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The Unique γ-min Labelings of Graphs

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To the Memory of Professor Narong Punnim

Abstract : Let G be a graph of order n and size m. A γ-*labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces an *edge-labeling* $f' : E(G) \to \{1, 2, \ldots, m\}$ on G defined by

$$f'(e) = |f(u) - f(v)|, \quad \text{for each edge } e = uv \text{ in } E(G).$$

The *value* of f is defined as

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

The *maximum value* of a γ-labeling of G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the *minimum value* of a γ-labeling of G is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if $\text{val}(g) = \text{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\text{val}(h) = \text{val}_{\min}(G)$.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\bar{f} : V(G) \to \{0, 1, \ldots, m\}$ of f is defined by

$$\bar{f}(v) = m - f(v) \text{ for } v \in V(G).$$

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Let G be a connected graph and f a γ-min labeling of G. Then G has a *unique* γ-min *labeling* if f and $\bar{f}$ are only two γ-min labelings of G.

In this paper, we study a connected graph having the unique γ-min labeling. The minimum value of a γ-labeling is determined for some classes of trees. Spontaneously, we are able to find that they have no unique γ-min labeling.

Keywords : γ-labeling; γ-min labeling; unique γ-min labeling.
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1 Introduction

Let G be a graph of order n and size m. A γ-*labeling* of G is defined in [1] as a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m\}$ that induces an *edge-labeling* $f'\colon E(G) \to \{1, \ldots, m\}$ on G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The *value* of f is defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

If the edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling*. Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a paper of Rosa [2], who used the term β-valuations. A few years later, Golomb [3] called these labelings "graceful" and this is the terminology that has been used since then.

Moreover, a more general vertex labeling of a graph was introduced by Hegde [4], in 2000, as follows. A vertex function f of a graph G is defined from $V(G)$ to the set of nonnegative integers that induces an edge function f' defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Such a function is called a *geodetic function* of G. A one-to-one geodetic function is a *geodetic labeling* of G if the induced edge function f' is also one-to-one. Gallian [5] has written an extensive survey on labelings of graphs.

Obviously, since γ-labeling f of a graph G of order n and size m is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\mathrm{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n-1$ and every connected graph has a γ-labeling.

The *maximum value* and the *minimum value* of a γ-labeling of G are defined in [1] as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}\,,$$

respectively. A γ-labeling g of G is a γ-max *labeling* if $\mathrm{val}(g) = \mathrm{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\mathrm{val}(h) = \mathrm{val}_{\min}(G)$.

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

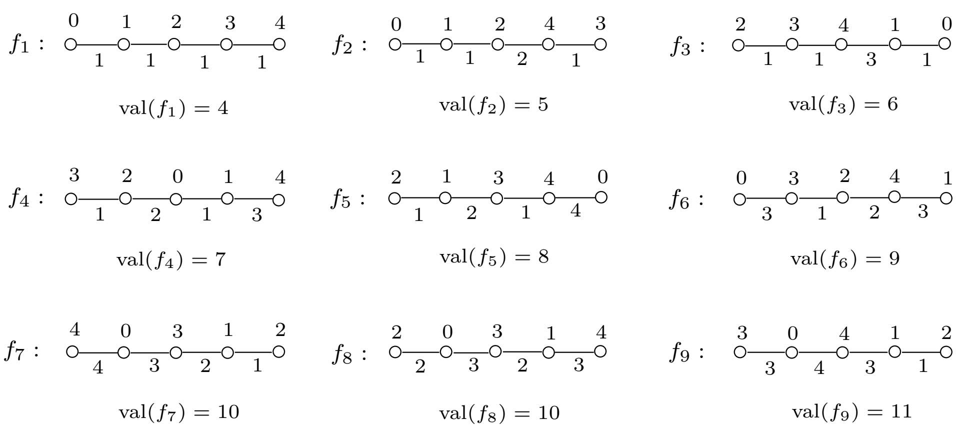

Figure 1: Some γ-labelings of P_5

Since $\mathrm{val}(f_1) = 4$ and the size of $P_5 = 4$, it follows that f_1 is a γ-min labeling of P_5. It is shown in [1] that the γ-labeling f_9 is a γ-max labeling of P_5.

