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Common Endpoints for Suzuki Mappings in Uniformly Convex Hyperbolic Spaces

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Abstract : The existence of endpoints for Suzuki mappings in uniformly convex hyperbolic spaces is proved. A common endpoint theorem for a commuting pair of single-valued and multivalued Suzuki mappings is also established.

Keywords : endpoint; fixed point; Suzuki mapping; commuting mapping; uniformly convex hyperbolic space.
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1 Introduction

Let (X, d) be a metric space. The *distance* from an element x in X to a nonempty subset E of X is defined by

$$\text{dist}(x, E) := \inf\{d(x, y) : y \in E\}.$$

We shall denote by $CB(E)$ the family of nonempty closed bounded subsets of E and by $K(E)$ the family of nonempty compact subsets of E. The *Pompeiu-*

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Hausdorff distance on $CB(E)$ is defined by

$$H(A,B) := \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\} \text{ for all } A,B \in CB(E).$$

A multivalued mapping $T : E \to CB(E)$ is said to be *contractive* if there exists a constant $k \in [0,1)$ such that

$$H(T(x),T(y)) \leq kd(x,y) \text{ for all } x,y \in E. \tag{1.1}$$

If (1.1) is valid when $k = 1$, then T is said to be *nonexpansive*. It is clear that every contractive mapping is nonexpansive and, in general, the converse is not true. The mapping T is called a *single-valued mapping* if $T(x)$ is a singleton for every x in E.

An element x in E is called a *fixed point* of T if $x \in T(x)$. Moreover, if $\{x\} = T(x)$, then x is called an *endpoint* of T. We denote by $Fix(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. It is clear that $End(T) \subseteq Fix(T)$ for every multivalued mapping T and $End(t) = Fix(t)$ for every single-valued mapping t.

The existence of endpoints for a special kind of contractive mappings was first studied by Aubin and Siegel [1] in 1980. They proved that every multivalued dissipative mapping on a complete metric space always has an endpoint. In 1986, Corley [2] proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multivalued mapping. Since then the endpoint results for several kinds of contractive mappings have been rapidly developed and many of papers have appeared (see, *e.g.*, [3–23]). Among other things, Panyanak [21] obtained the following result.

Theorem 1.1. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : E \to K(E)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Our purpose of this paper is threefold.

(i) To extend Theorem 1.1 from the class of nonexpansive mappings to a wider class of mappings, namely, the class of Suzuki mappings.

(ii) To extend Theorem 1.1 from the class of uniformly convex Banach spaces to a wider class of spaces, namely, the class of uniformly convex hyperbolic spaces.

(iii) To prove a common endpoint theorem for a commuting pair of single-valued and multivalued Suzuki mappings in uniformly convex hyperbolic spaces.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers.

Definition 2.1. [24] A *hyperbolic space* is a triple (X,d,W) where (X,d) is a metric space and $W : X \times X \times [0,1] \to X$ is a function such that for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$, we have

(W1) $d(z, W(x, y, \alpha)) \leq (1-\alpha)d(z, x) + \alpha d(z, y)$;

(W2) $d\left(W(x, y, \alpha), W(x, y, \beta)\right) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$.

If $x, y \in X$ and $\alpha \in [0,1]$, then we use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$. It is easy to see that for any $x, y \in X$ and $\alpha \in [0,1]$, one has

$$d(x, (1-\alpha)x \oplus \alpha y) = \alpha d(x, y) \text{ and } d(y, (1-\alpha)x \oplus \alpha y) = (1-\alpha)d(x, y).$$

We shall denote by $[x, y]$ the set $\{(1-\alpha)x \oplus \alpha y : \alpha \in [0,1]\}$. A nonempty subset C of X is said to be *convex* if $[x, y] \subseteq C$ for all $x, y \in C$.

Definition 2.2. [24] The hyperbolic space (X, d, W) is called *uniformly convex* if for any $r > 0$, and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq r\varepsilon$, it is the case that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *a modulus of uniform convexity.* The mapping δ is *monotone* (resp. *lower semi-continuous from the right*) if for every fixed ε it decreases (resp. is lower semi-continuous from the right) with respect to r.

Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces. CAT(0) spaces are also uniformly convex hyperbolic spaces, see [24, Proposition 8].

