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Some Remarks on the Large Deviation of the Visited Sites of Simple Random Walk in Random Scenery

Parkpoom Phetpradap

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai, Thailand, 50200
e-mail : parkpoom.phetpradap@cmu.ac.th

Abstract : For each $z \in \mathbb{Z}^d$, we define random scenery on the integer lattice $\mathbb{Z}^d$ as $\{\xi_z : z \in \mathbb{Z}^d\}$ where each ξ_z are identical and independent random variables with finite mean and variance. For a simple symmetric random walk on $\mathbb{Z}^d$ in dimension $d \geq 3$, we focus on $X_n := \sum_{z \in V_n} \xi_z$, where V_n is the lattice visited by the walk by time n. We investigate that X_n satisfies large deviation principle with explicitly given rate functions. The expectation and variance of X_n can also be calculated. This is an extended result from the large deviation result on the number of sites visited by a simple random walk.

Keywords : random walk; random scenery; large deviation principle; moderate deviation principle; sums of independent random variables.
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1 Introduction

Let $(S_n \colon n = 1, 2, \dots)$ be a simple symmetric random walk on the integer lattice $\mathbb{Z}^d$, i.e. $S_n = \sum_{i=1}^{n} X_n$ for $X_1, X_2, \dots$ a sequence of independent, identically distributed random vectors with $\mathbb{P}(X_1 = e_i) = \mathbb{P}(X_1 = -e_i) = \frac{1}{2d}$, where $e_1, e_2, \dots, e_d$ are the orthogonal unit vectors on $\mathbb{Z}^d$. Polya [1] show that the probability that the random walk will *not* return to the origin is positive for dimen-

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sion three or more. Later on, Dvoretzky and Erdös [2] investigate the number of lattice sites R_n visited at least once by the random walk up to time n, i.e. $R_n = \sharp\{S_1, \dots, S_n\}$, and they prove that, for dimension three or more, $\mathbb{E}(R_n)$ is associated with the non-return probability and of order n. Let V_n be the set of sites visited at least once by the random walk up to time n. Furthermore, define *random scenery* for each lattice $z \in \mathbb{Z}^d$ as $\{\xi_z : z \in \mathbb{Z}^d\}$ to be independent and identically distributed random variables with finite mean and variance, independent of (S_n). In this work, we focus on X_n, which is defined as

$$X_n \quad := \quad \sum_{x \in V_n} \xi_x.$$

Clearly, X_n depends on two randomnesses, (i) from the distribution of R_n, and (ii) from the distribution of ξ.
Note that, by setting $\xi_z = 1$ for all $z \in \mathbb{Z}^d$, X_n simply equals to R_n. The properties of R_n has been well-studied, such as expectation, variance, the strong law of large number [2], and the central limit theorem [3, 4]. In 1979, Donsker and Varadhan [5] discover the limit behaviour of R_n for $d \geq 3$ in an exponent form:

$$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{E} \exp(-\theta R_n) \quad = -k(\theta, d), \tag{1.1}$$

for any $\theta > 0$, where $k(\theta, d)$ is a constant. Later on, it turns out that the study coincides with *theory of large deviation* [6]. Large deviation theory deals with an exponential decay of the probability of increasingly unlikely events. It is one of the key techniques of modern probability. Applications of large deviation theory arise, for example, in statistical mechanics, information theory and insurance. Note that, large deviation theory compliments the well-known central limit theorem result in the sense of how fast the decay in the event that sample mean deviates from its expectation, which is very unlikely event by laws of large numbers. The definition of large deviation principle is in Section 2.1. To point out an example of large deviation theory, we can transform (1.1), by the Gärtner-Ellis theorem [7], into

$$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{P}\{R_n \leq \nu n^{d/(d+2)}\} = -J(\nu),$$

for explicitly given function $J(\nu)$. We can see that the decay depends on $n^{-d/(d+2)}$ and $J(\nu)$. It also turns out that central limit theorem can also be described by the large deviation theory [4, 8], see Theorem 2.4. More large deviation results on R_n are in [4, 5, 9, 10, 11, 12, 13].

