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Fixed Point and Coincidence Point Theorems for Expansive Mappings in Partial b-Metric Spaces

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Abstract : In this paper, we introduce a class of expansive mappings on partial b-metric spaces and prove fixed point and common fixed point theorems for a pair of those maps on partial b-metric spaces. We also establish a coincidence point theorem for two expansive maps on partial b-metric spaces. The results generalize and extend some results in literature.

Keywords : partial b-metric spaces; expansive mappings.
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1 Introduction

There are a number of generalizations of metric spaces and Banach contraction principle. In this sequel, Bakhtin [1] and Czerwik [2] introduced b-metric spaces as a generalization of metric spaces. They proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in such spaces. On the other hand, Matthews [3] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric space and can be applied in program verification. In [4], Shukla introduced the notion of a partial b-metric space as a generalization of partial metric spaces and b-metric spaces.

In 1981, Gillespie and Williams [5] introduced a new class of maps where the existing constant is greater than one.

Suppose (X, d) is a metric space, $T : X \to X$ and there exists a constant $k > 1$ such that $d(Tx, Ty) \geq kd(x, y)$, for all $x, y \in X$. Then T is called an expanding map.

In this work, we introduce the class of expanding maps in partial b-metric spaces and prove some fixed point theorems in the new setting. In 1999, Pant [6] introduced a new continuity condition known as reciprocal continuity and proved a common fixed point theorem by using the compatibility in metric spaces. The notion of reciprocal continuity is weaker than the continuity of one of the mappings.

Han and Xu [7] proved the existence of common fixed point for a pair of expanding mappings in cone metric spaces by assuming the surjectivity of the maps. Esakkiappan [8] later proved a common fixed point theorem using compatible and reciprocal continuous map in a cone metric space. Manro and Kumar [9] proved common fixed point theorems for expansion mapping using the concept of compatible maps and weakly reciprocal continuity in both metric and G-metric spaces. Huang et al. [10] proved the fixed point and common fixed point theorems for expansion mappings and pairs of weakly compatible expansion maps respectively in partial metric spaces. In this work, the existence of the fixed point of an expanding map and common fixed point for a pair of expanding mappings on partial b-metric spaces using the concept of compatible maps and reciprocal continuity are proved. Shatanawi and Awawdeh [11] proved some results for fixed and coincidence points for some expansive mappings in cone metric spaces in which the surjectivity of the two maps is not assumed in proving the coincidence point theorem. Also we prove the coincidence point theorem for expanding maps without assuming the surjectivity of the maps therein in partial b-metric spaces. Our results generalize the recent results of Huang et al. [10], Manro and Kumar [9] and an analogue results to the results of Han and Xu [7], Esakkiappan [8] and Shatanawi and Awawdeh [11] in the cone metric spaces.

2 Preliminaries

Throughout this paper the letters $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ will denote the set of real numbers, nonnegative real numbers and natural numbers, respectively.

First, we recall some definitions from b-metric and partial metric spaces

Definition 2.1. [1] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $b : X \times X \to \mathbb{R}^+$ is a *b-metric* on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $b(x, y) = 0$ if and only if $x = y$;

(b2) $b(x, y) = b(y, x)$;

(b3) $b(x,y) \le s\big(b(x,z) + b(z,y)\big)$.

In this case, the pair (X,d) is called a *b-metric space with coefficient s*.

Definition 2.2. [3] A *partial metric* on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$, such that for all $x,y,z \in X$:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;

(p2) $p(x,x) \le p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

In this case, the pair (X,p) is called *partial metric space.*

Now, we define the partial b-metric spaces.

Definition 2.3. [4] Let X be a nonempty set and $s \ge 1$ be a given real number. A mapping $p_b : X \times X \to \mathbb{R}^+$ is said to be a *partial b-metric* on X, if for all $x,y,z \in X$, the following conditions are satisfied:

(p_b1) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(p_b2) $p_b(x,x) \le p_b(x,y)$;

(p_b3) $p_b(x,y) = p_b(y,x)$;

(p_b4) $p_b(x,y) \le s\big(p_b(x,z) + p_b(z,y)\big) - p_b(z,z)$.

A *partial b-metric space* is a pair (X,p_b) such that X is a nonempty set and p_b is a *partial b-metric* on X. The number s is called the *coefficient* of (X,p_b).

Remark 2.4. *In a partial b-metric space (X,p_b), if $p_b(x,y) = 0$, then (p_b1) and (p_b2) imply that $x = y$. But the converse does not hold always. It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric is a partial b-metric space with same coefficient and zero distance. However, the converse of these facts need not hold.The following example shows that a partial b-metric on X might be neither a partial metric, nor a b-metric on X.*

Example 2.5. [4] Let $X = [0,\infty)$. Define a function $p_b : X \times X \to X$ such that $p_b(x,y) = \big\{\max\{x,y\}\big\}^2 + |x-y|^2$, for all $x,y \in X$. Then (X,p_b) is a partial b-metric space with the coefficient $s = 2 > 1$, but it is neither a b-metric nor a partial metric space.

In [4], S. Shukla defined Cauchy sequence and convergent sequence in partial b-metric spaces.

Definition 2.6. [4] Let (X,p_b) be a partial b-metric space with coefficient s. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be *convergent* and *converges to* x, if

$$\lim_{n\to\infty} p_b(x_n, x) = p_b(x, x).$$

(ii) The sequence $\{x_n\}$ is said to be *Cauchy sequence* in (X, p_b) if

$$\lim_{n,m\to\infty} p_b(x_n, x_m) \quad \text{exists (and is finite).}$$

(iii) (X, p_b) is said to be a *complete partial b-metric space* if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x, x).$$

Now, we define expanding and commuting mapping, coincidence point and weakly compatible in partial b-metric spaces.

Definition 2.7. Let (X, p_b) be a partial b-metric space and $T : X \to X$. Then T is called a *expanding mapping*, if for every $x, y \in X$ there exists a number $k > 1$ such that $p_b(Tx, Ty) \geq kp_b(x, y)$.

Definition 2.8. Two self mappings T and S of a partial b-metric space (X, p_b) are said to be *commuting* if $TSx = STx$ for all $x \in X$.

Definition 2.9. Let T and S be self-mappings on a set X. If $w = Tx = Sx$ for some $x \in X$, then the point x is called a *coincidence point* of T and S, and w is called a *point of coincidence* of T and S.

Definition 2.10. (i) Two self mappings T and S on a set X are said to be *compatible* if for $\{x_n\}$ in X, $Tx_n \to x$ and $Sx_n \to x$, for some $x \in X$.

