Thai **J**ournal of **M**athematics : 261–271
Special Issue: Annual Meeting in Mathematics 2017

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Connectedness of Cayley Graphs of Finite Transformation Semigroups

Chunya Tisklang[†,‡] **and Sayan Panma**[§,1]

[†]Thai Government Scholarships in the Area of Science and Technology
(Ministry of Science and Technology)
111 Thailang Science Park, Patumthani 12120, Thailand
[‡]Ph.D. Degree Program in Mathematics, Department of Mathematics
Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : `chunya.tis@gmail.com (C. Tisklang)`
[§]Center of Excellence in Mathematics and Applied Mathematics
Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : `panmayan@yahoo.com (S. Panma)`

Abstract : For a fixed non-empty subset Y of X, we denote by $T(X,Y)$ the semigroup consisting of all transformations on X whose range is contained in Y. In this paper, we investigate connectedness of Cayley graphs of finite transformation semigroups with restricted range. Necessary and sufficient conditions for Cayley graph of $T(X,Y)$ to be strongly connected, unilaterally connected, and weakly connected are given.

Keywords : Cayley graph; transformation semigroup; restricted range; strongly connected.

2010 Mathematics Subject Classification : 05C25; 05C75.

---

[1]Corresponding author.

Copyright © 2018 by the Mathematical Association of Thailand. All rights reserved.

1 Introduction

Let S be a semigroup with a subset A. The *Cayley graph* $\mathrm{Cay}(S, A)$ of S with the connection set A is defined as the digraph with vertex set S and edge set $E(\mathrm{Cay}(S, A))$ which consists of those ordered pairs (x, y) such that $y = xa$ for some $a \in A$. The various properties of Cayley graphs of semigroups have been considered by many authors, (see, [1–6]). In particular, A. V. Kelarev and C. E. Praeger in [5] studied the vertex transitive Cayley graphs of semigroups and showed that, for a semigroup S with a subset A such that $\langle A\rangle$ is completely simple and $SA = S$, every connected component of $\mathrm{Cay}(S, A)$ is strongly connected. Later Y. Lue et al. in [7] described all strongly connected bipartite Cayley graphs of completely simple semigroups. Recently, T. Suksumran and S. Panma in [8] considered the connectedness of Cayley graphs of semigroups and gave necessary and sufficient conditions for a Cayley graph of semigroup to be strongly connected and weakly connected.

The full transformation semigroup $T(X)$ on a set X, the set of all functions from X into itself, is the semigroup analogue of the symmetric group. Let Y be a non-empty subset of X and the *transformation semigroup with restricted range* denoted by $T(X, Y)$ is defined by $T(X, Y) = \{\alpha \in T(X) : X\alpha = Y\}$.

In this paper, we study the connectedness of Cayley graphs of $T(X, Y)$. We show under which conditions Cayley graphs of $T(X, Y)$ satisfy the property of being strongly connected, unilaterally connected, and weakly connected.

2 Preliminaries

Let S be a semigroup. The *subsemigroup generated by* A, denoted by $\langle A\rangle$, is a subsemigroup of S containing of the elements that can be expressed as a finite product of elements in A. A semigroup is said to be *completely simple* if it has no proper ideals and has a minimal idempotent with respect to the partial order $e \leq f \Leftrightarrow e = ef = fe$.

The monoid S^1 is a semigroup of adding an identity to S if S does not contain an identity and $S^1 = S$ if S contains an identity. The *Green's relations* on S are the equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on S. Here, we show only an $\mathcal{R}$-relation on S which is defined by, for any $a, b \in S$

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1.$$

Then $a\mathcal{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$.

We assume X is a finite set throughout this paper. Here we concerned only with the case where $X = X_n = \{1, 2, \ldots, n\}$ and $|Y| = r$, and we denote $T(X)$ by T_n.

For $\alpha \in T(X, Y)$ and $x \in X$, the image of x under α is written as $x\alpha$ and $Z\alpha$ denotes the set of all images of elements in Z, a subset of X, under α. The *rank* of α is the cardinal number of $\mathrm{im}(\alpha)$ and denoted by $\mathrm{rank}(\alpha)$. The *kernel* of α is given by

$$\ker(\alpha) = \{(x, y) \in X \times X : x\alpha = y\alpha\}.$$

The symbol π_α denotes the partition of X induced by the transformation α, namely

$$\pi_\alpha = \{y\alpha^{-1} \ : \ y \in \operatorname{im}(\alpha)\}$$

where $y\alpha^{-1}$ is the set of all $x \in X$ such that $x\alpha = y$. It is easily seen that, for all $\alpha, \beta \in T(X, Y)$,

$$\ker(\alpha) = \ker(\beta) \text{ if and only if } \pi_\alpha = \pi_\beta. \tag{2.1}$$