It is worth to say that X_n is *not* a random walk in random scenery, W_n, which is defined as $W_n = \sum_{i=1}^n \xi_{S_i}$. For the process W_n, random walk collects *energies* in every visit, while for the process X_n, the walk only collect energies at newly visited point. Clearly, $X_n \leq W_n$ when ξ_z is non negative random variables. Random walk in random scenery is a family of stationary processes exhibiting amazing rich behaviour, including large deviation behaviour. The large deviation results of random walk in random scenery are, for example, in [14, 15].

Our main aim of this article is to investigate expectation and variance, as well as the large deviation behavior of X_n. The rest of this paper is organised as follows: In Section 2, we provide background definitions, notations and related results on our random walk model. Then, Section 3 provides our main results.

2 Preliminaries

In this section, we first describe the large deviation principle. Then, we give definitions and notations of random walk, random scenery and the quantities we study. Finally, we quote the related results concerning our study.

2.1 Large Deviation Principle

2.1.1 Definition

Consider a sequence of random variable $X_1, X_2, \ldots$ in a general metric space M and consider events of the type $\{X_n \in A\}$ where $A \subset M$ is a Borel set. We now give the definitions of a rate function and a large deviation principle.

Definition 2.1. For a metric space M, the function $I : M \to [0, \infty]$ is called

- *a rate function* if it is lower semicontinuous, which means that the level sets $N_a := \{x \in M : I(x) \leq a\}$ are closed for any $a \geq 0$;
- *a good rate function* if the level sets N_a are compact for any $a \geq 0$.

Definition 2.2. A sequence of random variable $X_1, X_2, \ldots$ with values in a metric space is said to satisfy *a large deviation principle* with

- *speed* a_n (which tends to infinity as $n \to \infty$) and
- *rate function* I,

if, for all Borel sets $A \subset M$,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}\{X_n \in A\} \quad \leq \quad -\inf_{x\in cl(A)} I(x),$$

and

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}\{X_n \in A\} \quad \geq \quad -\inf_{x\in int(A)} I(x).$$

Remark 2.3. (a) *Loosely speaking, the probability of occurring of the event* $\{X_n \in A\}$ *decreases exponentially fast.*

(b) *In the case* $I(x) = +\infty$*, it implies that the probability of occurring of the event* $\{X_n \in A\}$ *decays slower than speed* a_n*.*

2.1.2 Central Limit Theorem as a Large Deviation Principle

Consider independent and identically distributed sequence of $Y, Y_1, Y_2, \ldots$ with mean μ and variance σ^2. Define $S_n = \sum_{i=1}^{n} Y_i$ as a partial sum up to time n. Note that by the weak law of large number $\frac{S_n}{n} \to \mu$ as $n \to \infty$. Also, by the central limit theorem, we have

$$\lim_{n\to\infty} \mathbb{P}\Big\{\frac{S_n - \mu n}{\sigma\sqrt{n}} \leq a\Big\} \to \Phi(a) \text{ as } n \to \infty$$

where Φ is the distribution of a standard normal random variable. To describe how the probabilities $\mathbb{P}\{\frac{1}{n}S_n > \mu + \epsilon\}$ decays as a function of ϵ, it turns out that finer features of the random variable Y *rather than the finiteness of its variance* is required, namely the existence of cumulant generating function condition:

$$\varphi(\lambda) := \log \mathbb{E}e^{\lambda Y_1} < \infty \text{ for all } \lambda \in \mathbb{R}. \tag{2.1}$$

If $(Y_i)_{i\geq 0}$ satisfy the above condition, the large deviation probability decays exponentially and Cramér's theorem tells us exactly how fast of the decay.

Theorem 2.4 (Cramér's theorem [7]). *Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with mean μ, $\log \mathbb{E}\big(e^{\lambda Y_1}\big) < \infty$ for all $\lambda \in \mathbb{R}$, and define $S_n = \sum_{i=1}^{n} Y_i$. Then, for any $y > \mu$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{\frac{1}{n}S_n \geq y\} = -\varphi^*(y),$$

where $\varphi^*(y)$ is given by

$$\varphi^*(y) := \sup_{\lambda\in\mathbb{R}} \{\lambda y - \varphi(\lambda)\}$$

is the Legendre transform of φ.

