.1) − (1.3). For this we impose following conditions:

(H8) $I_k, J_k : X \to X$, $k = 1, 2, ..., m$, are continuous and d_i, d_j are positive constants such that

$$\|I_k\| = d_i, \qquad \|J_k\| = d_j, \quad k = 1, 2, ..., m \quad and$$

$$\|I_k(u_1) - I_k(u_2)\| \leq d_i\|u_1 - u_2\|, \quad \|J_k(u_1) - J_k(u_2)\| \leq d_j\|u_1 - u_2\|,$$

for all $u_1, u_2 \in X$ and $k = 1, 2, ..., m$.

(H9) Let $\gamma, \widehat{\gamma}$ be any two constants, we take

$$\left.\begin{array}{l}\mathcal{R}_0\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_2]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_2+\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_1[\mathcal{L}_1 r_2+b\mathcal{L}_3 r_2]\\ +b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_2 r_2+c\mathcal{R}_0^2\|BE\|\mathcal{M}_2\Big[b\mathcal{M}_2\mathcal{L}_2 r_2+\mathcal{M}_1\mathcal{L}_g^0+\mathcal{M}_1 b[\mathcal{L}_1 r_2+b\mathcal{L}_3 r_2]\\ +\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_2]+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j\Big]+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j\end{array}\right\}\leq\gamma,$$

$$\left.\begin{array}{l}\mathcal{R}_0\mathcal{M}_2[\|v_0\|+\mathcal{M}_q r_2]+\mathcal{R}_0\|BEu_0\|\mathcal{M}_3+\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_2[\mathcal{L}_1 r_2+b\mathcal{L}_3 r_2]\\ +b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2 r_2+c\mathcal{R}_0^2\|BE\|\mathcal{M}_3\Big[b\mathcal{M}_2\mathcal{L}_2 r_2+\mathcal{M}_1\mathcal{L}_g^0+\mathcal{M}_1 b[\mathcal{L}_1 r_2+b\mathcal{L}_3 r_2]\\ +\mathcal{M}_1[\|v_0\|+\mathcal{M}_q r_2]+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j\Big]+\mathcal{M}_3\sum_{k=1}^m d_i+\mathcal{M}_2\sum_{k=1}^m d_j\end{array}\right\}\leq\widehat{\gamma}.$$

A continuous solution u of the integral equation

$$\begin{aligned}
&u(t)\\
&=B^{-1}S(t)[v_0-q(u)]+B^{-1}C(t)BEu_0+B^{-1}S(t)g(0,u(0))-\int_0^t B^{-1}C(t-s)g(s,u(s))ds\\
&-\sum_{i=1}^n c_iB^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds\\
&+\int_0^{t_i}B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]\\
&+\sum_{0<t_k<t_i}B^{-1}C(t_i-t_k)I_ku(t_k)+\sum_{0<t_k<t_i}B^{-1}S(t_i-t_k)J_ku(t_k)\Big\}\\
&+\int_0^t S(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\\
&+\sum_{0<t_k<t}B^{-1}C(t-t_k)I_ku(t_k)+\sum_{0<t_k<t}B^{-1}S(t-t_k)J_ku(t_k) \qquad (5.1)
\end{aligned}$$

is said to be a mild solution of problem (1.1) – (1.3) on I.

If $u(\cdot)$ is a mild solution of the problem (1.1) – (1.3), then by the properties of a second order differential equation and Lemma 2.2, we have

$$\begin{aligned}
&u'(t)\\
&=B^{-1}C(t)[v_0-q(u)]+B^{-1}AS(t)BEu_0+B^{-1}C(t)g(0,u(0))\\
&\quad-\int_0^t B^{-1}AS(t-s)g(s,u(s))ds\\
&\quad-\sum_{i=1}^n c_iB^{-1}AS(t)BE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i}B^{-1}C(t_i-s)g(s,u(s))ds\\
&\quad+\int_0^{t_i}B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]
\end{aligned}$$

$$+\sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_ku(t_k)+\sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_ku(t_k)\Big\}$$
$$+\int_0^t C(t-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds$$
$$+\sum_{0<t_k<t} B^{-1}AS(t-t_k)I_ku(t_k)+\sum_{0<t_k<t} B^{-1}C(t-t_k)J_ku(t_k).$$