In [1, 6, 7, 8, 9, 10, 11, 12], the maximum and minimum values of a γ-labeling of path P_n, cycle C_n, complete graph K_n, double star $S_{p,q}$, complete bipartite graph $K_{r,s}$, cycle with a triangle $C_n^{\triangle}$ and cycle with one chord $C_n + e$ are determined.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\bar{f} : V(G) \to \{0, 1, \ldots, m\}$ of f is defined by

$$\bar{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\bar{f}$ a γ-labeling of G but $\mathrm{val}(\bar{f}) = \mathrm{val}(f)$ as well. This gives us the following.

observation 1.1 ([1])**.** *Let f be a γ-labeling of a graph G. Then f is a γ-*max *labeling* (γ-min *labeling*) *of G if and only if $\bar{f}$ is a γ-*max *labeling* (γ-min *labeling*) *of G.*

For integers a and b with $a < b$, let

$$[a, b] = \{a, a + 1, \ldots, b\}$$

be a *consecutive set* of integers between a and b.

The following results appeared in [1], [7] and [13] are useful to us.

Theorem 1.2 ([1])**.** *If G is a connected graph of order n, then G has a γ-*min *labeling f such that $f(V(G)) = [0, n - 1]$.*

Theorem 1.3 ([7])**.** *Let G be a connected graph of order n and size m. Then* $\text{val}_{\min}(G) = m$ *if and only if* $G \cong P_n$.

Theorem 1.4 ([7])**.** *Let f be a γ-labeling of a connected graph G. If P is a $u-v$ path in G, then*

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)|.$$

Theorem 1.5 ([13])**.** *Let G be a nontrivial graph of order n and size m and f a γ-labeling of G. If f is a* γ-min *labeling of G, then $f(V(G))$ is a consecutive subset of $[0, m]$, that is, $f(V(G)) = [k, k+(n-1)]$ for some integer k with $0 \leq k \leq m-(n-1)$.*

A vertex of degree at least 3 in a graph G is called a *major vertex*. An end-vertex z of G is said to be a *terminal vertex* of a major vertex v of G if $d(v, z) < d(w, z)$ for every other major vertex w of G. A major vertex v of a graph G is an *exterior major vertex* of G if it has at least one terminal vertex.

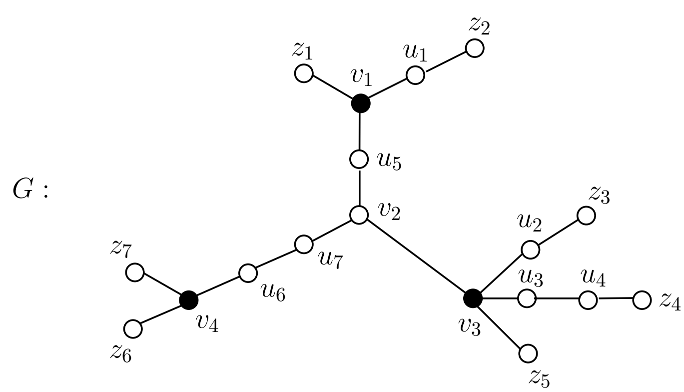

Figure 2: The graph G

For example, the graph G of Figure 2 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are z_1 and z_2, the terminal vertices of v_3 are z_3, z_4 and z_5, and the terminal vertices of v_4 are z_6 and z_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of G. Thus G has three exterior major vertices v_1, v_3 and v_4.

In [7] and [14], the minimum value and the maximum value of γ-labelings of some trees with exterior major vertices are determined.

Let G be a connected graph and f a γ-min labeling of G. Then G has a *unique* γ-min *labeling* if f and $\bar{f}$ are only two γ-min labelings of G. Consequently, since the γ-labelings f_1 and $\bar{f}_1$ are only two γ-min labelings of the path P_5 in Figure 1, it follows that the path P_5 has a unique γ-min labeling.

The goal of this paper is to study a connected graph having the unique γ-min labeling. We also determine the minimum values of γ-labelings of some generalized trees with exterior major vertices. It is shown that they have no unique γ-min labeling, but not so for a path.

The reader is referred to Chartrand and Zhang [15] for basic definitions and terminology not mentioned here.

2 Unique γ-min Labelings of Graphs

Let G be a connected graph of order n and size m and f a γ-labeling of G. For each integer k with $0 \leq k \leq m - \max\{f(v)\colon v \in V(G)\}$, let $f^k : V(G) \to \{0, 1, 2, \ldots, m\}$ be a γ-labeling of G defined by

$$f^k(v) = f(v) + k\,, \quad \text{for each } v \in V(G)\,.$$

Note that $f^k = f$ when $k = 0$.