A set is a *transversal* of partition if it intersects each class in exactly one element. For $y \in Y$, we let σ_y be a constant function with $\operatorname{im}(\sigma_y) = \{y\}$ in $T(X, Y)$. For convenience, we use the symbol $\alpha = [a_1, \ldots, a_n]$ instead of $\alpha = \begin{pmatrix} x_1 & \ldots & x_n \\ a_1 & \ldots & a_n \end{pmatrix}$. In [9], all Green's relations on $T(X, Y)$ were described and an $\mathcal{R}$-relation is presented as follows.

Lemma 2.1 ([9])**.** *Let $\alpha, \beta \in T(X, Y)$. Then $\beta = \alpha\mu$ for some $\mu \in T(X, Y)$ if and only if* $\ker(\alpha) \subseteq \ker(\beta)$. *Consequently, $\alpha\mathcal{R}\beta$ if and only if* $\ker(\alpha) = \ker(\beta)$.

Given nonnegative integers n and k, the *Stirling number of the second kind*, denoted by $S(n, k)$, is the number of ways to partition a set of n objects into k non-empty subsets. The recurrence relation is

$$S(n, k) = kS(n-1, k) + S(n-1, k-1)$$

for $n, k > 0$ with initial conditions $S(0, 0) = 1$ and $S(0, n) = 1 = S(n, 0)$ and the explicit formula is

$$S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \begin{pmatrix} k \\ i \end{pmatrix} (k-i)^n.$$

From Lemma 2.1 and (2.1), we get for $\alpha, \beta \in T(X, Y)$, $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$. This means α and β are in the same $\mathcal{R}$-class if and only if $\pi_\alpha = \pi_\beta$. Therefore,

$$S(n, k) = \text{ the number of } \mathcal{R}\text{-classes of } T(X, Y) \text{ with rank } k.$$

Let D be a digraph and u, v be distinct elements in D. A sequence $W : u = u_0, u_1, \ldots, u_k = v$ of vertices of D such that u_i is adjacent to u_{i+1} for all $i (0 \le i \le k-1)$ is called a (u, v)-*diwalk* in D. A (u, v)-*semi-diwalk* in D is a sequence $W : u = u_0, u_1, \ldots, u_k = v$ of vertices of D such that $u_iu_{i+1} \in E(D)$ or $u_{i+1}u_i \in E(D)$. A (u, v)-diwalk is a (u, v)-*dipath* if it has no repeated vertices. Similarly, a (u, v)-semi-diwalk is a (u, v)-*semi-dipath* if it has no repeated vertices.

A digraph D is *strongly connected* if a (u, v)-dipath exists for all distinct vertices u, v in D. A digraph D is *unilaterally connected* if D contains a (u, v)-dipath or a (v, u)-dipath for every pair u, v of distinct vertices of D. If, for all distinct vertices u, v in D, a (u, v)-semi-dipath exists, then D is called *weakly connected* (or *connected*). A maximal connected subgraph of a digraph D is called *component* of D. A subgraph H of D is called an *induced subgraph* of D if whenever $u, v \in V(H)$ and $(u, v) \in E(D)$, then $(u, v) \in E(H)$. Let A be a non-empty set of vertices of a digraph D. The *subgraph of D induced by A* is the induced subgraph with vertex set A and denoted by $D[A]$ or simply $[A]$. For a positive integer n, the nD is the disjoint union of n copies of D.

It is clear that if A is an empty set, $\mathrm{Cay}(S, A)$ is an empty graph. Therefore in the sequel we suppose that A is a non-empty set.

Some required known results are stated below.

Lemma 2.2 ([5]). *Let S be a semigroup with a subset A, let $s \in S$ and let C_s be the set of all vertices v of the Cayley graph $\mathrm{Cay}(S, A)$ such that there is a dipath from s to v. Then C_s is equal to the left coset $s\langle A\rangle$.*

Lemma 2.3 ([5]). *Let S be a semigroup with a subset A such that $\langle A\rangle$ is completely simple and $SA = S$. Then every connected component of the Cayley graph $\mathrm{Cay}(S, A)$ is strongly connected and, for each $v \in S$, the component containing v is $[v\langle A\rangle]$.*

Proposition 2.4 ([10]). *Let $A \subseteq T(X)$. Then $\langle A\rangle$ is a completely simple if and only if for all $\alpha, \beta \in A$, $\mathrm{im}(\alpha)$ is a transversal of π_β.*

It is obviously that the above proposition holds for a subset A of $T(X, Y)$.