Example 2.5. (a) If $Y_1 \sim ber(p)$, then it can be shown that

$$\varphi^*(y) = \begin{cases} y\log(y/p) + (1-y)\log(\frac{1-y}{1-p}), & \text{if } y \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

(b) If $Y_1 \sim Poi(\theta)$, then it can be shown that

$$\varphi^*(y) = \begin{cases} \theta - y + y\log(y/\theta)), & \text{if } y \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

(c) If $Y_1 \sim \mathcal{N}(0, \sigma^2)$, then $\varphi^*(y) = y^2/2\sigma^2$.

Remark 2.6. *Note that the central limit theorem tells us by how much the partial sum normally exceeds its average, namely by an order of $\sqrt{n}$. More precisely,*

$$\mathbb{P}\{S_n - \mu n \geq \sqrt{n}x\} \to 1 - \Phi(x/\sigma) > 0.$$

This implies that for any sequence a_n *with* $\sqrt{n} \ll a_n \ll n^*$*, we still have*

$$\mathbb{P}\{S_n - \mu n \geq a_n\} \to 0,$$

and neither the central limit theorem nor Cramér's theorem tells us how fast this convergence is. The answer of this question is by a moderate deviation principle *stated below*

Theorem 2.7 (Moderate deviation principle [7])**.** *Assume the same condition of* $Y_1, Y_2, \ldots$ *as in Theorem 2.4. If* $\sqrt{n} \ll a_n \ll n$*, we have, for all* $y > 0$*,*

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\{S_n - \mu n \geq y a_n\} = -\frac{y^2}{2\sigma^2}.$$

Remark 2.8. *(a) By Definition 2.2 and Theorem 2.4,* $\frac{1}{n}S_n$ *satisfy a large deviation principle with speed* n *and good rate function* φ^**.*

(b) By Definition 2.2 and Theorem 2.7, for any sequence $\sqrt{n} \ll a_n \ll n$*, the random variables*

$$\frac{S_n - \mu n}{a_n}$$

satisfy a large deviation principle with speed a_n^2/n *and good rate function* $I(y) = y^2/2\sigma^2$*.*

2.2 Random Walk and Random Scenery

Definition 2.9. A *simple symmetric random walk on the integer lattice* $\mathbb{Z}^d$ is the sequence $(S_n \colon n = 1, 2, \ldots)$, such that

$$S_n \quad = \quad \sum_{i=1}^{n} X_n$$

where $X_1, X_2, \ldots$ are a sequence of independent, identically distributed random vectors with $\mathbb{P}\{X_1 = e_i\} = \mathbb{P}\{X_1 = -e_i\} = \frac{1}{2d}$, where $e_1, e_2, \ldots, e_d$ are the orthogonal unit vectors on $\mathbb{Z}^d$.

Definition 2.10. For a simple symmetric random walk $(S_n \colon n = 1, 2, \ldots)$, we define

(a) $V_n := \{x \in \mathbb{Z}^d : S_i = x \text{ for some } 1 \leq i \leq n\}$ is the set of sites in $\mathbb{Z}^d$ that is visited at least once by the random walk up to time n, and,

(b) $R_n := \sharp\{S_1, \ldots, S_n\}$ is the number of lattice sites visited at least once by the random walk up to time n.

[*] we define $a_n \ll b_n$ when $\lim_{n\to\infty} b_n/a_n = \infty$

Definition 2.11 (Random scenery)**.** For $z \in \mathbb{Z}^d$, we define *random scenery* as $\{\xi_z : z \in \mathbb{Z}^d\}$ where ξ_z are identical and independent random variables with finite mean μ and finite variance σ^2, *independent of* (S_n).

Remark 2.12. *We may think of this random scenery as energies that embedded on each integer lattice.*

Next, we define X_n, our main quantity of interest.

Definition 2.13. Define the *total energies* collected by the random walk up to time n by

$$X_n \quad := \quad \sum_{x \in V_n} \xi_x.$$

Remark 2.14. (a) *From Definition 2.13, we can write*

$$X_n = \sum_{x \in \mathbb{Z}^d} \xi_x \mathbf{1}_{\{x \in V_n\}},$$

where $\mathbf{1}_{\{\cdot\}}$ *is an indicator function.*

(b) *In this set up, the random walk always collect energies from only newly-visit sites.*

2.3 Related Results on R_n

2.3.1 Expectation and Variance

One of the typical questions to be asked is what is the expected value and the variance of R_n. This question has been answered by Dvoretzky and Erdös [2]. Before we quote the theorem, we first need to define the non-return probability of random walk on $\mathbb{Z}^d$:

$$\kappa := \kappa(d) = \mathbb{P}\{S_i \neq 0 \text{ for all } i \geq 1\}, \tag{2.2}$$

i.e. κ is the probability of the event that the random will never return to the origin. Polya [1] prove that the non-return probability is positive for dimension three or more.