Theorem 2.1. *Let G be a connected graph of order n and size m and f a γ-labeling of G. Then for each integer k with $0 \leq k \leq m - \max\{f(v)\colon v \in V(G)\}$, $\mathrm{val}(f^k) = \mathrm{val}(f)$.*

Proof. Let k be an integer with $0 \leq k \leq m - \max\{f(v)\colon v \in V(G)\}$. Since $|f^k(u) - f^k(v)| = |(f(u)+k) - (f(v)+k)| = |f(u) - f(v)|$ for each $e = uv \in E(G)$, $\mathrm{val}(f^k) = \mathrm{val}(f)$. □

This also provides the following corollary.

Corollary 2.2. *Let G be a connected graph of order n and size m and f a γ-labeling of G. Then f is a γ-max labeling (γ-min labeling) of G if and only if f^k is a γ-max labeling (γ-min labeling) of G for each integer k with $0 \leq k \leq m - \max\{f(v)\colon v \in V(G)\}$.*

By Theorem 1.2 and Corollary 2.2, we can verify that none of graphs with cycle has a unique γ-min labeling.

Theorem 2.3. *If a connected graph G has the unique γ-min labeling, then G is a tree.*

Proof. Let G be a connected graph of order n and size m. Assume that G contains a cycle. Then $m \geq n$. By Theorem 1.2, G has a γ-min labeling f such that $f(V(G)) = [0, n-1]$. Since $m \geq n$, $m - (n-1) \geq 1$. Thus G has a γ- labeling f^1. By Corollary 2.2, f^1 is a γ-min labeling of G. Since $f^1(V(G)) = [1, n]$, $f^1 \neq f$ and $f^1 \neq \bar{f}$. Therefore G has no unique γ-min labeling. □

Next, we determine that every path P_n of order n has a unique γ-min labeling. This starts by characterizing γ-min labelings of a path P_n.

Theorem 2.4. *Let f be a γ-labeling of a path $P_n : v_1, v_2, \ldots, v_n$ defined by*

$$f(v_i) = i - 1\,, \quad \text{for each integer } i \text{ with } 1 \leq i \leq n\,.$$

Then f and $\bar{f}$ are only two γ-min labelings of P_n.

Proof. By Theorem 1.3, we have $\mathrm{val}_{\min}(P_n) = n-1$. Since $\mathrm{val}(f) = \mathrm{val}(\bar{f}) = n-1$, f and $\bar{f}$ are γ-min labelings of P_n. Let f_1 be a γ-min labelings of P_n. Then $\mathrm{val}(f_1) = \mathrm{val}_{\min}(P_n) = n-1$ which is the size of P_n. Since $f_1'(e) = 1$ for each edge e in P_n, it follows that $|f_1(v_{i+1}) - f_1(v_i)| = 1$ for each i, $1 \le i \le n-1$. Thus either $f_1 = f$ or $f_1 = \bar{f}$. Therefore f and $\bar{f}$ are only two γ-min labelings of P_n. □

Corollary 2.5. *A path has a unique* γ-min *labeling.*

The following result shows that there are many trees that fail to have unique γ-min labeling.

Theorem 2.6. *Let T be a tree with exterior major vertices. If there are at least two terminal vertices z_1 and z_2 of some exterior major vertex v of T such that $d(v, z_1) = d(v, z_2)$, then T has no unique* γ-min *labeling.*

Proof. Assume that there are at least two terminal vertices z_1 and z_2 of some exterior major vertex v of T such that $d(v, z_1) = d(v, z_2)$. By Theorem 1.2, T has a γ-min labeling f such that $f(V(T)) = [0, n-1]$. Let $P : v = u_0, u_1, \ldots, u_d = z_1$ be a $v - z_1$ path in T and $Q : v = w_0, w_1, \ldots, w_d = z_2$ be a $v - z_2$ path in T. Let f_1 be a γ-labeling of T defined by

$$f_1(a) = \begin{cases} f(a) & \text{if } a \in V(T) - \{u_i, w_j | 1 \le i, j \le d\} \\ f(w_i) & \text{if } a = u_i \quad \text{with } 1 \le i \le d \\ f(u_j) & \text{if } a = w_j \quad \text{with } 1 \le j \le d\,. \end{cases}$$

Then $\mathrm{val}(f_1) = \mathrm{val}(f) = \mathrm{val}_{\min}(T)$. Thus f_1 is a γ-min labeling of T such that $f_1 \ne f$ and $f_1 \ne \bar{f}$. Therefore T has no unique γ-min labeling. □

3 γ-min Labeling of a Tree with Exterior Major Vertices of Degree 3

The *maximum degree* of a graph G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$. A *caterpillar* is a tree of order at least 3, the removal of whose end-vertices produces a path. We recall the minimum value of a γ-labeling of a caterpillar with $\Delta(T) = 3$ having an arbitrary number of exterior major vertices as follows.