Lemma 2.5 ([11]). *Let $\alpha, \beta \in T(X, Y)$. Then $X\beta \subseteq Y\alpha$ if and only if there exists $\gamma \in T(X, Y)$ such that $\gamma\alpha = \beta$.*

Lemma 2.6 ([12]). *Let $\alpha \in T(X, Y)$. If $Y\alpha = Y$, then $T(X, Y) = T(X, Y)\alpha$.*

3 Main Results

In this section, we investigate connectedness of Cayley graph of $T(X, Y)$, i.e., necessary and sufficient conditions for Cayley graph of $T(X, Y)$ to be strongly connected, unilaterally connected and weakly connected. First, we give some properties to be used in following.

Lemma 3.1. *Let $A \subseteq T(X, Y)$. Then*

$$T(X, Y)A = T(X, Y) \text{ if and only if } Y\alpha = Y \text{ for some } \alpha \in A.$$

Proof. ($\Leftarrow$) Assume that $Y\alpha = Y$ for some $\alpha \in A$. By Lemma 2.6, we get $T(X, Y)\alpha = T(X, Y)$, and so $T(X, Y) = T(X, Y)\alpha \subseteq T(X, Y)A$. This means that $T(X, Y)A = T(X, Y)$.

($\Rightarrow$) Suppose that $Y\alpha \neq Y$ for all $\alpha \in A$. Let $\beta \in T(X, Y)$ such that $X\beta = Y$. Then $Y\alpha \subsetneq Y = X\beta$, this implies that there exists $z \in X\beta \backslash Y\alpha$. So $X\beta \nsubseteq Y\alpha$. By Lemma 2.5, $\gamma\alpha \neq \beta$ for all $\alpha \in A$ and for all $\gamma \in T(X, Y)$. From this we conclude that $T(X, Y)A \neq T(X, Y)$. □

From Proposition 2.4 and Lemma 3.1, we get the following result.

Corollary 3.2. *Let $A \subseteq T(X,Y)$. Then $\langle A\rangle$ is a completely simple semigroup and $T(X,Y)A = T(X,Y)$ if and only if $Y\alpha = Y$ for all $\alpha \in A$.*

Lemma 3.3. *Let $A \subseteq T(X,Y)$ and $Y\alpha = Y$ for all $\alpha \in A$. If elements β and γ are in the same component in* $\mathrm{Cay}(T(X,Y),A)$*, then $\beta\mathcal{R}\gamma$.*

Proof. Let $\beta,\gamma \in T(X,Y)$ such that β and γ are in the same component. Since $Y\alpha = Y$ for all $\alpha \in A$, $\langle A\rangle$ is a completely simple semigroup and $T(X,Y)A = T(X,Y)$. By Lemma 2.3, $[\alpha\langle A\rangle]$ is a strongly connected component of $\mathrm{Cay}(T(X,Y),A)$. Hence there exist dipaths from β to γ and from γ to β, i.e., $\gamma = \beta\beta_1\cdots\beta_m$ and $\beta = \gamma\gamma_1\cdots\gamma_k$ for some $\beta_1,\ldots,\beta_m,\gamma_1,\ldots,\gamma_k \in A$ which concludes that $\beta\mathcal{R}\gamma$. □

Define $A_Y = \{\alpha_{|_Y} : \alpha \in A\}$ where $A \subseteq T(X,Y)$. Now, we describe the relation of elements in each component of a Cayley graph of $T(X,Y)$ with a connection set A and $\langle A_Y\rangle$ is a symmetric group on Y.

Theorem 3.4. *Let $A \subseteq T(X,Y)$ be such that $\langle A_Y\rangle$ is a symmetric group on Y and $\alpha,\beta \in T(X,Y)$. Then α and β are in the same component in* $\mathrm{Cay}(T(X,Y),A)$ *if and only if $\alpha\mathcal{R}\beta$.*

Proof. If $\langle A_Y\rangle$ is a symmetric group on Y, $Y\alpha = Y$ for all $\alpha \in A$ and so $T(X,Y)A = T(X,Y)$ and $\langle A\rangle$ is a completely simple semigroup. By Lemma 2.3, $[\gamma\langle A\rangle]$ is a strongly connected component for all $\gamma \in T(X,Y)$. The if part is true by Lemma 3.3. Now, suppose that $\alpha\mathcal{R}\beta$. We show that there exists $\gamma \in \langle A\rangle$ such that $\beta\gamma = \alpha$. By assumption, $\pi_\alpha = \pi_\beta$ and it follows that $|X\alpha| = |X\beta| = m$ and so $|X\backslash X\alpha| = |X\backslash X\beta|$. From this, there exists a bijection from $X\backslash X\beta$ into $X\backslash X\alpha$, says f. For each $x \in X\beta, x = d_x\beta$ for some $d_x \in X$. Since $Y \subseteq X$, there are two possibilities.