(ii) Two self mappings T and S on a set X. Then T and S are said to be *weakly compatible* if they commute at each of their coincidence points; i.e., if $Tx = Sx$ for some $x \in X$, then $TSx = STx$.

The notion of reciprocal continuity defined as follow.

Definition 2.11. Two self-mappings T and S are called *reciprocally continuous* if $\lim_{n\to\infty} TSx_n = Tz$ and $\lim_{n\to\infty} STx_n = Sz$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

3 Main Results

3.1 Fixed Point Theorems

Theorem 3.1. *Let (X, p_b) be a complete partial b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a surjection. Suppose that there exist $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_3 > s(1 - a_2) \geq a_5$, such that*

$$\begin{aligned} p_b(Tx, Ty) \;\geq\; & a_1 p_b(x, y) + a_2 p_b(x, Tx) + a_3 p_b(y, Ty) \\ & + a_4 p_b(x, Ty) + a_5 p_b(y, Tx), \end{aligned} \tag{3.1}$$

for all $x, y \in X, x \neq y$. Then T has a fixed point in X. Moreover, if $a_1 + a_4 + a_5 > 1$, then the fixed point is unique.

Proof. Let $x_0 \in X$ be chosen. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Continuing the process, we can define a sequence $\{x_n\} \in X$ such that $x_{n-1} = Tx_n, n \in \mathbb{N}$. Without loss of generality, we suppose that $x_{n-1} \neq x_n$ for $n \geq 1$. From (3.1) we have

$$\begin{aligned} p_b(x_{n-1}, x_n) \;\geq\; & a_1 p_b(x_n, x_{n+1}) + a_2 p_b(x_n, Tx_n) + a_3 p_b(x_{n+1}, Tx_{n+1}) \\ & + a_4 p_b(x_n, Tx_{n+1}) + a_5 p_b(x_{n+1}, Tx_n) \\ =\; & a_1 p_b(x_n, x_{n+1}) + a_2 p_b(x_n, x_{n-1}) + a_3 p_b(x_{n+1}, x_n) \\ & + a_4 p_b(x_n, x_n) + a_5 p_b(x_{n+1}, x_{n-1}). \end{aligned}$$

From (p_b4) we have

$$p_b(x_{n-1}, x_{n+1}) \;\geq\; \tfrac{1}{s} p_b(x_{n-1}, x_n) - p_b(x_n, x_{n+1}) + \tfrac{1}{s} p_b(x_{n+1}, x_{n+1}).$$

Also

$$\begin{aligned} p_b(x_{n-1}, x_n) \;\geq\; & a_1 p_b(x_n, x_{n+1}) + a_2 p_b(x_{n-1}, x_n) + a_3 p_b(x_n, x_{n+1}) \\ & + a_5 \Big(\tfrac{1}{s} p_b(x_{n-1}, x_n) - p_b(x_n, x_{n+1}) \Big) \\ \geq\; & (a_1 + a_3 - a_5) p_b(x_n, x_{n+1}) + (a_2 + \tfrac{a_5}{s}) p_b(x_{n-1}, x_n). \end{aligned}$$

It follows that

$$(1 - a_2 - \tfrac{a_5}{s}) p_b(x_{n-1}, x_n) \;\geq\; (a_1 + a_3 - a_5) p_b(x_n, x_{n+1}).$$

Hence

$$p_b(x_n, x_{n+1}) \;\leq\; \tfrac{1 - a_2 - (a_5/s)}{a_1 + a_3 - a_5} p_b(x_{n-1}, x_n).$$

Let $k := \frac{1 - a_2 - (a_5/s)}{a_1 + a_3 - a_5}$. By $s(1 - a_2) \geq a_5,\ a_1 + a_3 > a_5$ and $a_1 + a_3 > s(1 - a_2)$ we have $k \in [0, \frac{1}{s})$. It follows that $p_b(x_n, x_{n+1}) \leq k p_b(x_{n-1}, x_n)$, by the induction, we have

$$p_b(x_n, x_{n+1}) \leq k p_b(x_{n-1}, x_n) \leq k^2 p_b(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n p_b(x_0, x_1),$$

and consequently $p_b(x_n, x_{n+1}) \leq k^n p_b(x_0, x_1)$ for all $n \in \mathbb{N}$. For $n > m$, we get

$$\begin{aligned} p_b(x_m, x_n) &\leq sp_b(x_m, x_{m+1}) + s^2 p_b(x_{m+1}, x_{m+2}) + \cdots + s^n p_b(x_{n-1}, x_n) \\ &\quad - p_b(x_{m+1}, x_{m+1}) - p_b(x_{m+2}, x_{m+2}) - \cdots - p_b(x_{n-1}, x_{n-1}) \\ &\leq (sk^m + s^2 k^{m+1} + \cdots + s^n k^{n-1}) p_b(x_0, x_1) \\ &= sk^m (1 + sk + \cdots + s^{n-1} k^{n-m-1}) p_b(x_0, x_1) \ ; \ (sk < 1) \\ &\leq \frac{sk^m}{1 - sk} p_b(x_0, x_1). \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $Tx_{n+1} = x_n \to p$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} p_b(x_n, p) = \lim_{n\to\infty} p_b(x_n, x_n) = \lim_{n,m\to\infty} p_b(x_m, x_n) = p_b(p, p).$$

Since T is a surjection, we find $q \in X$ such that $p = Tq$. Now we prove that $p = q$ is the fixed point of T. Using (3.1), we obtain

$$\begin{aligned} p_b(p, x_n) &= p_b(Tq, Tx_{n+1}) \\ &\geq a_1 p_b(q, x_{n+1}) + a_2 p_b(q, Tq) + a_3 p_b(x_{n+1}, Tx_{n+1}) \\ &\quad + a_4 p_b(q, Tx_{n+1}) + a_5 p_b(x_{n+1}, Tq). \end{aligned} \tag{3.2}$$

From (p_b4), we obtain

$$p_b(q, Tq) = p_b(q, p) \geq \tfrac{1}{s} p_b(q, x_{n+1}) - p_b(x_{n+1}, p) + \tfrac{1}{s} p_b(p, p), \tag{3.3}$$

$$p_b(q, x_n) \geq \tfrac{1}{s} p_b(q, x_{n+1}) - p_b(x_{n+1}, x_n) + \tfrac{1}{s} p_b(x_n, x_n) \tag{3.4}$$

and

$$p_b(p, x_n) \leq sp_b(p, x_{n+1}) + sp_b(x_{n+1}, x_n) - p_b(x_{n+1}, x_{n+1}). \tag{3.5}$$