Theorem 3.1 ([7])**.** *If T is a caterpillar of order $n \ge 4$ such that $\Delta(T) = 3$ and T has exactly k exterior major vertices, then*

$$\mathrm{val}_{\min}(T) = n + k - 1\,.$$

Note that if a tree T is a caterpillar, then $d(v, z) = 1$ for each terminal vertex z of an exterior major vertex v of T which does not lie on the path of length $\mathrm{diam}(T)$. Next, we generalize a caterpillar of Theorem 3.1 to a tree T having $\Delta(T) = 3$ and $d(v, z) \ge 1$ for each terminal vertex z of an exterior major vertex v of T, and then formulate $\mathrm{val}_{\min}(T)$.

Proposition 3.2. *Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \mathrm{diam}(T)$. Then*

$$\mathrm{val}_{\min}(T) \leq 2n - d - 2\,.$$

Proof. Let $P : v_0, v_1, \ldots, v_d$ be a path containing all exterior major vertices in T. Let $v_{l_1}, v_{l_2}, \ldots, v_{l_k}$ be all exterior major vertices in T such that $1 \leq l_1 < l_2 < \cdots < l_k \leq d-1$. For each $1 \leq j \leq k$, let z_j be the terminal vertices of v_{l_j} not on P and $Q_j : v_{l_j} = u_{j0}, u_{j1}, \ldots, u_{jd_j} = z_j$ the $v_{l_j} - z_j$ path in T. Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} i & \text{if } a = v_i \quad \text{with } 0 \leq i \leq l_1 \\ \left(\sum\limits_{r=1}^{s} d_r\right) + i & \text{if } a = v_i \quad \text{with } l_s + 1 \leq i \leq l_{s+1},\ 1 \leq s \leq k-1 \\ n - d - 1 + i & \text{if } a = v_i \quad \text{with } l_k + 1 \leq i \leq d \\ l_1 + i & \text{if } a = u_{1i} \ \text{with } 1 \leq i \leq d_1 \\ \left(\sum\limits_{r=1}^{j-1} d_r\right) + l_j + i & \text{if } a = u_{ji} \ \text{with } 1 \leq i \leq d_j,\ 2 \leq j \leq k. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f) &= \sum_{e \in E(P)} f'(e) + \left(\sum_{e \in E(Q_1)} f'(e) + \sum_{e \in E(Q_2)} f'(e) + \cdots + \sum_{e \in E(Q_k)} f'(e) \right) \\ &= 2n - d - 2\,. \end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \leq \mathrm{val}(f) = 2n - d - 2\,.$ □

We now establish the lower bound of the minimum value of a γ-labeling of a tree T with $\Delta(T) = 3$ having an arbitrary number of exterior major vertices of degree 3, as follows.

Proposition 3.3. *Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \mathrm{diam}(T)$. Then*

$$\mathrm{val}_{\min}(T) \geq 2n - d - 2\,.$$

Proof. Let g be an arbitrary γ-labeling of T. Since T has exactly $n-1$ edges, there are vertices $u, w \in V(T)$ with $g(u) = 0$ and $g(w) = n-1$. Let Q be a $u - w$ path in T. By Theorem 1.4,

$$\sum_{e \in E(Q)} g'(e) \geq |g(u) - g(w)| = n - 1.$$

Since the length of Q is at most $\text{diam}(T) = d$, there are at least $n-d-1$ edges of T not on Q, and hence

$$\sum_{e\in E(T)-E(Q)} g'(e) \geq n-d-1.$$

Thus

$$\begin{aligned}\text{val}(g) &= \sum_{e\in E(Q)} g'(e) + \sum_{e\in E(T)-E(Q)} g'(e)\\ &\geq 2n-d-2\,.\end{aligned}$$

Therefore $\text{val}_{\min}(T) \geq 2n-d-2\,.$ □

Combining Propositions 3.2 and 3.3, we have the following.

Theorem 3.4. *Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \text{diam}(T)$. Then*

$$\text{val}_{\min}(T) = 2n-d-2\,.$$

With aid of Theorem 3.4 we are able to show that a tree in Theorem 3.4 has no unique γ-min labeling.