Case 1: $Y = X$. Let $y \in Y$. Define $\tau \in T(X,Y)$ by

$$x\tau = \begin{cases} d_x\alpha & \text{if } x \in X\beta,\\ y & \text{if } x \in X\backslash X\beta.\end{cases}$$

Then τ is a function in $T(X,Y)$. Moreover, $Y\tau = Y$ and $\tau_{|_Y} \in \langle A_Y\rangle$. Hence

$$\tau_{|_Y} = \gamma_{1_{|_Y}}\gamma_{2_{|_Y}}\ldots\gamma_{k_{|_Y}} = (\gamma_1\gamma_2\ldots\gamma_k)_{|_Y} = \gamma_{|_Y}$$

where $\gamma_{1_{|_Y}},\gamma_{2_{|_Y}},\ldots,\gamma_{k_{|_Y}} \in \langle A_Y\rangle$ and $\gamma = \gamma_1\gamma_2\ldots\gamma_k$. Therefore, $\gamma_1,\gamma_2,\ldots,\gamma_k \in A$. For $x \in X$, $x\beta\gamma = (x\beta)\gamma = (x\beta)\tau = x\alpha$ and so $\beta\gamma = \alpha$. Thus $\alpha \in \beta\langle A\rangle$.

Case 2: $Y \subsetneq X$. We have $|X\backslash X\beta| = |X\backslash X\alpha| \geq |Y\backslash X\alpha|$. Let $l = r - m$, $X\backslash X\beta = \{z_1,z_2,\ldots,z_l,\ldots,z_{n-m}\}$ and $Y\backslash X\alpha = \{y_1,y_2,\ldots,y_l\}$. Set $h : X\backslash X\beta \to Y\backslash X\alpha$ defined by

$$z_ih = \begin{cases} y_i & \text{if } i = 1,2,\ldots,l,\\ y_1 & \text{if } i = l+1,\ldots,n-m.\end{cases}$$

We see that h is onto. Finally, define $\tau \in T(X, Y)$ by

$$x\tau = \begin{cases} d_x\alpha & \text{if} \quad x \in X\beta, \\ xh & \text{if} \quad x \in X \backslash X\beta. \end{cases}$$

Then τ is a function in $T(X, Y)$ and $Y\tau = Y$. Hence there exists $\gamma \in \langle A \rangle$ such that $\gamma_{|_Y} = \tau_{|_Y}$. It implies that $\beta\tau = \alpha$. Therefore, $\alpha \in \beta\langle A \rangle$ as required. □

Example 3.5. Let $Y = \{1, 2, 3\} \subseteq X_4$ and $A = \{[2, 1, 3, 3], [2, 3, 1, 3]\} \subseteq T(X, Y)$. Then $\langle A_Y \rangle$ is a symmetric group on Y. We have 6 $\mathcal{R}$-classes of $T(X, Y)$ with rank 3, 7 $\mathcal{R}$-classes of $T(X, Y)$ with rank 2, and 1 $\mathcal{R}$-class of $T(X, Y)$ with rank 1, and we see that all elements in each $\mathcal{R}$-class must be in the same component as shown in Figure 1.