Remark 5.1. A mild solution of the neutral integrodifferential (1.1) – (1.3) satisfies the condition (1.2). For, from (5.1)

$$u(0)=Eu_0-\sum_{i=1}^{n} c_iE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds$$
$$+\int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]$$
$$+\sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_ku(t_k)+\sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_ku(t_k)\Big\}$$

and

$$u(t_j)$$
$$=B^{-1}S(t_j)[v_0-q(u)]+B^{-1}C(t_j)BEu_0+B^{-1}S(t_j)g(0,u(0))$$
$$-\int_0^{t_j} B^{-1}C(t_j-s)g(s,u(s))ds$$
$$-\sum_{i=1}^{n} c_iB^{-1}C(t_j)BE\Big\{B^{-1}S(t_i)g(0,u(0))-\int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds$$
$$+\int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds+B^{-1}S(t_i)[v_0-q(u)]$$
$$+\sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_ku(t_k)+\sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_ku(t_k)\Big\}$$
$$+\int_0^{t_j} S(t_j-s)B^{-1}\Big[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau\Big]ds$$
$$+\sum_{0<t_k<t} B^{-1}C(t_j-t_k)I_ku(t_k)+\sum_{0<t_k<t} B^{-1}S(t_j-t_k)J_ku(t_k).$$

Therefore $\quad u(0)+\sum_{j=1}^{n} c_ju(t_j)=u_0.$

Theorem 5.2. *If assumptions* **(H1)-(H3)**, **(H5)-(H6)** *and* **(H8)-(H9)** *hold, then* (1.1) – (1.3) *has a mild solution on* I.

Proof. Let $\mathcal{F}_3$ be the subset of $\mathcal{PC}(I,X)$ defined by,

$$\mathcal{F}_3 = \{u : u(t) \in \mathcal{PC}(I,X), \|u(t)\| \leq r_2, \text{ for } t \in I\}.$$

We define a mapping $\Phi : \mathcal{F}_3 \to \mathcal{F}_3$ by

$$\begin{aligned}
&(\Phi u)(t)\\
&= B^{-1}S(t)[v_0 - q(u)] + B^{-1}C(t)BEu_0 + B^{-1}S(t)g(0,u(0)) - \int_0^t B^{-1}C(t-s)g(s,u(s))ds\\
&\quad - \sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds\\
&\quad + \int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds + B^{-1}S(t_i)[v_0 - q(u)]\\
&\quad + \sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_k u(t_k) + \sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_k u(t_k)\Big\}\\
&\quad + \int_0^t S(t-s)B^{-1}\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\\
&\quad + \sum_{0<t_k<t} B^{-1}C(t-t_k)I_k u(t_k) + \sum_{0<t_k<t} B^{-1}S(t-t_k)J_k u(t_k).
\end{aligned}$$

Now we show that the operator Φ maps $\mathcal{F}_3$ into itself.

$$\begin{aligned}
\|(\Phi u)(t)\| \leq\ & \|B^{-1}S(t)[v_0 - q(u)]\| + \|B^{-1}C(t)BEu_0\| + \|B^{-1}S(t)g(0,u(0))\|\\
&+ \Big\|\int_0^t B^{-1}C(t-s)g(s,u(s))ds\Big\|\\
&+ \Big\|\sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds\\
&+ \int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds + B^{-1}S(t_i)[v_0 - q(u)]\\
&+ \sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_k u(t_k) + \sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_k u(t_k)\Big\}\Big\|\\
&+ \Big\|\int_0^t S(t-s)B^{-1}\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\Big\|\\
&+ \Big\|\sum_{0<t_k<t} B^{-1}C(t-t_k)I_k u(t_k)\Big\| + \Big\|\sum_{0<t_k<t} B^{-1}S(t-t_k)J_k u(t_k)\Big\|\\
\leq\ & \mathcal{R}_0\mathcal{M}_1[\|v_0\| + \mathcal{M}_q r_2] + \mathcal{R}_0\|BEu_0\|\mathcal{M}_2 + \mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_2 r_2\\
&+ b\mathcal{R}_0\mathcal{M}_1[\mathcal{L}_1 r_2 + b\mathcal{L}_3 r_2] + c\mathcal{R}_0^2\|BE\|\mathcal{M}_2\Big[b\mathcal{M}_2\mathcal{L}_2 r_2 + \mathcal{M}_1\mathcal{L}_g^0\\
&+ \mathcal{M}_1 b[\mathcal{L}_1 r_2 + b\mathcal{L}_3 r_2] + \mathcal{M}_1[\|v_0\| + \mathcal{M}_q r_2] + \mathcal{M}_2\sum_{k=1}^{m} d_i + \mathcal{M}_1\sum_{k=1}^{m} d_j\Big]\\
&+ \mathcal{M}_2\sum_{k=1}^{m} d_i + \mathcal{M}_1\sum_{k=1}^{m} d_j
\end{aligned}$$