Theorem 3.5. *If T is a tree with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length* $\text{diam}(T)$*, then T has no unique γ-min labeling.*

Proof. Let T be a tree with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $\text{diam}(T)$. Let $P: v_0, v_1, \ldots, v_d$ be a path containing all exterior major vertices in T. Let $v_{l_1}, v_{l_2}, \ldots, v_{l_k}$ be all exterior major vertices in T such that $1 \leq l_1 < l_2 < \cdots < l_k \leq d-1$. For each $1 \leq j \leq k$, let z_j be the terminal vertices of v_{l_j} not on P and $Q_j : v_{l_j} = u_{j0}, u_{j1}, \ldots, u_{jd_j} = z_j$ the $v_{l_j} - z_j$ path in T. Let f_1 be a γ-labeling of T defined by

$$f_1(a) = \begin{cases} i & \text{if } a = v_i \text{ with } 0 \leq i \leq l_1 - 1\\ \left(\sum_{r=1}^{s} d_r\right) + i & \text{if } a = v_i \text{ with } l_s + 1 \leq i \leq l_{s+1},\ 1 \leq s \leq k-1\\ n-d-1+i & \text{if } a = v_i \text{ with } l_k + 1 \leq i \leq d\\ l_1 + d_1 - i & \text{if } a = u_{1i} \text{ with } 0 \leq i \leq d_1\\ \left(\sum_{r=1}^{j-1} d_r\right) + l_j + i & \text{if } a = u_{ji} \text{ with } 1 \leq i \leq d_j,\ 2 \leq j \leq k. \end{cases}$$

Then

$$\begin{aligned}\text{val}(f_1) &= \sum_{e\in E(P)} f_1'(e) + \left(\sum_{e\in E(Q_1)} f_1'(e) + \sum_{e\in E(Q_2)} f_1'(e) + \cdots + \sum_{e\in E(Q_k)} f_1'(e)\right)\\ &= n-1+\sum_{i=1}^{k} d_i\\ &= \text{val}_{\min}(T) \qquad \text{(by Theorem 3.4).}\end{aligned}$$

Thus not only f_1 is a γ-min labeling of T, but the γ-labeling f in Proposition 3.2 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$. Therefore T has no unique γ-min labeling. □

4 γ-min Labeling of a Tree with a Unique Exterior Major Vertex

In this section, we establish a minimum value of a γ-labeling of a tree with a unique exterior major vertex of an arbitrary degree. In order to do this, we first present the minimum value of a γ-labeling of a tree with a unique exterior major vertex of degree 3 shown in [7].

Theorem 4.1 ([7]). *Let T be a tree of order n with a unique exterior major vertex v of degree 3. If $d = \min\{d(v,z) \mid z$ is a terminal vertex of $v\}$, then*

$$\mathrm{val}_{\min}(T) = n + d - 1\,.$$

Next, we generalize Theorem 4.1 to a tree T with a unique exterior major vertex of an arbitrary degree. We are now prepared to present the upper bound of the minimum value of a γ-labeling of such a tree.

Proposition 4.2. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\mathrm{val}_{\min}(T) \leq \begin{cases} n - 1 + \sum\limits_{j=1}^{\frac{\Delta(T)}{2}-1} \sum\limits_{i=1}^{2j} d_i & \text{if } \Delta(T) \text{ is even} \\ n - 1 + \sum\limits_{j=1}^{\frac{\Delta(T)-1}{2}} \sum\limits_{i=1}^{2j-1} d_i & \text{if } \Delta(T) \text{ is odd}\,. \end{cases}$$

Proof. Let $z_1, z_2, \ldots, z_{\Delta(T)}$ be the terminal vertices of an exterior major vertex v. For each $1 \leq i \leq \Delta(T)$, let $Q_i : v = v_{i0}, v_{i1}, \ldots, v_{id_i} = z_i$ be the $v - z_i$ path in T.

Case 1. *$\Delta(T)$ is even.*

Let f be a γ-labeling of T defined by

$$f(v_{ij}) = \begin{cases} \left(\sum\limits_{\substack{i \leq k \leq \Delta(T) \\ k \text{ is even}}} d_k \right) - j & \text{if } i \text{ is even, } 2 \leq i \leq \Delta(T) \text{ and } 1 \leq j \leq d_i \\ n - 1 + j - \sum\limits_{\substack{i \leq k \leq \Delta(T)-1 \\ k \text{ is odd}}} d_k & \text{if } i \text{ is odd, } 1 \leq i \leq \Delta(T) - 1 \text{ and } 1 \leq j \leq d_i \\ \sum\limits_{\substack{1 \leq k \leq \Delta(T) \\ k \text{ is even}}} d_k & \text{if } v_{ij} = v\,. \end{cases}$$