Theorem 2.15. *As* $n \to \infty$,

$$\mathbb{E}(R_n) = \begin{cases} \kappa n + O(n^{1/2}), & \text{if } d = 3, \\ \kappa n + O(\log n), & \text{if } d = 4, \\ \kappa n + c_d + O(n^{2-d/2}), & \text{if } d \geq 5, \end{cases}$$

where c_d *are positive constants depending on the dimension* $d \geq 5$*. Furthermore, it also satisfies the strong law of large numbers,*

$$\lim_{n\to\infty} \frac{R_n}{\mathbb{E}R_n} = 1 \text{ almost surely.}$$

We can see that the expected value of R_n is of order n. Next, we quote the exact order of the variance of R_n proved by Jain and Orey [3] for $d \geq 5$ and by Jain and Pruitt [4] for $d \geq 3$:

Theorem 2.16.

$$var(R_n) \asymp \begin{cases} O(n \log n), & \text{for } d = 3, \\ O(n), & \text{for } d \geq 4, \end{cases}$$

where we write $f(n) \asymp O\big(g(n)\big)$ implies that, for some positive constants c_1, c_2,

$$c_1 g(n) \leq f(n) \leq c_2 g(n),$$

for all large n.

2.3.2 Large Deviation Principle for R_n

Phetpradap [11] proves the large deviation result for deviations *below* the typical value of R_n.

Theorem 2.17. *Let $d \geq 3$. For every $b > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\{R_n \leq bn\} = -\frac{1}{d} I^{\kappa}(b), \tag{2.3}$$

where

$$I^{\kappa}(b) = \inf_{\phi \in \Phi^{\kappa}(b)} \Big[\frac{1}{2}\int_{\mathbb{R}^d} |\nabla \phi|^2(x)\, dx\Big], \tag{2.4}$$

with

$$\Phi^{\kappa}(b) = \big\{\phi \in H^1(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa \phi^2(x)}) dx \leq b\big\}, \tag{2.5}$$

where $H^1 \equiv W^{1,2}$ is the Hilbert-Sobolev space.

Remark 2.18. *As the deviations we are interested in are on the scale of the mean, some may consider this as a moderate deviation result. A large deviation result (where the range is assumed to be on a much smaller scale than the mean) was given by Donsker and Varadhan in [5]. They show that for any $b > 0$,*

$$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{P}\big\{R_n \leq bn^{d/(d+2)}\big\} = -J(b),$$

where

$$J(b) = \Big(\frac{d+2}{2}\Big)\Big(\frac{d}{2\alpha_d}\Big)^{\frac{2(b-1)}{d-2}} - b\Big(\frac{b}{2\alpha_d}\Big)^{b},$$

and α_d is the lowest eigenvalue of $-(1/2)\triangle$ for the sphere of unit volume in $\mathbb{R}^d$ with zero boundary values. Note that this result has been transformed from the original result by the Gärtner-Ellis Theorem.

Let us finally mention the result for deviations *above* the typical value on the scale of the mean, which is due to Hamana and Kesten [10]. They show that the large deviation behaviour in upward direction exists.

Theorem 2.19. *For $d \geq 3$, the function*

$$\psi(\theta) = \lim_{n\to\infty} \frac{-1}{n} \log \mathbb{P}\{R_n \geq \theta n\}$$

exists for all θ, and satisfies

$$\begin{aligned} \psi(\theta) &= 0, \quad \text{for } \theta \leq \kappa \\ 0 < \psi(\theta) &< \infty, \quad \text{for } \kappa < \theta \leq 1 \\ \psi(\theta) &= \infty, \quad \text{for } \theta > 1. \end{aligned}$$

3 Main Results

In this section, we make some remarks for the behaviour of X_n. Theorem 3.1 provides expectation and variance of the process X_n, while Theorem 3.2 shows its large deviation behaviour.