Using (3.3), (3.4) and (3.5) in (3.2), we find that

$$\begin{aligned} & sp_b(p, x_{n+1}) + sp_b(x_n, x_{n+1}) - p_b(x_{n+1}, x_{n+1}) \\ &\quad \geq a_1 p_b(q, x_{n+1}) + \left(\tfrac{a_2}{s} p_b(q, x_{n+1}) - a_2 p_b(x_{n+1}, p) + \tfrac{a_2}{s} p_b(p, p) \right) \\ &\qquad + a_3 p_b(x_n, x_{n+1}) + \left(\tfrac{a_4}{s} p_b(q, x_{n+1}) - a_4 p_b(x_n, x_{n+1}) + \tfrac{a_4}{s} p_b(x_n, x_n) \right) \\ &\qquad + a_5 p_b(x_{n+1}, p) \\ &\quad \geq (a_1 + \tfrac{a_2 + a_4}{s}) p_b(q, x_{n+1}) + (a_5 - a_2) p_b(x_{n+1}, p) + (a_3 - a_4) p_b(x_n, x_{n+1}). \end{aligned}$$

Taking the limit as $n \to \infty$ yields $0 \geq (a_1 + \frac{a_2 + a_4}{s}) p_b(q, p)$. Since $a_1 + \frac{a_2 + a_4}{s} \geq 0$, we have $p_b(q, p) \leq 0$. But $p_b(q, p) \geq 0$. Hence $q = p$. That is $q = p = Tq$. This gives that p is the fixed point of T. Now we suppose that $a_1 + a_4 + a_5 > 1$ and let p and v are fixed points of T, then

$$\begin{aligned} p_b(p, v) = p_b(Tp, Tv) &\geq a_1 p_b(p, v) + a_2 p_b(p, Tp) + a_3 p_b(v, Tv) \\ &\quad + a_4 p_b(p, Tv) + a_5 p_b(v, Tp), \\ &= (a_1 + a_4 + a_5) p_b(p, v), \end{aligned}$$

which implies that

$$p_b(p, v) \leq \tfrac{1}{a_1 + a_4 + a_5} p_b(p, v).$$

Since $a_1 + a_4 + a_5 > 1$, we have $p_b(p, v) = 0$ i.e., $p = v$. Therefore, T has a unique fixed point in X. □

Corollary 3.2. *Let (X, p_b) be a complete partial b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a surjection. Suppose that there exists $a_1, a_2, a_3 \geq 0$ with $a_1 + a_3 > s(1 - a_2)$ such that*

$$p_b(Tx, Ty) \geq a_1 p_b(x, y) + a_2 p_b(x, Tx) + a_3 p_b(y, Ty) , \quad x, y \in X, x \neq y. \tag{3.6}$$

Then T has a fixed point in X. Moreover, if $a_1 > 1$, then the fixed point is unique.

Proof. It follows by taking $a_4 = a_5 = 0$ in Theorem 3.1. □

Corollary 3.3. *Let (X, p_b) be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ is onto and satisfies the condition*

$$p_b(Tx, Ty) \geq k p_b(x, y)$$

for all $x, y \in X$, where $k > s$ is a constant. Then T has a unique fixed point in X.

Proof. It follows by taking $a_2 = a_3 = 0$ in Corollary 3.2. □

3.2 Coincidence Point Theorems

We prove a theorem on the coincidence point of two expansive type mappings in the partial b-metric spaces in which the surjectivity condition of the maps is not assumed.

Theorem 3.4. *Let (X, p_b) be a partial b-metric space with coefficient $s \geq 1$. Let $T, S : X \to X$ be mappings satisfying:*

$$\begin{aligned} p_b(Tx, Ty) &\geq a_1 p_b(Sx, Sy) + a_2 p_b(Sx, Tx) + a_3 p_b(Sy, Ty) \\ &\quad + a_4 p_b(Sx, Ty) + a_5 p_b(Sy, Tx), \end{aligned} \tag{3.7}$$

for all $x, y \in X$ where $a_1, a_2, a_3, a_4, a_5 \geq 0$ which are not all zero. Suppose the following hypotheses are also satisfy: (1) $a_1 + a_3 > s(1 - a_2) \geq a_5$ or $a_1 + a_2 > s(1 - a_3) \geq a_4$; (2) $S(X) \subseteq T(X)$ and (3) $T(X)$ is a complete subspace of X. Then T and S have a coincidence point. Moreover, if $a_1 + a_4 + a_5 > 1$, then the point of coincidence is unique.

Proof. Let $x_0 \in X$ be chosen. We choose $x_1 = Sx_0$ and $x_2 = Tx_1$. Since $S(X) \subseteq T(X)$ and $x_1 \neq x_2$ then there exists a sequence $\{x_n\}$ such that $Sx_n = Tx_{n+1} (n \geq 2)$. Without loss of generality, we claim that $x_{n-1} \neq x_n$ for $n \geq 1$. From (3.7) with $x = x_n$ and $y = x_{n+1}$ we have the following.

Case (i)

$$\begin{aligned}
p_b(Sx_{n-1}, Sx_n) &= p_b(Tx_n, Tx_{n+1}) \\
&\geq a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_n, Sx_{n-1}) + a_3 p_b(Sx_{n+1}, Sx_n) \\
&\quad + a_4 p_b(Sx_n, Sx_n) + a_5 p_b(Sx_{n+1}, Sx_{n-1}) \\
&\geq a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_{n-1}, Sx_n) + a_3 p_b(Sx_n, Sx_{n+1}) \\
&\quad + a_4 p_b(Sx_n, Sx_n) + \tfrac{a_5}{s} p_b(Sx_{n-1}, Sx_n) - a_5 p_b(Sx_n, Sx_{n+1}) \\
&\qquad + \tfrac{a_5}{s} p_b(Sx_{n+1}, Sx_{n+1}) \\
&\geq a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_{n-1}, Sx_n) + a_3 p_b(Sx_n, Sx_{n+1}) \\
&\quad + \tfrac{a_5}{s} p_b(Sx_{n-1}, Sx_n) - a_5 p_b(Sx_n, Sx_{n+1}) \\
&\geq (a_1 + a_3 - a_5) p_b(Sx_n, Sx_{n+1}) + (a_2 + \tfrac{a_5}{s}) p_b(Sx_{n-1}, Sx_n).
\end{aligned} \tag{3.8}$$

If $a_1 + a_3 > s(1 - a_2) \geq a_5$, then (3.8) becomes

$$\begin{aligned}
(1 - a_2 - \tfrac{a_5}{s}) p_b(Sx_{n-1}, Sx_n) &\geq (a_1 + a_3 - a_5) p_b(Sx_n, Sx_{n+1}), \\
p_b(Sx_n, Sx_{n+1}) &\leq \tfrac{1 - a_2 - (a_5/s)}{a_1 + a_3 - a_5} p_b(Sx_{n-1}, Sx_n).
\end{aligned} \tag{3.9}$$