Then

$$\begin{aligned}\mathrm{val}(f) &= \left(\sum_{e\in E(Q_1)} f'(e) + \sum_{e\in E(Q_3)} f'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)-1})} f'(e)\right)\\ &\quad + \left(\sum_{e\in E(Q_2)} f'(e) + \sum_{e\in E(Q_4)} f'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)})} f'(e)\right)\\ &= n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i\,.\end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \le \mathrm{val}(f) = n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i\,.$

Case 2. $\Delta(T)$ *is odd.*
Let f be a γ-labeling of T defined by

$$f(v_{ij}) = \begin{cases} \left(\sum\limits_{\substack{i\le k\le \Delta(T)\\ k \text{ is odd}}} d_k\right) - j & \text{if } i \text{ is odd, } 1\le i\le \Delta(T) \text{ and } 1\le j\le d_i\\ n-1+j-\sum\limits_{\substack{i\le k\le \Delta(T)-1\\ k \text{ is even}}} d_k & \text{if } i \text{ is even, } 2\le i\le \Delta(T)-1 \text{ and } 1\le j\le d_i\\ \sum\limits_{\substack{1\le k\le \Delta(T)\\ k \text{ is odd}}} d_k & \text{if } v_{ij}=v\,.\end{cases}$$

Then

$$\begin{aligned}\mathrm{val}(f) &= \left(\sum_{e\in E(Q_1)} f'(e) + \sum_{e\in E(Q_3)} f'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)})} f'(e)\right)\\ &\quad + \left(\sum_{e\in E(Q_2)} f'(e) + \sum_{e\in E(Q_4)} f'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)-1})} f'(e)\right)\\ &= n-1+\sum_{j=1}^{\frac{\Delta(T)-1}{2}}\sum_{i=1}^{2j-1} d_i\,.\end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \le \mathrm{val}(f) = n-1+\sum_{j=1}^{\frac{\Delta(T)-1}{2}}\sum_{i=1}^{2j-1} d_i\,.$ □

We are able to show the lower bound of the minimum value of a γ-labeling of a tree with a unique exterior major vertex of an arbitrary degree.

Proposition 4.3. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\operatorname{val}_{\min}(T) \geq \begin{cases} n-1+\displaystyle\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i & \text{if } \Delta(T) \text{ is even} \\ n-1+\displaystyle\sum_{j=1}^{\frac{\Delta(T)-1}{2}}\sum_{i=1}^{2j-1} d_i & \text{if } \Delta(T) \text{ is odd.} \end{cases}$$

Proof. Let g be an arbitrary γ-labeling of T. Since T has exactly $n-1$ edges, there are vertices $u_1, w_1 \in V(T)$ with $g(u_1) = 0$ and $g(w_1) = n-1$. Let Q_1 be a $u_1 - w_1$ path in T. By Theorem 1.4,

$$\sum_{e \in E(Q_1)} g'(e) \geq |g(u_1) - g(w_1)| = n-1.$$

Let $u_2, w_2 \in V(T)$ with

$$g(u_2) = \min\{g(x) \mid x \notin V(Q_1)\} \text{ and } g(w_2) = \max\{g(x) \mid x \notin V(Q_1)\}.$$

Let Q_2 be a $u_2 - w_2$ path in T. By Theorem 1.4,

$$\sum_{e \in E(Q_2)} g'(e) \geq |g(u_2) - g(w_2)| = g(w_2) - g(u_2).$$

Since the length of Q_1 is at most $\operatorname{diam}(T) = d_{\Delta(T)} + d_{\Delta(T)-1}$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1}$ edges of T not on Q_1, and hence

$$g(w_2) - g(u_2) \geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1}.$$

Thus

$$\begin{aligned} \sum_{e \in E(Q_2)} g'(e) &\geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} \\ &= d_1 + d_2 + \cdots + d_{\Delta(T)-2}. \end{aligned}$$

Let $u_3, w_3 \in V(T)$ with

$$g(u_3) = \min\{g(x) \mid x \notin V(Q_1) \cup V(Q_2)\}$$

and

$$g(w_3) = \max\{g(x) \mid x \notin V(Q_1) \cup V(Q_2)\}.$$

Let Q_3 be a $u_3 - w_3$ path in T. By Theorem 1.4,

$$\sum_{e \in E(Q_3)} g'(e) \geq |g(u_3) - g(w_3)| = g(w_3) - g(u_3).$$

Since the sum of the length of Q_1 and Q_2 is at most $d_{\Delta(T)} + d_{\Delta(T)-1} + d_{\Delta(T)-2} + d_{\Delta(T)-3}$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3}$ edges of T not on Q_1 and Q_2, and hence

$$g(w_3) - g(u_3) \geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3}.$$

Thus

$$\begin{aligned} \sum_{e\in E(Q_3)} g'(e) &\geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3} \\ &= d_1 + d_2 + \cdots + d_{\Delta(T)-4}. \end{aligned}$$

