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The Eigenvalue and Fixed-Point Theorem for some Nonlinear Mapping on Near-algebra and Banach Algebra

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Abstract : Let a (p, q)-additive selfmap f on near-algebra or Banach algebra X satisfy $f(e) = e$ and $f(u) = \phi(u)f(u^{-1})\varphi(u)$,where $\phi : X \to X$ and $\varphi : X \to DES(X)$ be an automorphism and antiautomorphism respectively such that $\phi(u) = u\varphi(u^{-1})u$ for each invertible u of X. Then the selfmap f has the common eigenvalue and fixed point all of the normal invertibles of X.

Keywords : fixed point, near-algebra, Banach algebra, automorphism, antiautomorphism, normal invertible, eigenvalue.

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1 Introduction

The materials [1-7] show that the study to near-ring is surpassed the close range of pure abstract algebra. In this paper we introduce the concept of near-algebra, and try to explore a special way to research some fixed-point and eigenvalue questions in range of functional analysis by means of abstract algebraic method [8]. J. Vukman [9] studied derivations in prime rings and Banach algebras and some functional equation in Banach algebras [8], and Wang [10] had discussion for derivations in prime near-rings. J. E. Andersen [11] researched fixed points of the mapping class group in the $SU(n)$ moduli spaces. Now we are interested in the following questions: 1. Which mapping has fixed point in near-algebras and Banach algebras? 2. What mapping has eigenvalue in near-algebras and Banach algebras? The study to fixed points should be introduced into near-algebras and Banach algebras. This paper tries give one way to answer the questions above.

2 Definitions

A set X, together with two binary operations $+$ and $\cdot$, is called a (left) near-ring, denoted $(X,+,\cdot)$, if: (i) $(X,+)$ is a group (not necessarily abelian); (ii) $(X,\cdot)$ is a semigroup; (iii) $x(y+z)=xy+xz$ for all $x,y,z\in X$. An element $d\in X$ is called distributive if $(x+y)d=xd+yd$ for all $x,y\in X$. The subset $DES(X)$ of the distributive elements in X forms a subsemigroup of $(X,\cdot)$[11]. A (left) near-ring $(X,+,\cdot)$ with identity e over scalars K is called a (left) near-algebra over K, denoted $(X,K,+,\cdot)$, if for any x in X and λ in K, a product $\lambda x\in X$ is defined in such a way that the following rule holds: $\lambda(xy)=(\lambda x)y=x(\lambda y), y\in X$. Clearly, Banach algebra is a special case of near-algebra .

Definition 1. A mapping $f: X\to X$ is called (p,q)-additive, if there are $p,q\in K, q\neq 0$, such that $f(e\pm u)=pf(e)\pm qf(u)$ for any invertible element u in X, where X is a near-algebra, with identity e, over scalars K.Obviously,(p,q)-additive mappings must not be additive hence they are nonlinear.

Definition 2. An invertible element u in X is called to be normal if $(e-u)^{-1}$ existing in X.

3 Eigenvalue and Fixed-Point Theorems on Near-algebra and Banach Algebra

Theorem 1. Let $(X,K,+,\cdot)$ be a left near-algebra, with identity e, over scalars K. Let a (p,p)-additive selfmap f of X satisfy $f(e)=e$ and $f(u)=\phi(u)f(u^{-1})\varphi(u)$ where $\phi: X\to X$ and $\varphi: X\to DES(X)$ be an automorphism and antiautomorphism (i.e.,$\varphi(uv)=\varphi(v)\varphi(u)$) respectively such that $\phi(u)=u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X have the common eigenvalue, of map f, $\lambda=2p/(1+p)$ where $p\neq -1$.