and

$$
\begin{aligned}
&\|(\Phi u)'(t)\| \\
&\leq \|B^{-1}C(t)[v_0 - q(u)]\| + \|B^{-1}AS(t)BEu_0\| + \|B^{-1}C(t)g(0,u(0))\| \\
&\quad + \|\int_0^t B^{-1}AS(t-s)g(s,u(s))ds\| \\
&\quad + \Big\| \sum_{i=1}^{n} c_i B^{-1}AS(t)BE\Big\{B^{-1}S(t_i)g(0,u(0)) - \int_0^{t_i} B^{-1}C(t_i-s)g(s,u(s))ds \\
&\quad + \int_0^{t_i} B^{-1}S(t_i-s)\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds + B^{-1}S(t_i)[v_0 - q(u)] \\
&\quad + \sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)I_k u(t_k) + \sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)J_k u(t_k)\Big\}\Big\| \\
&\quad + \|\int_0^t C(t-s)B^{-1}\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds\| \\
&\quad + \|\sum_{0<t_k<t} B^{-1}C(t-t_k)I_k u(t_k)\| + \|\sum_{0<t_k<t} B^{-1}S(t-t_k)J_k u(t_k)\| \\
&\leq \mathcal{R}_0\mathcal{M}_2[\|v_0\| + \mathcal{M}_q r_2] + \mathcal{R}_0\|BEu_0\|\mathcal{M}_3 + \mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0 + b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2 r_2 \\
&\quad + b\mathcal{R}_0\mathcal{M}_2[\mathcal{L}_1 r_2 + b\mathcal{L}_h r_2] + c\mathcal{R}_0^2\|BE\|\mathcal{M}_3\Big[b\mathcal{M}_2\mathcal{L}_2 r_2 + \mathcal{M}_1\mathcal{L}_g^0 \\
&\quad + \mathcal{M}_1 b[\mathcal{L}_1 r_2 + b\mathcal{L}_3 r_2] + \mathcal{M}_1[\|v_0\| + \mathcal{M}_q r_2] + \mathcal{M}_2\sum_{k=1}^{m} d_i + \mathcal{M}_1\sum_{k=1}^{m} d_j\Big] \\
&\quad + \mathcal{M}_3\sum_{k=1}^{m} d_i + \mathcal{M}_2\sum_{k=1}^{m} d_j.
\end{aligned}
$$

From the assumption (**H9**), $\|(\Phi u)(t)\| \leq \gamma$ and $\|(\Phi u)'(t)\| \leq \widehat{\gamma}$. Therefore Φ maps $\mathcal{S}$ into itself. Also, let $u_1, u_2 \in \mathcal{S}$ then

$$
\begin{aligned}
\|(\Phi u_1)(t) - (\Phi u_2)(t)\| \quad \leq \quad & \|B^{-1}S(t)[g(0,u_1(0)) - g(0,u_2(0))]\| \\
& + \|\int_0^t B^{-1}C(t-s)[g(s,u_1(s)) - g(s,u_2(s))]ds\| \\
& + \Big\|\sum_{i=1}^{n} c_i B^{-1}C(t)BE\Big\{B^{-1}S(t_i)[g(0,u_1(0)) - g(0,u_2(0))] \\
& - \int_0^{t_i} B^{-1}C(t_i-s)[g(s,u_1(s)) - g(s,u_2(s))]ds \\
& + \int_0^{t_i} B^{-1}S(t_i-s)\Big[\Big(f(s,u_1(s)) - f(s,u_2(s))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s\Big(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\\
&+\sum_{0<t_k<t_i}B^{-1}C(t_i-t_k)[I_ku_1(t_k)-I_ku_2(t_k)]\\
&+\sum_{0<t_k<t_i}B^{-1}S(t_i-t_k)[J_ku_1(t_k)-J_ku_2(t_k)]\Big\}\Big\|\\
&+\|\int_0^t S(t-s)B^{-1}\Big[\Big(f(s,u_1(s))-f(s,u_2(s))\Big)\\
&+\int_0^s\Big(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\|\\
&+\|\sum_{0<t_k<t}B^{-1}C(t-t_k)[I_ku_1(t_k)-I_ku_2(t_k)]\|\\
&+\|\sum_{0<t_k<t}B^{-1}S(t-t_k)[J_ku_1(t_k)-J_ku_2(t_k)]\|\\
&\le\Big[\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_2\sum_{k=1}^m d_i\\
&+\mathcal{M}_1\sum_{k=1}^m d_j+c\mathcal{R}_0^2\|BE\|\mathcal{M}_2[\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{M}_2\mathcal{L}_2+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}\\
&+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j]\Big]\|u_1-u_2\|\\
&\le\Omega_1\|u_1-u_2\| \qquad (5.2)
\end{aligned}
$$