Definition 2.3. [25] Let E be a nonempty subset of a metric space (X, d). A multivalued mapping $T : E \to CB(E)$ is said to be *Suzuki* if for each $x, y \in E$,

$$\frac{1}{2}\operatorname{dist}(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq d(x, y).$$

The mapping T is said to satisfy *condition (E_μ)* if there exists $\mu \geq 1$ such that for each $x, y \in E$, we have

$$\operatorname{dist}(x, T(y)) \leq \mu \operatorname{dist}(x, T(x)) + d(x, y).$$

The mapping T is said to be *quasi-nonexpansive* if for each $x \in E$ and $y \in Fix(T)$, one has

$$H(T(x), T(y)) \leq d(x, y).$$

A single-valued mapping $t : E \to E$ and a multivalued mapping $T : E \to CB(E)$ are said to be *commuting* [26] if for $x, y \in E$ such that $x \in T(y)$, we have $t(x) \in T(t(y))$.

Definition 2.4. [27] Let E be a nonempty subset of a metric space (X, d) and $x \in X$. The *radius* of E relative to x is defined by

$$r_x(E) := \sup\{d(x, y) : y \in E\}.$$

The *diameter* of E is defined by

$$\operatorname{diam}(E) := \sup\{d(x, y) : x, y \in E\}.$$

The set E is said to be *bounded* if $\operatorname{diam}(E) < \infty$.

Definition 2.5. [12] Let $T : E \to CB(E)$ be a multivalued mapping. A sequence $\{x_n\}$ in E is called *an approximate fixed point sequence* (resp. *an approximate endpoint sequence*) for T if $\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0$ (resp. $\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0$). The mapping T is said to have *the approximate fixed point property* (resp. *the approximate endpoint property*) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in E.

Let E be a nonempty subset of a metric space (X, d) and $\{x_n\}$ be a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ relative to E is defined by

$$r(E, \{x_n\}) = \inf\big\{\limsup_{n\to\infty} d(x_n, x) : x \in E\big\}.$$

The *asymptotic center* of $\{x_n\}$ relative to E is defined by

$$A(E, \{x_n\}) = \big\{x \in E : \limsup_{n\to\infty} d(x_n, x) = r(E, \{x_n\})\big\}.$$

The sequence $\{x_n\}$ is called *regular* relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_k}\})$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence (see, *e.g.*, [27, p. 166]). The proof is metric in nature and carries over to the present setting without change.

Before proving our main results we collect some basic facts about uniformly convex hyperbolic spaces. From now on, X stands for a complete uniformly convex hyperbolic space with monotone (or lower semi-continuous from the right) modulus of uniform convexity.

Lemma 2.6. *The following statements hold:*

(i) [28, Theorem 2.6] *if E is a nonempty bounded closed convex subset of X and $t : E \to E$ is a single-valued Suzuki mapping, then $End(t)$ is nonempty closed and convex;*

(ii) [29, Proposition 2] *if E is a nonempty subset of X and $t : E \to E$ is a single-valued Suzuki mapping with $End(t) \neq \emptyset$, then t is a quasi-nonexpansive mapping;*

(iii) [28, Lemma 3.2] *if E is a nonempty closed convex subset of X and $T : E \to K(E)$ is a multivalued Suzuki mapping, then T satisfies condition (E_3);*

(iv) [30, Proposition 2.4] *if E is a nonempty closed convex subset of X and $x \in X$, then there exists a unique point $x_0 \in E$ such that*

$$d(x, x_0) = dist(x, E).$$

Lemma 2.7. [21, Proposition 2.4] *Let E be a nonempty subset of X, $\{x_n\}$ be a sequence in E, and $T : E \to K(E)$ be a multivalued mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and only if dist$(x_n, T(x_n)) \to 0$ and diam$(T(x_n)) \to 0$.*

3 Main Results

We begin this section by proving the existence of endpoints for multivalued Suzuki mappings in uniformly convex hyperbolic spaces. Our proof follows the ideas of Panyanak [21] and Abkar and Eslamian [31].