Theorem 3.1. *Expectation and variance of X_n*

(*a*)

$$\mathbb{E}(X_n) = \begin{cases} \kappa\mu n + O(n^{1/2}), & \text{if } d = 3, \\ \kappa\mu n + O(\log n), & \text{if } d = 4, \\ \kappa\mu n + c_d + O(n^{2-d/2}), & \text{if } d \geq 5, \end{cases}$$

(*b*)

$$var(X_n) \asymp \begin{cases} O(n \log n), & \text{for } d = 3, \\ O(n), & \text{for } d \geq 4, \end{cases}$$

Proof. We write $\mathbb{E}_R(var_R)$ as the expectation(variance) with respect to R_n, and $\mathbb{E}_\xi(var_\xi)$ as expectation(variance) with respect to $\{\xi_z\}$. Since $X_n = \sum_{x\in V_n} \xi_x$, we can write this as a random sum

$$X_n = \sum_{i=1}^{R_n} \hat{\xi}_i,$$

where $\hat{\xi}_1, \hat{\xi}_2, \ldots$ are i.i.d. random variables with the same distribution as ξ_z.

(a) By the tower rule, we have

$$\mathbb{E}(X_n) = \mathbb{E}_R(\mathbb{E}_\xi(X_n|R_n)) = \mathbb{E}_R(\mu R_n).$$

Hence, the result follows by Theorem 2.15. Note that the order of error terms in each dimension remains the same.

(b) By the conditional variance formula, we have

$$\begin{aligned} var(X_n) &= var_R\big(\mathbb{E}_\xi(X_n|R_n)\big) + \mathbb{E}_R\big(var_\xi(X_n|R_n)\big) \\ &= var_R(\mu R_n) + \mathbb{E}_R(\sigma^2 R_n) \\ &= \mu^2 var_R(R_n) + \sigma^2 \mathbb{E}_R(R_n) \end{aligned}$$

Note that, by Theorem 2.15 and Theorem 2.16, the orders of expectation and variance are, respectively, n and $n \log n$ in $d = 3$, and n and n in $d \geq 4$. □

Theorem 3.2 (Large deviation for X_n). *Let $d \geq 3$. For every $b > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(\{X_n \leq b\mu n\} = -\frac{1}{d} I^\kappa(b), \tag{3.1}$$

where $I^\kappa(b)$ is the same rate function given in (2.4).

Proof. Note that by Theorem 3.1, we have $\mathbb{E}(X_n)$ is approximately equal $\mu\kappa n$. So, it is sensible to consider the large deviation on the scale of order n. Hence, we will now consider the event of the type $\mathbb{P}\{\frac{1}{\mu n} X_n < b\}$. Note that, we only consider only the case $b \leq \kappa$, otherwise, the event become typical and the rate function will be infinite. Now, by conditional probability, we can write

$$\mathbb{P}\Big\{\frac{1}{\mu n} X_n < b\Big\} = \sum_{R_n} \mathbb{P}_R\Big\{\frac{1}{n} R_n < b\Big\} \mathbb{P}_\xi\Big\{\frac{1}{n} \sum_{y \in V_n} \xi_y < \mu | R_n\Big\},$$

where $\mathbb{P}_R$ and $\mathbb{P}_\xi$ are probabilities with respect to R_n and ξ_z respectively. Note that, by Theorem 2.17, $\frac{1}{n} R_n$ satisfies the large deviation principle with speed $n^{(d-2)/d}$ and rate function $\frac{1}{d} I^\kappa(b)$. Next, the average value of i.i.d. random variables $\{\xi_y\}$ for $y \in V_n$ tends to its expectation as n tends to infinity due to the central limit theorem. This is because the order of R_n is n by Theorem 2.15. Hence, condition on R_n, $\frac{1}{n} \sum_{y \in V_n} \xi_y$ satisfies the large deviation principle with speed n and rate function $\varphi^*(\mu)$. Since the speed from the first probability is slower, the decay rate of X_n is influenced by the large deviation behaviour from R_n. The rate function of event $\{X_n \leq b\kappa n\}$ is the same as in Theorem 2.17 as R_n is the dominated term. □

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