where $\Omega_1=\Big[\mathcal{R}_0\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_2\mathcal{L}_2+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j$
$+c\mathcal{R}_0^2\|BE\|\mathcal{M}_2[\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{M}_2\mathcal{L}_2+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_2\sum_{k=1}^m d_i+\mathcal{M}_1\sum_{k=1}^m d_j]\Big]$

and

$$
\begin{aligned}
&\|(\Phi u_1)'(t)-(\Phi u_2)'(t)\|\\
&\quad\le\ \|B^{-1}C(t)[g(0,u_1(0))-g(0,u_2(0))]\|\\
&\quad+\|\int_0^t B^{-1}AS(t-s)[g(s,u_1(s))-g(s,u_2(s))]ds\|\\
&\quad+\Big\|\sum_{i=1}^n c_iB^{-1}AS(t)BE\Big\{B^{-1}S(t_i)[g(0,u_1(0))-g(0,u_2(0))]\\
&\quad-\int_0^{t_i}B^{-1}C(t_i-s)[g(s,u_1(s))-g(s,u_2(s))]ds
\end{aligned}
$$

$$+\int_0^{t_i} B^{-1}S(t_i-s)\Big[\Big(f(s,u_1(s))-f(s,u_2(s))\Big)+\int_0^s\Big(h(s,\tau,(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds$$
$$+\sum_{0<t_k<t_i} B^{-1}C(t_i-t_k)[I_k u_1(t_k)-I_k u_2(t_k)]+\sum_{0<t_k<t_i} B^{-1}S(t_i-t_k)[J_k u_1(t_k)-J_k u_2(t_k)]\Big\}\Big\|$$
$$+\Big\|\int_0^t C(t-s)B^{-1}\Big[\Big(f(s,u_1(s))-f(s,u_2(s))\Big)+\int_0^s\Big(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau))\Big)d\tau\Big]ds\Big\|$$
$$+\Big\|\sum_{0<t_k<t} B^{-1}AS(t-t_k)[I_k u_1(t_k)-I_k u_2(t_k)]\Big\|+\Big\|\sum_{0<t_k<t} B^{-1}C(t-t_k)[J_k u_1(t_k)-J_k u_2(t_k)]\Big\|$$
$$\le \Big[\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2+b\mathcal{M}_2\{\mathcal{L}_f+b\mathcal{L}_h\}+\mathcal{M}_3\sum_{k=1}^m d_i+\mathcal{M}_2\sum_{k=1}^m d_j$$
$$+c\mathcal{R}_0^2\|BE\|\mathcal{M}_3[\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{M}_2\mathcal{L}_2+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_2\sum_{k=1}^m d_i$$
$$+\mathcal{M}_1\sum_{k=1}^m d_j]\Big]\|u_1-u_2\|.$$

Thus

$$\|(\Phi u_1)'(t)-(\Phi u_2)'(t)\|\le \Omega_2\|u_1-u_2\|, \tag{5.3}$$

where $\Omega_2=\Big[\mathcal{R}_0\mathcal{M}_2\mathcal{L}_g^0+b\mathcal{R}_0\mathcal{M}_3\mathcal{L}_2+b\mathcal{M}_2\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_3\sum_{k=1}^m d_i+\mathcal{M}_2\sum_{k=1}^m d_j$

$$+c\mathcal{R}_0^2\|BE\|\mathcal{M}_3[\mathcal{M}_1\mathcal{L}_g^0+b\mathcal{M}_2\mathcal{L}_2+b\mathcal{M}_1\{\mathcal{L}_1+b\mathcal{L}_3\}+\mathcal{M}_2\sum_{k=1}^m d_i$$
$$+\mathcal{M}_1\sum_{k=1}^m d_j]\Big].$$

Since $\Omega_1<1$ and $\Omega_2<1$, (5.2) and (5.3) show that the operator Φ is contraction on $\mathcal{PC}(I,X)$ and hence by Banach fixed point theorem there exists a unique fixed point $u\in\mathfrak{F}_3$ such that $(\Psi u)(t)=u(t)$. This fixed point is then the solution of the problem (1.1) – (1.3). □