Theorem 3.1. *Let E be a nonempty bounded closed convex subset of X and $T : E \to K(E)$ be a multivalued Suzuki mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof. It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property. Then there exists a sequence $\{x_n\}$ in E such that $r_{x_n}(T(x_n)) \to 0$. It follows from Lemma 2.7 that

$$\operatorname{dist}(x_n, T(x_n)) \to 0 \ \text{ and } \ \operatorname{diam}(T(x_n)) \to 0. \tag{3.1}$$

For each $n \in \mathbb{N}$, select $y_n \in T(x_n)$ so that $d(x_n, y_n) = \operatorname{dist}(x_n, T(x_n))$. By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to E. Let $A(E, \{x_n\}) = \{x\}$ and $r = r(E, \{x_n\})$. We show that x is an endpoint of T.

Case 1. For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\frac{1}{2}d(x_m, y_m) > d(x_m, x)$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}d(x_{n_k}, y_{n_k}) > d(x_{n_k}, x) \ \text{ for all } \ k \in \mathbb{N}. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\lim_{k\to\infty} x_{n_k} = x$. By Lemma 2.6 (iii), we have

$$\begin{aligned} \operatorname{dist}(x, T(x)) &\leq d(x, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x)) \\ &\leq 2d(x, x_{n_k}) + 3\operatorname{dist}(x_{n_k}, T(x_{n_k})) \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

Hence $x \in T(x)$. Notice also that $\frac{1}{2}\operatorname{dist}(x, T(x)) = 0 \leq d(x_{n_k}, x)$ for all $k \in \mathbb{N}$. Since T is Suzuki, we have

$$H(T(x_{n_k}), T(x)) \leq d(x_{n_k}, x) \to 0 \ \text{ as } \ k \to \infty. \tag{3.3}$$

We now let $v \in T(x)$ and choose $u_{n_k} \in T(x_{n_k})$ so that $d(v, u_{n_k}) = \operatorname{dist}(v, T(x_{n_k}))$. From (3.1) and (3.3) we have

$$\begin{aligned} d(x, v) &\leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) + d(y_{n_k}, u_{n_k}) + d(u_{n_k}, v) \\ &\leq d(x, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) + \operatorname{diam}(T(x_{n_k})) + H(T(x_{n_k}), T(x)) \to 0 \end{aligned}$$

as $k \to \infty$. Hence $v = x$ for all $v \in T(x)$. Therefore, $x \in End(T)$.

Case 2. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{2}d(x_n, y_n) \leq d(x_n, x)$ for all $n \geq n_0$. This implies that $\frac{1}{2}\text{dist}(x_n, T(x_n)) \leq d(x_n, x)$ and so $H(T(x_n), T(x)) \leq d(x_n, x)$. For each $n \in \mathbb{N}$, select $z_n \in T(x)$ so that $d(y_n, z_n) = \text{dist}(y_n, T(x))$. Since $T(x)$ is compact, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to w \in T(x)$. For j sufficiently large, we have

$$\begin{aligned} d(x_{n_j}, w) &\leq d(x_{n_j}, y_{n_j}) + d(y_j, z_{n_j}) + d(z_{n_j}, w) \\ &\leq d(x_{n_j}, y_{n_j}) + H(T(x_{n_j}), T(x)) + d(z_{n_j}, w) \\ &\leq \text{dist}(x_{n_j}, T(x_{n_j})) + d(x_{n_j}, x) + d(z_{n_j}, w). \end{aligned}$$

This implies by the regularity of $\{x_n\}$ that $\limsup_{j\to\infty} d(x_{n_j}, w) \leq \limsup_{j\to\infty} d(x_{n_j}, x) = r$. Hence $w \in A(E, \{x_{n_j}\}) = \{x\}$. Therefore $x = w \in T(x)$. Let $v \in T(x)$ and choose $u_{n_j} \in T(x_{n_j})$ so that $d(v, u_{n_j}) = \text{dist}(v, T(x_{n_j}))$. Thus

$$\begin{aligned} d(x_{n_j}, v) &\leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, u_{n_j}) + d(u_{n_j}, v) \\ &\leq d(x_{n_j}, y_{n_j}) + \text{diam}(T(x_{n_j})) + H(T(x), T(x_{n_j})) \\ &\leq \text{dist}(x_{n_j}, T(x_{n_j})) + \text{diam}(T(x_{n_j})) + d(x_{n_j}, x). \end{aligned}$$