Proof of Theorem 1. Obviously, $N=$ {all of the normal invertibles of X}$\neq\Phi$, for example, $e/2\in N$.Let

$$g(u)=f(u)-u \text{ ....(1)}$$

then

$$g(e)=f(e)-e=0 \text{ ....(2)}$$

Now we assume that f is (p,q)-additive and

$$e=(e-u)^{-1}(e-u)=(e-u)^{-1}-(e-u)^{-1}u \text{ ....(3)}$$

hence

$$g(e\pm u)=pf(e)\pm qf(u)-(e\pm u)=\pm qg(u)+(p-1)e\pm(q-1)u \text{ ......(4)}$$

$$g((e-u)^{-1})=g(e+(e-u)^{-1}u)=qg((e-u)^{-1}u)+(p-1)e+(q-1)(e-u)^{-1}u \text{ ......(5)}$$

$$g(u^{-1}) = g(e + u^{-1}(e-u)) = qg(u^{-1}(e-u)) + (p-q)e + (q-1)u^{-1} \text{ ......(6)}$$

On the other side, from $\varphi(u) \in DES(X)$ and other conditions of the theorem, we obtain

$$\phi(u)g(u^{-1})\varphi(u) = \phi(u)f(u^{-1})\varphi(u) - \phi(u)u^{-1}\varphi(u) = f(u) - u\varphi(u^{-1})uu^{-1}\varphi(u) = g(u) \text{ ...(7)}$$

Now we have from (1)-(7)

$g(u) = \phi(u)g(u^{-1})\varphi(u)$
$= \phi(u)[qg(u^{-1}(e-u)) + (p-q)e + (q-1)u^{-1}]\varphi(u)$
$= q\phi(u)g(u^{-1}(e-u))\varphi(u) + (p-q)\phi(u)\varphi(u) + (q-1)u\varphi(u^{-1})uu^{-1}\varphi(u)$
$= q\phi(u)\phi(u^{-1}(e-u))g((e-u)^{-1}u)\varphi(u^{-1}(e-u))\varphi(u) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= q\phi(uu^{-1}(e-u))g((e-u)^{-1}u)\varphi(uu^{-1}(e-u)) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= q\phi(e-u)g((e-u)^{-1}u)\varphi(e-u) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= \phi(e-u)[g((e-u)^{-1}) + (1-p)e + (1-q)(e-u)^{-1}u]\varphi(e-u) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= \phi(e-u)\phi((e-u)^{-1})g(e-u)\varphi((e-u)^{-1})\varphi(e-u) + r(u)$
$= \phi((e-u)(e-u)^{-1})g(e-u)\varphi((e-u)(e-u)^{-1}) + r(u)$
$= g(e-u) + r(u)$
$= -qg(u) + (p-1)e + (1-q)u + 2(q-1)u + (1-p)e + (p-q)(\phi(u) + \varphi(u))$
$= -qg(u) + (q-1)u + (p-q)(\phi(u) + \varphi(u))$

where

$r(u) = (1-p)\phi(e-u)\varphi(e-u) + (1-q)\phi(e-u)(e-u)^{-1}u\varphi(e-u) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= (1-p)\phi(e-u)\varphi(e-u) + (1-q)\phi(e-u)[(e-u)^{-1} - e]\varphi(e-u) + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= (1-p)\phi(e-u)\varphi(e-u) + (1-q)[\phi(e-u)(e-u)^{-1}\varphi(e-u) - \phi(e-u)\varphi(e-u)] + (p-q)\phi(u)\varphi(u) + (q-1)u$
$= (1-q)(e-u)\varphi((e-u)^{-1})(e-u)(e-u)^{-1}\varphi(e-u) + (q-1)u + (q-p)[\phi(e-u)\varphi(e-u) - \phi(u)\varphi(u)]$
$= (1-q)(e-u)\varphi((e-u)(e-u)^{-1}) + (q-p)[(e-\phi(u))(e-\varphi(u)) - \phi(u)\varphi(u)] + (q-1)u$
$= (1-q)(e-u)\varphi(e) + (q-p)[e - \phi(u) - \varphi(u)) + \phi(u)\varphi(u) - \phi(u)\varphi(u)] + (q-1)u$
$= (1-p)e + 2(q-1)u + (p-q)(\phi(u) + \varphi(u))$

there

$(1+q)g(u) = (p-q)(\phi(u) + \varphi(u)) - (1-q)u$......(8)

Obviously,(8) and (1) yields the following conclusions:

$(i) f(u) = \lambda u, \lambda = 2q/(1+q), if p = q \neq -1;$
$(ii) f(u) = u, if (p-q)(\phi(u) + \varphi(u)) = (1-q)u, and q \neq -1;$
$(iii) \phi(u) + \varphi(u) = \lambda u, \lambda = 2/(1+q), if p \neq q = -1.$ □

Since a Banach algebra is a near-algebra and $X = DES(X)$, we can get the following result by Theorem 1.