Continue until we have for each $1 \leq j \leq \left\lfloor \frac{\Delta(T)}{2} \right\rfloor$, let $u_j, w_j \in V(T)$ with

$$g(u_j) = \min\{g(x) \mid x \notin \bigcup_{i=1}^{j-1} V(Q_i)\} \text{ and } g(w_j) = \max\{g(x) \mid x \notin \bigcup_{i=1}^{j-1} V(Q_i)\}$$

and let Q_j be a $u_j - w_j$ path in T. Then

$$\begin{aligned} \sum_{e\in E(Q_j)} g'(e) &\geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - \cdots - d_{\Delta(T)-2j+4} - d_{\Delta(T)-2j+3} \\ &= d_1 + d_2 + \cdots + d_{\Delta(T)-2j+2}. \end{aligned}$$

Case 1. $\Delta(T)$ *is even.*

Then $\left\lfloor \frac{\Delta(T)}{2} \right\rfloor = \frac{\Delta(T)}{2}$. We have $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) = \emptyset$ or $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) \neq \emptyset$. If $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) = \emptyset$, then

$$\begin{aligned} \text{val}(g) &= \sum_{e\in E(Q_1)} g'(e) + \sum_{e\in E(Q_2)} g'(e) + \cdots + \sum_{e\in E\left(Q_{\frac{\Delta(T)}{2}}\right)} g'(e) \\ &\geq n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i\,. \end{aligned}$$

If $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) \neq \emptyset$, then

$$\begin{aligned}
\mathrm{val}(g) &= \left(\sum_{e\in E(Q_1)} g'(e) + \sum_{e\in E(Q_2)} g'(e) + \cdots + \sum_{e\in E\left(Q_{\frac{\Delta(T)}{2}}\right)} g'(e) \right) \\
&\quad + \sum_{e\in E(T)-\bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j)} g'(e) \\
&\geq n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i + 1 \\
&> n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i\,.
\end{aligned}$$

In general, $\mathrm{val}(g) \geq n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i$. Therefore $\mathrm{val}_{\min}(T) \geq n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i$.

Case 2. $\Delta(T)$ *is odd.*

Then $\left\lfloor \frac{\Delta(T)}{2} \right\rfloor = \frac{\Delta(T)-1}{2}$, and so $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)-1}{2}} E(Q_j) \neq \emptyset$.

Since the sum of the length of Q_j for all $1 \leq j \leq \frac{\Delta(T)-1}{2}$ is at most $d_{\Delta(T)} + d_{\Delta(T)-1} + d_{\Delta(T)-2} + \cdots + d_3 + d_2$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - \cdots - d_3 - d_2 = d_1$ edges of T not on Q_j for all $1 \leq j \leq \frac{\Delta(T)-1}{2}$.
Thus

$$\begin{aligned}
\mathrm{val}(g) &= \left(\sum_{e\in E(Q_1)} g'(e) + \sum_{e\in E(Q_2)} g'(e) + \cdots + \sum_{e\in E\left(Q_{\frac{\Delta(T)-1}{2}}\right)} g'(e) \right) \\
&\quad + \sum_{e\in E(T)-\bigcup_{j=1}^{\frac{\Delta(T)-1}{2}} E(Q_j)} g'(e) \\
&\geq n-1+\sum_{j=1}^{\frac{\Delta(T)-1}{2}}\sum_{i=1}^{2j-1} d_i\,.
\end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \geq n-1+\sum_{j=1}^{\frac{\Delta(T)-1}{2}}\sum_{i=1}^{2j-1} d_i\,.$ □

We compute the minimum value of a γ-labeling of a tree with a unique exterior major vertex of an arbitrary degree by combining Propositions 4.2 and 4.3 as follows.

Theorem 4.4. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\mathrm{val}_{\min}(T) = n - 1 + \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} \left(\left\lfloor \frac{\Delta(T)}{2} \right\rfloor - i \right) (d_{2i-1} + d_{2i}) + \delta_\Delta \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} d_{2i-1}$$

where

$$\delta_\Delta = \begin{cases} 0 & \text{if } \Delta(T) \text{ is even} \\ 1 & \text{if } \Delta(T) \text{ is odd}. \end{cases}$$

We are now able to apply Theorem 4.4 to show that a tree with a unique exterior major vertex of an arbitrary degree has no unique γ-min labeling.