Case (ii)

$$\begin{aligned}
p_b(Sx_n, Sx_{n-1}) &= p_b(Tx_{n+1}, Tx_n) \\
&\geq a_1 p_b(Sx_{n+1}, Sx_n) + a_2 p_b(Sx_{n+1}, Tx_{n+1}) + a_3 p_b(Sx_n, Tx_n) \\
&\quad + a_4 p_b(Sx_{n+1}, Tx_n) + a_5 p_b(Sx_n, Tx_{n+1}) \\
&= a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_{n+1}, Sx_n) + a_3 p_b(Sx_n, Sx_{n-1}) \\
&\quad + a_4 p_b(Sx_{n+1}, Sx_{n-1}) + a_5 p_b(Sx_n, Sx_n) \\
&\geq a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_n, Sx_{n+1}) + a_3 p_b(Sx_{n-1}, Sx_n) \\
&\quad + \tfrac{a_4}{s} p_b(Sx_{n-1}, Sx_n) - a_4 p_b(Sx_n, Sx_{n+1}) \\
&\qquad + \tfrac{a_4}{s} p_b(Sx_{n+1}, Sx_{n+1}) + a_5 p_b(Sx_n, Sx_n) \\
&\geq a_1 p_b(Sx_n, Sx_{n+1}) + a_2 p_b(Sx_n, Sx_{n+1}) + a_3 p_b(Sx_{n-1}, Sx_n) \\
&\quad + \tfrac{a_4}{s} p_b(Sx_{n-1}, Sx_n) - a_4 p_b(Sx_n, Sx_{n+1}) \\
&= (a_1 + a_2 - a_4) p_b(Sx_n, Sx_{n+1}) + (a_3 + \tfrac{a_4}{s}) p_b(Sx_{n-1}, Sx_n).
\end{aligned}$$

If $a_1 + a_2 > s(1 - a_3) \geq a_4$, then the above inequality becomes

$$\begin{aligned}
(1 - a_3 - \tfrac{a_4}{s}) p_b(Sx_{n-1}, Sx_n) &\geq (a_1 + a_2 - a_4) p_b(Sx_n, Sx_{n+1}), \\
p_b(Sx_n, Sx_{n+1}) &\leq \tfrac{1 - a_3 - (a_4/s)}{a_1 + a_2 - a_4} p_b(Sx_{n-1}, Sx_n).
\end{aligned} \tag{3.10}$$

In both cases, we put $k := \frac{1-a_2-(a_5/s)}{a_1+a_3-a_5}$ in (3.9) and $k := \frac{1-a_3-(a_4/s)}{a_1+a_2-a_4}$ in (3.10). Thus in both cases, we have $k < 1$. Hence $p_b(Sx_n, Sx_{n+1}) \leq kp_b(Sx_{n-1}, Sx_n)$ for all $n \in \mathbb{N}$. Consequently, we have

$$p_b(Sx_n, Sx_{n+1}) \leq k^n p_b(Sx_0, Sx_1) \; ; \quad \text{for all } n \in \mathbb{N}.$$

For $n > m$, we obtain

$$\begin{aligned} p_b(Sx_m, Sx_n) &\leq sp_b(Sx_m, Sx_{m+1}) + s^2 p_b(Sx_{m+1}, Sx_{m+2}) + ... + s^n p_b(Sx_{n-1}, Sx_n) \\ &\quad - p_b(Sx_{m+1}, Sx_{m+1}) - \cdots - p_b(Sx_{n-1}, Sx_{n-1}) \\ &\leq (sk^m + s^2 k^{m+1} + \cdots + s^n k^{n-1}) p_b(Sx_0, Sx_1) \\ &\leq \frac{sk^m}{1-sk} p_b(Sx_0, Sx_1). \end{aligned}$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since $T(X)$ is a complete subspace of X, there exists a point $z \in X$ such that $Tx_n \to Tz$ as $n \to \infty$. Likewise, $Sx_n \to Tz$ as $n \to \infty$. Also $p_b(Tz, Tz) = \lim_{n\to\infty} p_b(Tx_n, Tz) = \lim_{n,m\to\infty} p_b(Tx_m, Tx_n) = 0$. Since a_1, a_2, a_3, a_4, a_5 are not all zero, from (3.7) we obtain the following cases:

$$\begin{aligned} &\text{if } a_1 \neq 0, \; p_b(Tx_n, Tz) \geq a_1 p_b(Sx_n, Sz); \\ &\text{if } a_2 \neq 0, \; p_b(Tz, Tx_n) \geq a_2 p_b(Sz, Tz); \\ &\text{if } a_3 \neq 0, \; p_b(Tx_n, Tz) \geq a_3 p_b(Sz, Tz); \\ &\text{if } a_4 \neq 0, \; p_b(Tz, Tx_n) \geq a_4 p_b(Sz, Tx_n); \\ &\text{if } a_5 \neq 0, \; p_b(Tx_n, Tz) \geq a_5 p_b(Sz, Tx_n). \end{aligned}$$

In all cases, as $n \to \infty$, we have $p_b(Tz, Tz) \geq a_i p_b(Tz, Sz)$, $i = 1, 2, 3, 4, 5$. Thus $p_b(Tz, Sz) \leq 0$. But $p_b(Tz, Sz) \geq 0$. Therefore $p_b(Tz, Sz) = 0$, which implies that $Tz = Sz$. Thus S and T have coincidence point which is z.

Now we suppose that $a_1 + a_4 + a_5 > 1$. Let v, w are points of coincidence of T and S. So $Tx = Sx = v, Ty = Sy = w$ for some $x, y \in X$. Then

$$\begin{aligned} p_b(v, w) = p_b(Tx, Ty) &\geq a_1 p_b(Sx, Sy) + a_2 p_b(Sx, Tx) + a_3 p_b(Sy, Ty) \\ &\quad + a_4 p_b(Sx, Ty) + a_5 p_b(Sy, Tx) \\ &= a_1 p_b(v, w) + a_2 p_b(v, v) + a_3 p_b(w, w) \\ &\quad + a_4 p_b(v, w) + a_5 p_b(w, v) \\ &= (a_1 + a_4 + a_5) p_b(v, w). \end{aligned}$$

which implies that

$$p_b(v, w) \leq \frac{1}{a_1 + a_4 + a_5} p_b(v, w).$$

Since $a_1 + a_4 + a_5 > 1$, we have $p_b(v, w) = 0$ i.e., $v = w$. Therefore, T and S have a unique point of coincidence in X. □

Corollary 3.5. *Let (X, p_b) be a partial b-metric space with coefficient $s \geq 1$. Let $T, S : X \to X$ be mappings satisfying:*

$$p_b(Tx, Ty) \;\geq\; a_1 p_b(Sx, Sy) + a_2 p_b(Sx, Tx) + a_3 p_b(Sy, Ty) \tag{3.11}$$

for all $x, y \in X$ where $a_1, a_2, a_3 \geq 0$ which are not all zero. Suppose the following hypotheses are also satisfy: (1) $a_1 + a_3 > s(1 - a_2)$ or $a_1 + a_2 > s(1 - a_3)$, (2) $S(X) \subseteq T(X)$ and (3) $T(X)$ is a complete subspace of X. Then T and S have a coincidence point. Moreover, if $a_1 > 1$, then the point of coincidence is unique.