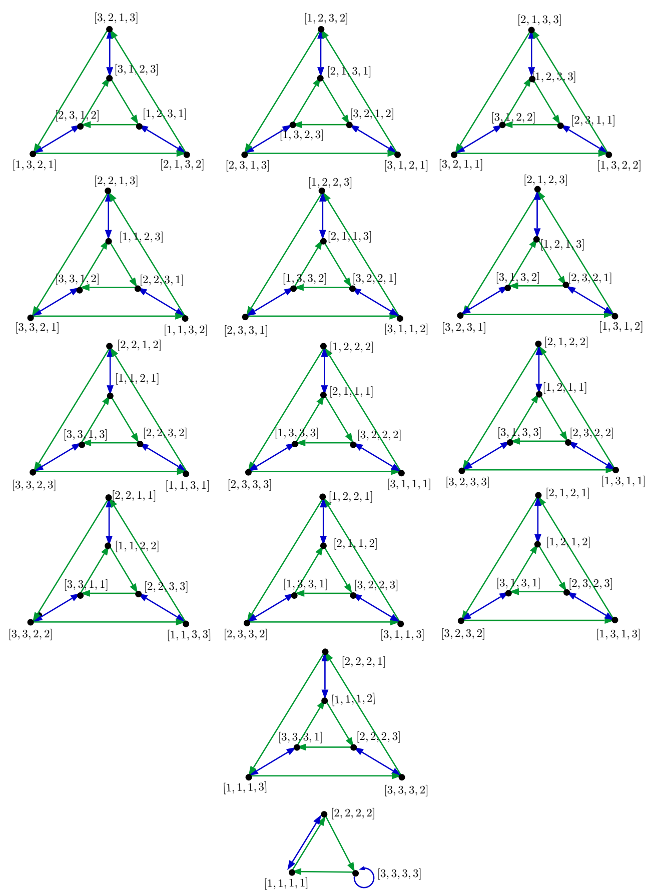

Figure 1: Cay$(T(X, Y), A)$ where $A = \{[2, 1, 3, 3], [2, 3, 1, 3]\}$

From the observation in Section 2, i.e.,

$$S(n,k) = \text{ the number of } \mathcal{R}\text{-classes of } T(X,Y) \text{ with rank } k,$$

we get the characterizations of Cayley graphs of $T(X,Y)$ with some connection sets as shown below.

Theorem 3.6. *Let $A \subseteq T(X,Y)$ be such that $\langle A_Y \rangle$ is a symmetric group on Y and $\{\beta_1, \beta_2, \ldots, \beta_r\} \subseteq T(X,Y)$ where β_i is an element of rank i in $T(X,Y)$. Then*

$$\mathrm{Cay}(T(X,Y),A) \cong \bigcup_{k=1}^{r} S(n,k)[\beta_k\langle A\rangle].$$

Proof. Let $\{R_1, R_2, \ldots, R_m\}$ be the set of all distinct $\mathcal{R}$-classes of $T(X,Y)$. By Theorem 3.4, $\{[R_1], [R_2], \ldots, [R_m]\}$ is the set of all distinct components of $\mathrm{Cay}(T(X,Y),A)$. This means that

$$\mathrm{Cay}(T(X,Y),A) = \bigcup_{i\in I}[R_i]$$

where $I = \{1, 2, \ldots, m\}$. By Lemma 2.3, $[R_i]$ is strongly connected component of $\mathrm{Cay}(T(X,Y),A)$ and $[R_i] = [\alpha_i\langle A\rangle]$ for some $\alpha_i \in T(X,Y)$. Therefore, $\mathrm{Cay}(T(X,Y),A) = \bigcup_{i\in I}[\alpha_i\langle A\rangle]$. Let $\mu, \gamma \in T(X,Y)$. If $\mathrm{rank}(\mu) = \mathrm{rank}(\gamma)$, then the function $\phi : V([\mu\langle A\rangle]) \to V([\gamma\langle A\rangle])$ defined by, for all $\rho \in \langle A\rangle$, $\phi(\mu\rho) = \gamma\rho$ is an isomorphism and so $[\mu\langle A\rangle] \cong [\gamma\langle A\rangle]$. Hence $\mathrm{Cay}(T(X,Y),A) = \bigcup_{i\in I}[\alpha_i\langle A\rangle] \cong \bigcup_{k=1}^{r} S(n,k)[\beta_k\langle A\rangle]$. □

In Example 3.5, $\langle A_Y\rangle$ is a symmetric group on Y which is a subset of $T(Y)$. We see that

$$\begin{aligned}\mathrm{Cay}(T(X,Y),A) \cong\; & S(4,1)\big[[1,1,1,1]\langle A\rangle\big] \cup S(4,2)\big[[1,1,1,2]\langle A\rangle\big] \cup \\ & S(4,3)\big[[1,1,2,3]\langle A\rangle\big] \\ =\; & \big[[1,1,1,1]\langle A\rangle\big] \cup 7\big[[1,1,1,2]\langle A\rangle\big] \cup 6\big[[1,1,2,3]\langle A\rangle\big].\end{aligned}$$

Next, we give necessary and sufficient conditions for each component of $\mathrm{Cay}(T(X,Y),A)$ to be strongly connected.