6 Example

Consider the partial integrodifferential equation of the form

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\frac{\partial}{\partial t}z(t,y)-z_{yy}(t,y)-\int_{-\infty}^t\int_0^\pi b_1(s,y)z(s,y)ds\Big]-z_{yy}(t,y)&\\ = b_2(s,y)\Big(t,\sin z(t,y)\Big)+\int_0^t \sin z(s,y)e^{-z(\sin s,y)}ds,& \quad (6.1)\\ z(t,0) = z(t,\pi)=0,\quad t\in I,& \quad (6.2)\end{aligned}$$

$$z(0,y)+\sum_{i=1}^{m}e_i\Phi_{t_i}(s,y) \quad = \quad z_0(y) \quad 0<y<1,\ t\in I; \tag{6.3}$$

$$\Delta z(t)|_{t=t_k} = I_k(z(y)) = (\delta_i(z(y))+t_k)^{-1},\ z\in X, 1\le i\le m, \tag{6.4}$$

$$\Delta z'(t)|_{t=t_k} = J_k(z'(y)) = (\widehat{\delta_i}(z(y))+t_k)^{-1},\ z\in X, 1\le i\le m, \tag{6.5}$$

where $a(t,y)$ is continuous on $0\le y\le \pi, t\in I$ and the constant $e_i, \delta_i, \widehat{\delta_i}$ are small.

Let us take $X=\mathcal{L}^2[0,\pi]$ to be endowed with the usual norm $\|\cdot\|_{\mathcal{L}_2}$. and let

$$g(t,u(t)) = \int_{-\infty}^{t}\int_0^{\pi} b_1(s,y)z(s,y)ds$$

$$f(t,u(t)) = b_2(s,y)\Big(t,\sin z(t,y)\Big)$$

$$\int_0^t h(t,s,u(s))ds = \int_0^t \sin z(s,y)e^{-z(\sin s,y)}ds$$

$$\sum c_i u(t_i) = \sum_{i=1}^{m} e_i\Phi_{t_i}(s,y)$$

$$I_k(z(u)) = (\delta_i(z(y))+t_k)^{-1}$$

$$J_k(z'(u)) = (\widehat{\delta_i}(z(y))+t_k)^{-1}.$$

Define the operator $A:\mathcal{D}(A)\subset X\to X$ and $E:\mathcal{D}(E)\subset X\to X$ by

$$Az=-z_{xx}, \quad Ez=z-z_{xx},$$

where each domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ is given by

$$\{z\in X: z,\ z_x \text{ are absolutely continuous}, z_{xx}\in X, z(0)=z(\pi)=0\}.$$

It is well-known that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in R}$ on X. Then A and B can be written, respectively, as

$$Az=\sum_{n=1}^{\infty} n^2\langle z,z_n\rangle z_n,\ z\in\mathcal{D}(A)$$

$$Bz=\sum_{n=1}^{\infty}(1+n^2)\langle z,z_n\rangle z_n,\ z\in\mathcal{D}(B),$$

where $z_n(x)=\sqrt{2/\pi}\sin(nx)$, $n=1,2,\ldots,$ is the orthogonal set of vectors of A.

Furthermore for $z \in X$, we have

$$B^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1+n^2} \langle z, z_n \rangle z_n,$$
$$-AB^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} \langle z, z_n \rangle z_n,$$
$$C(t)z = \sum_{n=1}^{\infty} \frac{\cos(nt)}{1+n^2} \langle z, z_n \rangle,$$
$$\text{and} \quad S(t)z = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n(1+n^2)} \langle z, z_n \rangle z_n.$$

For this $S(t), C(t), B, B^{-1}$, we assume that the operator E exists. With the choice of A, f, g, h, I_k and J_k, we see that (6.1)-(6.5) problem can be formulated abstractly as

$$\frac{d}{dt}\big[(Bu(t))' + g(t, u(t))\big] = Au(t) + f(t, u(t)) + \int_0^t h(t, s, u(s))ds,$$
$$t \in (0, b], \; t \neq t_k,$$
$$u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0, \qquad u'(0) + q(u) = v_0$$
$$\Delta u(t_k) = I_k(u_{t_k}), \quad \Delta u'(t_k) = J_k(u_{t_k}), \quad k = 1, 2, \ldots, m,$$

Further, all the conditions stated in the Theorem 5.2 are satisfied and it is possible choose $b_1, b_2, \delta_i, \widehat{\delta_i}, e_i$. Hence by the Theorem 5.2, the equation (6.1)-(6.5) has a mild solution on I.

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