Theorem 4.5. *If T is a tree with a unique exterior major vertex, then T has no unique γ-*min *labeling.*

Proof. Let T be a tree with a unique exterior major vertex v. Let $z_1, z_2, \ldots, z_{\Delta(T)}$ be the terminal vertices of v. Let $Q_i : v = v_{i0}, v_{i1}, \ldots, v_{id_i} = z_i$ be the $v - z_i$ path of T for each $1 \leq i \leq \Delta(T)$.

Case 1. $\Delta(T)$ *is even.*
Let f_1 be a γ-labeling of T defined by

$$f_1(v_{ij}) = \begin{cases} \left(\sum\limits_{\substack{i \leq k \leq \Delta(T) \\ k \text{ is even}}} d_k \right) - j & \text{if } i \text{ is even, } 4 \leq i \leq \Delta(T) \text{ and } 1 \leq j \leq d_i \\ n - 1 + j - \sum\limits_{\substack{i \leq k \leq \Delta(T)-1 \\ k \text{ is odd}}} d_k & \text{if } i \text{ is odd, } 3 \leq i \leq \Delta(T) - 1 \text{ and } 1 \leq j \leq d_i \\ n - 1 - d_2 + j - \sum\limits_{\substack{3 \leq k \leq \Delta(T)-1 \\ k \text{ is odd}}} d_k & \text{if } i = 2 \text{ and } 1 \leq j \leq d_2 \\ d_1 + \left(\sum\limits_{\substack{4 \leq k \leq \Delta(T) \\ k \text{ is even}}} d_k \right) - j & \text{if } i = 1 \text{ and } 1 \leq j \leq d_1 \\ d_1 + \sum\limits_{\substack{4 \leq k \leq \Delta(T) \\ k \text{ is even}}} d_k & \text{if } v_{ij} = v. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_1) &= \left(\sum_{e\in E(Q_1)} f_1'(e) + \sum_{e\in E(Q_3)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)-1})} f_1'(e) \right) \\ &\quad + \left(\sum_{e\in E(Q_2)} f_1'(e) + \sum_{e\in E(Q_4)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)})} f_1'(e) \right) \\ &= n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i \\ &= \mathrm{val}_{\min}(T) \qquad \text{(by Theorem 4.4).} \end{aligned}$$

Thus f_1 is a γ-min labeling of T. Since the γ-labeling f in Case 1 of Proposition 4.2 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$, it follows that T has no unique γ-min labeling.

Case 2. $\Delta(T)$ *is odd.*
Let f_1 be a γ-labeling of T defined by

$$f_1(v_{ij}) = \begin{cases} \left(\sum\limits_{\substack{i\le k\le \Delta(T) \\ k \text{ is odd}}} d_k \right) - j & \text{if } i \text{ is odd, } 3\le i\le \Delta(T) \text{ and } 1\le j\le d_i \\ n-1+j-\sum\limits_{\substack{i\le k\le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } i \text{ is even, } 2\le i\le \Delta(T)-1 \text{ and } 1\le j\le d_i \\ n-1-d_1+j-\sum\limits_{\substack{2\le k\le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } i=1 \text{ and } 1\le j\le d_1 \\ n-1-d_1-\sum\limits_{\substack{2\le k\le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } v_{ij}=v\,. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_1) &= \left(\sum_{e\in E(Q_1)} f_1'(e) + \sum_{e\in E(Q_3)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)})} f_1'(e) \right) \\ &\quad + \left(\sum_{e\in E(Q_2)} f_1'(e) + \sum_{e\in E(Q_4)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)-1})} f_1'(e) \right) \\ &= n-1+\sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i \\ &= \mathrm{val}_{\min}(T) \qquad \text{(by Theorem 4.4).} \end{aligned}$$

Thus f_1 is a γ-min labeling of T, however the γ-labeling f in Case 2 of Proposition 4.2 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$. Therefore T has no unique γ-min labeling. □

5 Open Question

Theorems 2.6, 3.5 and 4.5 show that some trees with exterior major vertices have no unique γ-min labeling. However, Corollary 2.5 shows that a path has a unique γ-min labeling. All such results lead us to the conjecture:

"A connected graph G has the unique γ-min labeling if and only if G is a path."

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