Proof. It follows by taking $a_4 = a_5 = 0$ in Theorem 3.4. □

Corollary 3.6. *Let (X, p_b) be a partial b-metric space with coefficient $s \geq 1$. Let $T, S : X \to X$ be mappings satisfying:*

$$p_b(Tx, Ty) \;\geq\; a_1 p_b(Sx, Sy), \tag{3.12}$$

for all $x, y \in X$ where $a_1 > s$. Suppose the following hypotheses are also satisfy: (1) $S(X) \subseteq T(X)$ and (2) $T(X)$ is a complete subspace of X. Then T and S have a unique coincidence point.

Proof. It follows by taking $a_2 = a_3 = 0$ in Corollary 3.5. □

The following Corollary is the partial b-metric version of Banach contraction principle.

Corollary 3.7. *Let (X, p_b) be a partial b-metric space with coefficient $s \geq 1$. Let $S : X \to X$ be mapping satisfying:*

$$p_b(Sx, Sy) \;\leq\; k p_b(x, y), \tag{3.13}$$

for all $x, y \in X$ where $k \in (0, \frac{1}{s})$. Then S has a unique fixed point in X. Furthermore, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Setting $k := 1/a_1$ and $T = I$, the identity mapping on X, in Corollary 3.6. □

Example 3.8. Let $X = \mathbb{R}^+$ and $p_b : X \times X \to [0, \infty)$ defined by $p_b(x, y) = \{\max\{x, y\}\}^2$ for all $x, y \in X$. Then (X, p_b) is a complete partial b-metric space with $s = 2$. Define $T, S : X \to X$ by $Tx = \frac{x}{2}$ and $Sx = \frac{x}{5}$ for all $x \in X$. Then for every $x, y \in X$ we have $p_b(Tx, Ty) \geq 6 p_b(Sx, Sy)$ i.e. the condition (3.7) holds for $a_1 = 6, a_2 = a_3 = a_4 = a_5 = 0$. Therefore we have all the hypothesis of Theorem 3.4 satisfied and 0 is the coincidence point of T and S.

3.3 Common Fixed Point Theorems

In the next theorem, we prove the existence of the common fixed point for a pair of weakly compatible maps satisfying certain conditions in partial b-metric spaces in which the surjectivity of the two maps is assumed.

Theorem 3.9. *Let T and S be two weakly compatible and surjective self mappings of a complete partial b-metric space (X, p_b) satisfying the following conditions: for any $x, y \in X$ and $a_1 + a_3 > s(1 - a_2) \geq a_5, a_1 + a_2 > s(1 - a_3) \geq a_4$ we have that*

$$\begin{aligned} p_b(Tx, Sy) \ &\geq \ a_1 p_b(x, y) + a_2 p_b(x, Tx) + a_3 p_b(y, Sy) \\ &\quad + a_4 p_b(x, Sy) + a_5 p_b(y, Tx). \end{aligned} \tag{3.14}$$

If S and T are compatible pair of reciprocal continuous maps, then S and T have a common fixed point in X. Moreover, if $a_1 + a_4 + a_5 > 1$, then the common fixed point is unique.

Proof. Let $x_0 \in X$ be chosen. Since T and S are surjective then there exist $x_1, x_2 \in X$ such that $x_0 = Tx_1$ and $x_1 = Sx_2$. Continuing the process, we can define a sequence $\{x_n\} \in X$ such that $x_{2n} = Tx_{2n+1},\ x_{2n+1} = Sx_{2n+2}$. Using (3.14), we have

$$\begin{aligned} p_b(x_{2n}, x_{2n+1}) \ &= \ p_b(Tx_{2n+1}, Sx_{2n+2}) \\ &\geq \ a_1 p_b(x_{2n+1}, x_{2n+2}) + a_2 p_b(x_{2n+1}, Tx_{2n+1}) + a_3 p_b(x_{2n+2}, Sx_{2n+2}) \\ &\quad + a_4 p_b(x_{2n+1}, Sx_{2n+2}) + a_5 p_b(x_{2n+2}, Tx_{2n+1}) \\ &= \ a_1 p_b(x_{2n+1}, x_{2n+2}) + a_2 p_b(x_{2n+1}, x_{2n}) + a_3 p_b(x_{2n+2}, x_{2n+1}) \\ &\quad + a_4 p_b(x_{2n+1}, x_{2n+1}) + a_5 p_b(x_{2n+2}, x_{2n}). \end{aligned}$$

By (p_b4), the above inequality becomes

$$\begin{aligned} p_b(x_{2n}, x_{2n+1}) &\geq a_1 p_b(x_{2n+1}, x_{2n+2}) + a_2 p_b(x_{2n}, x_{2n+1}) + a_3 p_b(x_{2n+1}, x_{2n+2}) \\ &\quad + a_4 p_b(x_{2n+1}, x_{2n+1}) + \tfrac{a_5}{s} p_b(x_{2n}, x_{2n+1}) - a_5 p_b(x_{2n+1}, x_{2n+2}) \\ &\qquad + \tfrac{a_5}{s} p_b(x_{2n+2}, x_{2n+2}) \\ &\geq a_1 p_b(x_{2n+1}, x_{2n+2}) + a_2 p_b(x_{2n}, x_{2n+1}) + a_3 p_b(x_{2n+1}, x_{2n+2}) \\ &\quad + \tfrac{a_5}{s} p_b(x_{2n}, x_{2n+1}) - a_5 p_b(x_{2n+1}, x_{2n+2}) \\ &\geq (a_1 + a_3 - a_5) p_b(x_{2n+1}, x_{2n+2}) + (a_2 + \tfrac{a_5}{s}) p_b(x_{2n}, x_{2n+1}) \\ &\geq (a_1 + a_3 - a_5) p_b(x_{2n+1}, x_{2n+2}) + (a_2 + \tfrac{a_5}{s}) p_b(x_{2n}, x_{2n+1}). \end{aligned}$$