Theorem 2. Let $(X, K, +, \cdot)$ be a Banach algebra, with identity e, over scalars K. Let a (p,p)-additive selfmap f of X satisfy $f(e) = e$ and $f(u) = \phi(u)f(u^{-1})\varphi(u)$ where $\phi : X \to X$ and $\varphi : X \to DES(X)$ be an automorphism and antiautomorphism (i.e.,$\varphi(uv) = \varphi(v)\varphi(u)$) respectively such that $\phi(u) = u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X have the common eigenvalue, of map f, $\lambda = 2p/(1+p)$ where $p \neq -1$.

By Theorem 1, we see that all of the normal invertibles of X are the eigenvalue points of map f. Thus, The following result is naturally obtained

Theorem 3. Let $(X, K, +, \cdot)$ be a left near-algebra, with identity e, over scalars K. Let an additive selfmap f,i.e.,$f(x+y) = f(x) + f(y)$, of X satisfy $f(e) = e$ and $f(u) = \phi(u)f(u^{-1})\varphi(u)$ where $\phi : X \to X$ and $\varphi : X \to DES(X)$ be an automorphism and antiautomorphism (i.e.,$\varphi(uv) = \varphi(v)\varphi(u)$) respectively such that $\phi(u) = u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X are the fixed points of map f.

Theorem 4. Let $(X, K, +, \cdot)$ be a Banach algebra, with identity e, over scalars K. Let an additive selfmap f,i.e.,$f(x+y) = f(x) + f(y)$, of X satisfy $f(e) = e$ and $f(u) = \phi(u)f(u^{-1})\varphi(u)$ where $\phi : X \to X$ and $\varphi : X \to DES(X)$ be an automorphism and antiautomorphism (i.e.,$\varphi(uv) = \varphi(v)\varphi(u)$) respectively such that $\phi(u) = u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X are the fixed points of map f.

Theorem 5. Let $(X, K, +, \cdot)$ be a left near-algebra, with identity e, over scalars K. Let a (p, q)-additive selfmap f of X satisfy $f(e) = e$ and $f(u) = \phi(u)f(u^{-1})\varphi(u)$ where $\phi : X \to X$ and $\varphi : X \to DES(X)$ be an automorphism and antiautomorphism (i.e.,$\varphi(uv) = \varphi(v)\varphi(u)$) respectively such that $\phi(u) = u\varphi(u^{-1})u$ for each invertible u of X. Then all of the normal invertibles of X have the common eigenvalue, of map $\phi + \varphi$, $\lambda = 2/(1+p)$ where $p \neq q = -1$.

4 Remark

Remark The main result in [8] is the especial case of Theorem 4. Now let $F(x) = f(x) + x$. Clearly, $F(x)$ is additive if f is additive. When

$f(u) = \phi(u)f(u^{-1})\varphi(u)$.......(9)

We have

$\phi(u)F(u^{-1})\varphi(u) = \phi(u)(f(u^{-1})+u^{-1})\varphi(u) = \phi(u)f(u^{-1})\varphi(u)+\phi(u)u^{-1}\varphi(u)$...(10)

On the other hand

$\phi(u)u^{-1}\varphi(u) = u\varphi(u^{-1})uu^{-1}\varphi(u) = u\varphi(u^{-1})\varphi(u) = u\varphi(uu^{-1}) = u\varphi(e) = u$...(11)

From (9)-(11), we have

$\phi(u)F(u^{-1})\varphi(u) = \phi(u)f(u^{-1})\varphi(u) + \phi(u)u^{-1}\varphi(u) = f(u) + u = F(u)$...(12)

Therefore $F(x)$ satisfies Theorem 4. Thus, all of the normal invertibles u of X are the fixed points of map $F(x)$, i.e., $u = F(u) = f(u) + u$, hence $f(u) = 0$. So we get the main result of J. Vukman [8].

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