Theorem 3.7. *Let $A \subseteq T(X,Y)$. Then each component of $\mathrm{Cay}(T(X,Y),A)$ is a strongly connected component if and only if $Y\alpha = Y$ for all $\alpha \in A$.*

Proof. Assume that $Y\alpha \neq Y$ for some $\alpha \in A$. Let $\beta \in T(X,Y)$ be such that $\beta_{|_Y}$ is a permutation on Y. Then there is a $(\beta, \beta\alpha)$-dipath. Since $\mathrm{im}(\beta\alpha) = (\mathrm{im}(\beta) \cap \mathrm{dom}(\alpha))\alpha = Y\alpha \subsetneq Y$, $\pi_{\beta\alpha} \subsetneq \pi_\beta$. It follows that there is no dipath from $\beta\alpha$ to β and hence the component which contains β is not strongly connected. Conversely, suppose that $Y\alpha = Y$ for all $\alpha \in A$. Then $\langle A\rangle$ is a completely simple semigroup and $T(X,Y)A = T(X,Y)$. By Lemma 2.3, every connected component of $\mathrm{Cay}(T(X,Y),A)$ is strong. □

Corollary 3.8. *Let $A \subseteq T(X,Y)$. Then* $\mathrm{Cay}(T(X,Y),A)$ *is a strongly connected digraph if and only if $r = 1$.*

Proof. Suppose that $\mathrm{Cay}(T(X,Y),A)$ is a strongly connected digraph. By Theorem 3.7 and Lemma 3.3, it implies that $\beta\mathcal{R}\gamma$ for all $\beta,\gamma \in T(X,Y)$. Hence $r = 1$. □

Theorem 3.9. *Let A be a subset of $T(X,Y)$. Then* $\mathrm{Cay}(T(X,Y),A)$ *is weakly connected if and only if $\sigma_z \in \langle A\rangle$ for some $z \in Y$.*

Proof. ($\Leftarrow$) Suppose that $\sigma_z \in \langle A\rangle$ for some $z \in Y$. Let $\beta \in T(X,Y)$. We get that $x(\beta\sigma_z) = (x\beta)\sigma_z = z = x\sigma_z$ for all $x \in X$. Thus $\beta\sigma_z = \sigma_z$. This means there is a dipath from β to σ_z for all $\beta \in T(X,Y)$. Then, for $\gamma,\lambda \in T(X,Y)$, there exist (γ,σ_z)-dipath and (λ,σ_z)-dipath. Hence there is a semi-dipath from γ to λ. Therefore, $\mathrm{Cay}(T(X,Y),A)$ is weakly connected.

($\Rightarrow$) Suppose that $\mathrm{Cay}(T(X,Y),A)$ is weakly connected. Then $\mathrm{Cay}(T(X,Y),A)$ has only one component. We will prove that there exists $\sigma_z \in \langle A\rangle$ for some $z \in Y$. It clearly suffices to prove this for $r \neq 1$. By Lemma 2.1, there are at least two $\mathcal{R}$-classes. If $Y\alpha = Y$ for all $\alpha \in A$, by Lemma 3.3, it implies that $\mathrm{Cay}(T(X,Y),A)$ has at least two components which is a contradiction. Therefore, $Y\alpha \neq Y$ for some $\alpha \in A$. Let $\beta \in T(X,Y)$ be such that $X\beta = Y$. Thus $\beta\alpha = \mu$ where $\ker(\beta) \subsetneq \ker(\mu)$ for some $\mu \in T(X,Y)$. Since $\mathrm{Cay}(T(X,Y),A)$ is weakly connected and $\ker(\beta) \subseteq \ker(\sigma_z)$ but $\ker(\sigma_z) \nsubseteq \ker(\beta)$ for all $z \in Y$, there is a dipath from β to σ_y for some $\sigma_y \in T(X,Y)$. This means $\beta\alpha' = \sigma_y$ where $\alpha' \in \langle A\rangle$ and for $x \in X$, $(x\beta)\alpha' = x(\beta\alpha') = x\sigma_y = y$. Hence we can suppose that $\alpha' = \begin{pmatrix} Y & a_1 & \dots & a_k \\ y & y_1 & \dots & y_k \end{pmatrix}$ where $X \backslash Y = \{a_1, a_2, \dots, a_k\}$, $y_1, y_2, \dots, y_k \in Y$ and $Y\alpha' = \{y\}$. Thus $\sigma_y = \alpha'\alpha' \in \langle A\rangle$ as desired. □

Example 3.10. Let $Y = \{1,2,3\} \subseteq X_4$ and $A = \{[2,3,1,3],[1,2,2,2]\}$. So $\sigma_2 = ([2,3,1,3] \circ [1,2,2,2])^2 \in \langle A\rangle$ and $Cay(T(X,Y),A)$ is weakly connected. Moreover, we see that there is no edges between $[2,1,1,2]$ and $[1,2,1,1]$. Hence $Cay(T(X,Y),A)$ is not unilaterally connected (see Figure 2).