Therefore, we have $(1 - a_2 - \frac{a_5}{s}) p_b(x_{2n}, x_{2n+1}) \geq (a_1 + a_3 - a_5) p_b(x_{2n+1}, x_{2n+2})$ and

$$p_b(x_{2n+1}, x_{2n+2}) \ \leq \ \tfrac{1 - a_2 - (a_5/s)}{a_1 + a_3 - a_5} p_b(x_{2n}, x_{2n+1}).$$

Let $M := \frac{1 - a_2 - (a_5/s)}{a_1 + a_3 - a_5}$. Since $a_1 + a_3 > s(1 - a_2) \geq a_5$, we have $M \in [0, \frac{1}{s})$, and

$$p_b(x_{2n+1}, x_{2n+2}) \ \leq \ M p_b(x_{2n}, x_{2n+1}). \tag{3.15}$$

Similarly, we have

$$\begin{aligned}
p_b(x_{2n}, x_{2n-1}) &= p_b(Tx_{2n+1}, Sx_{2n}) \\
&\geq a_1 p_b(x_{2n+1}, x_{2n}) + a_2 p_b(x_{2n+1}, Tx_{2n+1}) + a_3 p_b(x_{2n}, Sx_{2n}) \\
&\quad + a_4 p_b(x_{2n+1}, Sx_{2n}) + a_5 p_b(x_{2n}, Tx_{2n+1}) \\
&= a_1 p_b(x_{2n}, x_{2n+1}) + a_2 p_b(x_{2n+1}, x_{2n}) + a_3 p_b(x_{2n}, x_{2n-1}) \\
&\quad + a_4 p_b(x_{2n+1}, x_{2n-1}) + a_5 p_b(x_{2n}, x_{2n}).
\end{aligned}$$

By (p_b4), the above inequality becomes

$$\begin{aligned}
p_b(x_{2n-1}, x_{2n}) &\geq a_1 p_b(x_{2n}, x_{2n+1}) + a_2 p_b(x_{2n}, x_{2n+1}) + a_3 p_b(x_{2n-1}, x_{2n}) \\
&\quad + \tfrac{a_4}{s} p_b(x_{2n-1}, x_{2n}) - a_4 p_b(x_{2n}, x_{2n+1}) \\
&\qquad + a_4 p_b(x_{2n+1}, x_{2n+1}) + a_5 p_b(x_{2n}, x_{2n}) \\
&\geq a_1 p_b(x_{2n}, x_{2n+1}) + a_2 p_b(x_{2n}, x_{2n+1}) + a_3 p_b(x_{2n-1}, x_{2n}) \\
&\quad + \tfrac{a_4}{s} p_b(x_{2n-1}, x_{2n}) - a_4 p_b(x_{2n}, x_{2n+1}) \\
&\geq (a_1 + a_2 - a_4) p_b(x_{2n}, x_{2n+1}) + (a_3 + \tfrac{a_4}{s}) p_b(x_{2n-1}, x_{2n}).
\end{aligned}$$

Therefore, $(1 - a_3 - \frac{a_4}{s}) p_b(x_{2n-1}, x_{2n}) \geq (a_1 + a_2 - a_4) p_b(x_{2n}, x_{2n+1})$ and

$$p_b(x_{2n}, x_{2n+1}) \leq \frac{1 - a_3 - (a_4/s)}{a_1 + a_2 - a_4} p_b(x_{2n-1}, x_{2n}).$$

Let $L := \frac{1-a_3-(a_4/s)}{a_1+a_2-a_4}$. Since $a_1 + a_2 > s(1 - a_3) \geq a_4$, we have that $L \in [0, \frac{1}{s})$ and

$$p_b(x_{2n}, x_{2n+1}) \leq L p_b(x_{2n-1}, x_{2n}). \tag{3.16}$$

Let $\lambda := ML \in [0, \frac{1}{s^2})$. Then by induction, we have

$$\begin{aligned}
p_b(x_{2n+1}, x_{2n+2}) &\leq M p_b(x_{2n}, x_{2n+1}) \\
&\leq M\big(L p_b(x_{2n-1}, x_{2n})\big) \\
&\leq M\lambda p_b(x_{2n-2}, x_{2n-1}) \\
&\quad \vdots \\
&\leq M\lambda^n p_b(x_0, x_1)
\end{aligned}$$

and

$$\begin{aligned}
p_b(x_{2n}, x_{2n+1}) &\leq L p_b(x_{2n-1}, x_{2n}) \\
&\leq \lambda p_b(x_{2n-2}, x_{2n-1}) \\
&\quad \vdots \\
&\leq \lambda^n p_b(x_0, x_1).
\end{aligned}$$

For $n > m$, we get

$$\begin{aligned}
p_b(x_{2m+1}, x_{2n+1}) &\leq sp_b(x_{2m+1}, x_{2m+2}) + s^2 p_b(x_{2m+2}, x_{2m+3}) + \\
&\quad \cdots + s^{2(n-m)} p_b(x_{2n}, x_{2n+1}) - p_b(x_{2m+2}, x_{2m+2}) \\
&\qquad - p_b(x_{m+3}, x_{m+3}) - \cdots - p_b(x_{2n}, x_{2n}) \\
&\leq sp_b(x_{2m+1}, x_{2m+2}) + s^2 p_b(x_{2m+2}, x_{2m+3}) + \\
&\quad \cdots + s^{2(n-m)} p_b(x_{2n}, x_{2n+1}) \\
&\leq sM\lambda^m p_b(x_0, x_1) + s^2 \lambda^{m+1} p_b(x_0, x_1) + s^3 M \lambda^{m+1} p_b(x_0, x_1) \\
&\quad + s^4 \lambda^{m+2} p_b(x_0, x_1) + \cdots + s^{2(n-m)} \lambda^n p_b(x_0, x_1) \\
&\leq \Big(sM\lambda^m \big(1 + s^2\lambda + \ldots\big) + s^2 \lambda^{m+1} \big(1 + s^2 \lambda + \ldots\big)\Big) p_b(x_0, x_1) \\
&\leq \left(\frac{sM\lambda^m + s^2\lambda^{m+1}}{1 - s^2\lambda}\right) p_b(x_0, x_1) \\
&\leq (sL + 1)\Big(\frac{sM\lambda^m}{1 - s^2\lambda}\Big) p_b(x_0, x_1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
p_b(x_{2m}, x_{2n+1}) &\leq sp_b(x_{2m}, x_{2m+1}) + s^2 p_b(x_{2m+1}, x_{2m+2}) + \\
&\quad \cdots + s^{2(n-m)+1} p_b(x_{2n}, x_{2n+1}) - p_b(x_{2m+1}, x_{2m+1}) \\
&\qquad - p_b(x_{m+2}, x_{m+2}) - \cdots - p_b(x_{2n}, x_{2n}) \\
&\leq sp_b(x_{2m}, x_{2m+1}) + s^2 p_b(x_{2m+1}, x_{2m+2}) + \\
&\quad \cdots + s^{2(n-m)+1} p_b(x_{2n}, x_{2n+1}) \\
&\leq \Big(\big(s\lambda^m + s^3\lambda^{m+1} + ...\big) + \big(s^2 M\lambda^m + s^4 M \lambda^{m+1} + ...\big)\Big) p_b(x_0, x_1) \\
&\leq (sM + 1)\left(\frac{s\lambda^m}{1 - s^2\lambda}\right) p_b(x_0, x_1).
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. It is equivalent to $x_{2n} = Tx_{2n+1} \to z, x_{2n+1} = Sx_{2n+2} \to z$ as $n \to \infty$. Also