It follows from (3.1) that $\limsup_{j\to\infty} d(x_{n_j}, v) \leq \limsup_{j\to\infty} d(x_{n_j}, x) = r$. Hence $v \in A(E, \{x_{n_j}\}) = \{x\}$, and so $v = x$ for all $v \in T(x)$. Therefore, $x \in End(T)$. □

Finally, we prove the existence of common endpoints for a commuting pair of single-valued and multivalued Suzuki mappings in uniformly convex hyperbolic spaces. Here $KC(E)$ denotes the family of nonempty compact convex subsets of E.

Theorem 3.2. *Let E be a nonempty bounded closed convex subset of X, $t : E \to E$ be a single-valued mapping and $T : E \to KC(E)$ be a multivalued mapping. Suppose that t and T are Suzuki commuting mappings such that T has an approximate endpoint sequence in End(t). Then t and T have a common endpoint in E.*

Proof. This proof is patterned after the proof of Theorem 3.1 in [32]. Let $A = \text{End}(t)$. By Lemma 2.6 (i), A is nonempty closed and convex. Since t and T are commuting mappings,

$$t(T(x)) \subseteq T(x) \text{ for all } x \in A. \tag{3.4}$$

Again by Lemma 2.6 (i), $T(x) \cap A \neq \emptyset$ for all $x \in A$. Therefore, the mapping $F(\cdot) := T(\cdot) \cap A : A \to KC(A)$ is well defined. Let $x, y \in A$ and z be the unique point in $T(y)$ such that $d(x, z) = \text{dist}(x, T(y))$. It follows from (3.4) that $t(z) \in T(y)$. By Lemma 2.6 (ii),

$$d(x, t(z)) \leq d(x, z) = \text{dist}(x, T(y)).$$

By the uniqueness of z we have $z \in A$. This implies that

$$\operatorname{dist}(x, T(y)) = \operatorname{dist}(x, F(y)) \text{ for all } x, y \in A. \tag{3.5}$$

Since T has an approximate endpoint sequence in A, F has the approximate endpoint property. Next, we show that F is a Suzuki mapping. Let $x, y \in A$ be such that $\frac{1}{2}\operatorname{dist}(x, F(x)) \leq d(x, y)$. Thus $\frac{1}{2}\operatorname{dist}(x, T(x)) \leq d(x, y)$ and hence

$$H(T(x), T(y)) \leq d(x, y). \tag{3.6}$$

From (3.5) and (3.6) we have

$$\begin{aligned} H(F(x), F(y)) &= \max\left\{\sup_{u \in F(x)} \operatorname{dist}(u, F(y)), \sup_{v \in F(y)} \operatorname{dist}(v, F(x))\right\} \\ &= \max\left\{\sup_{u \in F(x)} \operatorname{dist}(u, T(y)), \sup_{v \in F(y)} \operatorname{dist}(v, T(x))\right\} \\ &\leq \max\left\{\sup_{u \in T(x)} \operatorname{dist}(u, T(y)), \sup_{v \in T(y)} \operatorname{dist}(v, T(x))\right\} \\ &= H(T(x), T(y)) \\ &\leq d(x, y). \end{aligned}$$

Therefore, F is a Suzuki mapping. By Theorem 3.1, F has an endpoint in A which in turn implies that t and T have a common endpoint in E. □

In Theorem 3.2, the condition that "T has an approximate endpoint sequence in $\operatorname{End}(t)$" seems to be strong but the following example shows that it is necessary. Notice also that this condition cannot be replaced by "T has an approximate endpoint sequence in E".

Example 3.3. Let $X = \mathbb{R}$, $E = [0, 1]$ and $t : E \to E$ be defined by

$$t(x) = \frac{x}{2} \text{ for all } x \in E.$$

Let $T : E \to KC(E)$ be defined by

$$T(x) = [x, 1] \text{ for all } x \in E.$$

Then t and T are Suzuki commuting mappings. Notice that T does not have an approximate endpoint sequence in $\operatorname{End}(t)$. Since $\operatorname{End}(t) = \{0\}$ and $\operatorname{End}(T) = \{1\}$, t and T do not have a common endpoint in E.

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