Example 3.11. Let $A = \{[2,1],[1,1]\}$ which is a subset of T_2. The Cayley graph $Cay(T_2,A)$ as shown in Figure 3. We observe that $Cay(T_2,A)$ is a unilaterally connected digraph.

Some properties of unilaterally connected of $\mathrm{Cay}(T(X,Y),A)$ are therefore provided.

Theorem 3.12. *Let $A \subseteq T(X,Y)$. Then* $\mathrm{Cay}(T(X,Y),A)$ *is a unilaterally connected digraph if and only if $r = 1$ or ($r = n = 2$, $[2,1] \in A$ and ($[1,1] \in A$ or $[2,2] \in A$)).*

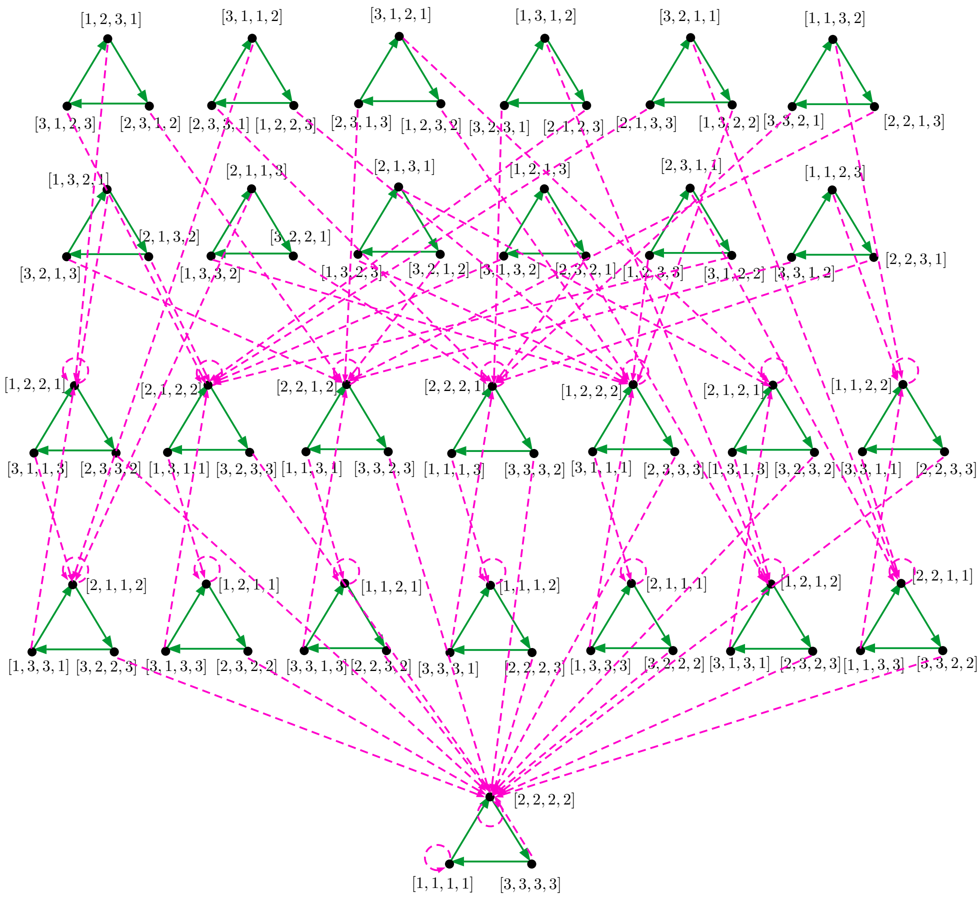

Figure 2: $\mathrm{Cay}(T(X,Y),A)$ where $A = \{[2,3,1,3],[1,2,2,2]\}$

Proof. ($\Rightarrow$) Suppose that $\mathrm{Cay}(T(X,Y),A)$ is a unilaterally connected digraph and $r \neq 1$. Then $r, n \geq 2$. Assume that $n \geq 3$. Since $S(3,1) = 1$ and $S(3,2) = 3$, the number of $\mathcal{R}$-classes is greater than or equal to 4 and there exist $\beta, \gamma \in T(X,Y)$ such that $\ker(\beta) \nsubseteq \ker(\gamma)$ and $\ker(\gamma) \nsubseteq \ker(\beta)$. If there is a (β,γ)-dipath, then $\gamma = \beta\alpha_1\alpha_2\ldots\alpha_k$ for some $\alpha_1, \alpha_2, \ldots, \alpha_k \in A$. Thus $\ker(\beta) \subseteq \ker(\gamma)$ which is a contradiction. Similarly, there is no (γ,β)-dipath. Hence $n = 2 = r$ and so $T_2 = \{[1,1],[2,2],[1,2],[2,1]\}$. If $[2,1] \notin A$, there is no dipath from $[2,1]$ to $[1,2]$, a contradiction. Consequently, $[2,1] \in A$. By assumption, we get that $\mathrm{Cay}(T_2,A)$ is a weakly connected digraph. Therefore, there is $\alpha \in A$ such that $Y\alpha \neq Y$. It implies that $[1,1] \in A$ or $[2,2] \in A$.