$$p_b(z, z) = \lim_{n\to\infty} p_b(x_n, z) = \lim_{n,m\to\infty} p_b(x_n, x_m) = 0.$$

Suppose T and S are compatible and reciprocal continuous. By reciprocal continuity of T and S, $\lim_{n\to\infty} TSx_n = Tz$ and $\lim_{n\to\infty} STx_n = Sz$. By compatibility of T and S, $Tz = Sz$. Since T and S are weakly compatible, $Tz = Sz$ implies $TTz = TSz = STz = SSz$.

Next we show that z is a common fixed point of S and T. From (3.14) we have

$$\begin{aligned}
p_b(Tz, Sx_{2n+2}) &\geq a_1 p_b(z, x_{2n+2}) + a_2 p_b(z, Tz) + a_3 p_b(x_{2n+2}, Sx_{2n+2}) \\
&\quad + a_4 p_b(z, Sx_{2n+2}) + a_5 p_b(x_{2n+2}, Tz).
\end{aligned}$$

From the following three inequalities,

$$\begin{aligned} p_b(z,Tz) &\geq \tfrac{1}{s}p_b(z,x_{2n+2}) - p_b(x_{2n+2},Tz) + \tfrac{1}{s}p_b(Tz,Tz), \\ p_b(z,x_{2n+1}) &\geq \tfrac{1}{s}p_b(z,x_{2n+2}) - p_b(x_{2n+2},x_{2n+1}) + \tfrac{1}{s}p_b(x_{2n+1},x_{2n+1}) \end{aligned}$$

and

$$p_b(Tz,x_{2n+1}) \leq sp_b(Tz,x_{2n+2}) + sp_b(x_{2n+2},x_{2n+1}) - p_b(x_{2n+2},x_{2n+2}),$$

we obtain

$$\begin{aligned} & sp_b(Tz,x_{2n+2}) + sp_b(x_{2n+1},x_{2n+2}) - p_b(x_{2n+2},x_{2n+2}) \\ &\quad \geq a_1p_b(z,x_{2n+2}) + \tfrac{a_2}{s}p_b(z,x_{2n+2}) - a_2p_b(x_{2n+2},Tz) + \tfrac{a_2}{s}p_b(Tz,Tz) \\ &\qquad + a_3p_b(x_{2n+2},x_{2n+1}) + \tfrac{a_4}{s}p_b(z,x_{2n+2}) - a_4p_b(x_{2n+1},x_{2n+2}) \\ &\qquad + \tfrac{a_4}{s}p_b(x_{2n+1},x_{2n+1}) + a_5p_b(x_{2n+2},Tz) \\ &\quad \geq (a_1 + \tfrac{a_2}{s} + \tfrac{a_4}{s})p_b(z,x_{2n+2}) + (a_3 - a_4)p_b(x_{2n+1},x_{2n+2}) \\ &\qquad + (a_5 - a_2)p_b(Tz,x_{2n+2}). \end{aligned}$$

Therefore, we have

$$(a_5-a_2-1)p_b(Tz,x_{2n+2}) \leq (1-a_3+a_4)p_b(x_{n+2},x_{2n+1})-(a_1+\tfrac{a_2}{s}+\tfrac{a_4}{s})p_b(z,x_{2n+2}),$$

or

$$p_b(Tz,x_{2n+2}) \leq \frac{1-a_3+a_4}{a_5-a_2-1}p_b(x_{n+2},x_{2n+1}) - \frac{a_1+(a_2/s)+(a_4/s)}{a_5-a_2-1}p_b(z,x_{2n+2}).$$

As $n \to \infty$, we get $p_b(Tz,z) \leq 0$. $p_b(Tz,z) \geq 0$ implies that $p_b(Tz,z) = 0$. Therefore $Tz = Sz = z$. Suppose there exists $u \in X$ such that u is another common fixed point of T and S then we show that $u = z$. On the contrary, letting $u \neq z$ and using (3.14) we have

$$\begin{aligned} p_b(u,z) &= p_b(Tu,Sz) \\ &\geq a_1p_b(u,z) + a_2p_b(u,Tu) + a_3p_b(z,Sz) + a_4p_b(u,Sz) + a_5p_b(z,Tu) \\ &= a_1p_b(u,z) + a_4p_b(u,z) + a_5p_b(z,u) \\ &= (a_1 + a_4 + a_5)p_b(u,z). \end{aligned}$$

Since $a_1 + a_4 + a_5 > 1$, then we have $p_b(u,z) = 0$ i.e., $u = z$. The uniqueness of fixed points is proved. □

Corollary 3.10. *Let (X,p_b) be a complete partial b-metric space. Suppose mappings $T,S : X \to X$ are onto, compatible, reciprocally continuous, and satisfy*

$$p_b(Tx,Sy) \geq a_1p_b(x,y) + a_2p_b(x,Tx) + a_3p_b(y,Sy) \tag{3.17}$$

for all $x,y \in X$, with $a_1 + a_3 > s(1 - a_2), a_1 + a_2 > s(1 - a_3)$ and $a_1 > 1$. Then S and T have a unique common fixed point.

Corollary 3.11. *Let (X, p_b) be a complete partial b-metric space. Suppose mappings $T, S : X \to X$ are onto, compatible, reciprocally continuous, and satisfy*

$$p_b(Tx, Sy) \;\geq\; a_1 p_b(x, y) \tag{3.18}$$

for all $x, y \in X$, with $a_1 > s$. Then S and T have a unique common fixed point.

Example 3.12. Let $X = \mathbb{R}^+$ and $p_b(x,y) = \big\{\max\{x,y\}\big\}^2$; then (X, p_b) is a complete partial b-metric space. Let $T, S : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx \;=\; Sx \;=\; \tfrac{3}{2}x\sqrt{1 + \tfrac{1}{1+x^2}}\,, \quad \text{for all } x \in X.$$

Then T and S are surjective, reciprocally continuous and compatible. Without loss of generality, we assume that $x \leq y$.

$$\begin{aligned} p_b(Tx, Sy) \;&=\; \left(\max\left\{\tfrac{3}{2}x\sqrt{1+\tfrac{1}{1+x^2}}, \tfrac{3}{2}y\sqrt{1+\tfrac{1}{1+y^2}}\right\}\right)^2 = \left(\tfrac{3}{2}y\sqrt{1+\tfrac{1}{1+y^2}}\right)^2 \\ &=\; \tfrac{9}{4}\left(y^2 + \tfrac{y^2}{1+y^2}\right) \;\geq\; 2\left(y^2 + \tfrac{y^2}{1+y^2}\right) \;=\; 2p_b(x,y). \end{aligned}$$

Also T and S satisfy the inequality of Theorem 3.9 with $a_1 = 2$ and $a_2 = a_3 = a_4 = a_5 = 0$. Hence T and S have a unique common fixed point 0 in X.

Theorem 3.13. *Let T and S be two continuous and surjective self mappings of a complete partial b-metric space (X, p_b) satisfying the following conditions: for any $x \in X$ and k, c_1, c_2 are nonnegative real numbers with $c_1, c_2 > s(k+1) + k$ we have that*

$$p_b(TSx, Sx) + \tfrac{k}{s}p_b(TSx, x) \;\geq\; c_1 p_b(Sx, x) \tag{3.19}$$

and

$$p_b(STx, Tx) + \tfrac{k}{s}p_b(STx, x) \;\geq\; c_2 p_b(Tx, x). \tag{3.20}$$

Then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be chosen. Since T and S are surjective then there exist $x_1, x_2 \in X$ such that $x_0 = Tx_1$ and $x_1 = Sx_2$. Continuing the process, we can define a sequence $\{x_n\} \in X$ such that $x_{2n} = Tx_{2n+1},\ x_{2n+1} = Sx_{2n+2}$. Using (3.19), we have

$$p_b(TSx_{2n+2}, Sx_{2n+2}) + \tfrac{k}{s}p_b(TSx_{2n+2}, x_{2n+2}) \;\geq\; c_1 p_b(Sx_{2n+2}, x_{2n+2}),$$

which implies that

$$p_b(x_{2n}, x_{2n+1}) + \tfrac{k}{s}p_b(x_{2n}, x_{2n+2}) \;\geq\; c_1 p_b(x_{2n+1}, x_{2n+2}).$$

Hence, we have

$$\begin{aligned} c_1 p_b(x_{2n+1}, x_{2n+2}) \;&\leq\; p_b(x_{2n}, x_{2n+1}) + kp_b(x_{2n}, x_{2n+1}) + kp_b(x_{2n+1}, x_{2n+2}) \\ &\quad - \tfrac{k}{s}p_b(x_{2n+1}, x_{2n+1}) \\ &\leq\; p_b(x_{2n}, x_{2n+1}) + kp_b(x_{2n}, x_{2n+1}) + kp_b(x_{2n+1}, x_{2n+2}) \end{aligned}$$

Therefore,

$$p_b(x_{2n+1}, x_{2n+2}) \ \leq \ \tfrac{1+k}{c_1-k} p_b(x_{2n}, x_{2n+1}). \tag{3.21}$$

Similarly, by using (3.20), we obtain that

$$p_b(x_{2n}, x_{2n+1}) \ \leq \ \tfrac{1+k}{c_2-k} p_b(x_{2n-1}, x_{2n}). \tag{3.22}$$

Let $\lambda := \max\{\frac{1+k}{c_1-k}, \frac{1+k}{c_2-k}\}$. From $c_1, c_2 > s(k+1)+k$, we obtain $\lambda \in (0, \frac{1}{s})$. Combining (3.21) and (3.22), we get

$$p_b(x_n, x_{n+1}) \ \leq \ \lambda p_b(x_{n-1}, x_n) \ , \quad \text{for all } n \in \mathbb{N}.$$

By an argument similar to that used in Theorem 3.1, it follows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Also, $x_{2n+1} \to p$ and $x_{2n+2} \to p$ as $n \to \infty$. The continuity of S and T imply that $Tx_{2n+1} \to Tp$ and $Sx_{2n+2} \to Sp$ as $n \to \infty$ i.e., $x_{2n} \to Tp$ and $x_{2n+1} \to Sp$ as $n \to \infty$. The uniqueness of limit yields that $p = Sp = Tp$. Hence, p is a common fixed point of S and T. □

Corollary 3.14. *Let T be a continuous surjective mapping of a complete partial b-metric space (X, p_b) satisfying the following conditions: for any $x \in X$ and k, C are nonnegative real numbers with $C > s(k+1)+k$ we have*

$$p_b(T^2x, Tx) + \tfrac{k}{s} p_b(T^2x, x) \ \geq \ C p_b(Tx, x). \tag{3.23}$$

Then T has a fixed point in X.

Proof. It follows from Theorem 3.13 by taking $S = T$ and set $C := c_1 = c_2$. □

Corollary 3.15. *Let T be a continuous surjective mapping of a complete partial b-metric space (X, p_b) satisfying the following conditions: for any $x \in X$ and k, C are nonnegative real numbers with $C > s$ we have*

$$p_b(T^2x, Tx) \ \geq \ C p_b(Tx, x). \tag{3.24}$$

Then T has a fixed point in X.

Proof. It follows from Corollary 3.14 by taking $k = 0$. □

Example 3.16. Let $X = \mathbb{R}^+$ and $p_b(x, y) = \big\{\max\{x, y\}\big\}^2$; then (X, p_b) is a complete partial b-metric space with $s = 2$. Let $T, S : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = 2x$ and $Sx = 3x$. T and S are surjective, reciprocally continuous, compatible. For all $x \in X$,

$$\big\{\max\{6x, 3x\}\big\}^2 + \tfrac{1}{2}\big\{\max\{6x, x\}\big\}^2 = 54x^2 \ \geq \ 54x^2 = 6\big\{\max\{3x, x\}\big\}^2$$

and

$$\big\{\max\{6x, 2x\}\big\}^2 + \tfrac{1}{2}\big\{\max\{6x, x\}\big\}^2 = 54x^2 \ \geq \ 24x^2 = 6\big\{\max\{2x, x\}\big\}^2.$$

Also satisfy the inequalities of Theorem 3.13 with $c_1 = c_2 = 6 > s(k+1)+k$, where $k = 1$. Then T and S have a common fixed point 0 in X.

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