($\Leftarrow$) (i) Let $r = 1$. It is obviously that $\mathrm{Cay}(T_1,A)$ is a unilaterally connected digraph.

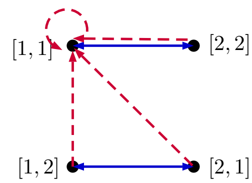

Figure 3: $\mathrm{Cay}(T_2, A)$ where $A = \{[2,1],[1,1]\}$

(ii) Assume that $r = n = 2$, $\{[2,1],[1,1]\} \subseteq A$. According to Example 3.11, we get that $\mathrm{Cay}(T_2, \{[2,1],[1,1]\})$, a subdigraph of $\mathrm{Cay}(T_2, A)$, is a unilaterally connected digraph. Hence $\mathrm{Cay}(T_2, A)$ is also.

(iii) Assume that $r = n = 2$, $\{[2,1],[2,2]\} \subseteq A$. Since $\mathrm{Cay}(T_2, \{[2,1],[2,2]\})$ is isomorphic to $\mathrm{Cay}(T_2, \{[2,1],[1,1]\})$, it is implies that $\mathrm{Cay}(T_2, A)$ is unilaterally connected. □

4 Summary

This paper studies the connectedness of Cayley graphs of $T(X,Y)$. In the following table we collect our results and present the necessary and sufficient conditions of $T(X,Y)$ which their Cayley graphs are strongly connected, weakly connected and unilaterally connected.

Properties of Cay$(T(X,Y),A)$	**Necessary and Sufficient Conditions**
strongly connected	$r = 1$
weakly connected	$\sigma_z \in \langle A \rangle$ for some $z \in Y$
unilaterally connected	(i) $r = 1$ or (ii) $r = n = 2$ and $\{[2,1],[1,1]\} \subseteq A$ or (iii) $r = n = 2$ and $\{[2,1],[2,2]\} \subseteq A$

Acknowledgement : The authors thank the referees for some useful suggestions and comments. This research was supported by Chiang Mai University.

References

[1] Sr. Arworn, U. Knauer, N. Na Chiangmai, Characterization of digraphs of right (left) zero unions of groups, Thai J. Math. 1 (1) (2003) 131–140.

[2] S. Fan, Y. Zeng, On Cayley graphs of bands, Semigroup Forum 74 (1) (2007) 99–105.

[3] A.V. Kelarev, On undirected Cayley graphs, Australas. J. Combin. 25 (2002) 73–78.

[4] A.V. Kelarev, Graph Algebras and Automata, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2003.

[5] A.V. Kelarev, C.E. Praeger, On transitive Cayley graphs of groups and semigroups, European J. Combin. 24 (1) (2003) 59–72.

[6] B. Khosravi, Some properties of Cayley graphs of cancellative semigroups, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 17 (1) (2016) 3–10.

[7] Y. Luo, Y. Hao, G.T. Clarke, On the Cayley graphs of completely simple semigroups, Semigroup Forum 82 (2) (2011) 288–295.

[8] T. Suksumran, S. Panma, On connected Cayley graphs of semigroups, Thai J. Math. 13 (3) (2015) 641–652.

[9] J. Sanwong, W. Sommanee, Regularity and Green's relations on a semigroup of transformations with restricted range, Int. J. Math. Math. Sci. (2008) ID 794013.

[10] R. Gray, J.D. Mitchell, Largest subsemigroups of the full transformation monoid, Discrete Math. 308 (20) (2008) 4801–4810.

[11] K. Sangkhanan, J. Sanwong, Partial orders on semigroups of partial transformations with restricted range, Bull. Aust. Math. Soc. 86 (1) (2012) 100–118.

[12] J. Sanwong, B. Singha, R.P. Sullivan, Maximal and minimal congruences on some semigroups, Acta Math. Sin. (Engl. Ser.) 25 (3) (2009) 455–466.

(Received 18 April 2017)
(Accepted 30 June 2017)