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Hyperideals in *EL*-Semihyperrings

Warud Nakkhasen and Bundit Pibaljommee[1]

Department of Mathematics, Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayuttaya Rd.
Bangkok 10400, Thailand
e-mail : w.nakkhasen@gmail.com (W. Nakkhasen)
banpib@kku.ac.th (B. Pibaljommee)

Abstract : We construct the EL-semihyperrings from ordered semirings using the concept of Ends lemma. Moreover, we study the connection between many types of ideals in ordered semirings and hyperideals of its associated EL-semihyperrings.

Keywords : Ends lemma; EL-semihyperring; hyperideal; bi-hyperideal; interior hyperideal; (m, n)-hyperideal.
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1 Introduction

The concept of a semiring was introduced by Vandiver [1] in 1934 as a generalization of a ring. In 2011, Gan and Jiang [2] introduced the notion of an ordered semiring as a semiring with a partially relation on its universe set such that the relation is compatible with both operations of the semiring.

In 1934, Marty [3] gave the notion of an algebraic hyperstructure published in the eighth congress of scandinavian mathematicians. This theory has been studied in the following decades and nowadays by many mathematicians among whom, for example, see [4–7]. Vougiouklis [8] generalized the concept of a hyperring $(R, +, \cdot)$ by droppings the reproduction axiom where $+$ and $\cdot$ are associative hyperoperations and $\cdot$ distributive over $+$ and named it as a semihyperring.

[1] Corresponding author.

EL-hyperstructures, which were first introduced by Chvalina in [9], are hypercompositional structures constructed, from a partially/quasi (semi)group using a construction known as Ending lemma or Ends lemma. In [10], Račková extended Ends lemma to investigate transposition hypergroups. Then Novák [11] studied a theoretical background for algebraic hyperstructures such as hypergroups and semihypergroups using "Ends lemma". Later, he discussed a construction of hyperstructures from quasi or partially ordered semigroups and studied some of their properties, see [12]. In 2016, Ghazavi, Anvariyeh and Mirvakili [13] investigated various kinds of ideals in a quasi-ordered semigroup and its EL-(semi)hypergroup.

The purpose of this paper is to investigate the connection between many types of ideals in ordered semirings and hyperideals of its associated EL-semihyperrings.

2 Preliminaries

In this section, we review some definitions and some results which will be used in the next section.

A *semiring* S is a nonempty set which is closed under two binary associative operations $+$ and $\cdot$ such that $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in S$. A nonempty subset I of S is called a *left* (resp. *right*) *ideal* of S if I is closed under addition and $S \cdot I \subseteq I$ (resp. $I \cdot S \subseteq I$). We call I a *two-sided ideal* or an *ideal* of S if it is both a left and a right ideal of S.

A semiring S is called *simple* if it does not contain proper ideals. An element e of S is said to be an *identity* if $x = e \cdot x$ and $x = x \cdot e$ for all $x \in S$. A semiring S is called *idempotent* if $a = a + a$ and $a = a \cdot a$ for all $a \in S$. An *ordered semiring* is a semiring S equipped with a partial order relation $\leq$ on S such that it is compatible with the operations on S.

We denote $[x) = \{s \in S \mid x \leq s\}$ and also $[A) = \bigcup_{x \in A}[x)$. Oppositely, $(x] = \{s \in S \mid s \leq x\}$ and $(A] = \bigcup_{x \in A}(x]$. A nonempty subset A of S is called an *upper end* of S if for every $a \in A$, $[a) \subseteq A$ (i.e., $[A) = A$).

Remark 2.1. *Let A and B be nonempty subsets of an ordered semiring $(S, +, \cdot, \leq)$. Then the following statements hold:*

- (i) $A \subseteq [A)$;
- (ii) $[[A)) = [A)$ *(i.e., $[A)$ is an upper end of S);*
- (iii) *if $A \subseteq B$, then $[A) \subseteq [B)$;*
- (iv) $[A \cdot B) = [A) \cdot [B)$.

Let $(S, +, \cdot, \leq)$ be an ordered semiring and a nonempty subset I of S be closed under addition. Then

- (i) I is called a *left* (resp. *right*) *ideal* of S if $S \cdot I \subseteq I$ (resp. $I \cdot S \subseteq I$);
- (ii) I is called a *two-sided ideal* or an *ideal* of S if it is both a left and a right ideal of S;

(*iii*) I is called a *bi-ideal* of S if $I^2 \subseteq I$ and $I \cdot S \cdot I \subseteq I$;

(*iv*) I is called an *interior ideal* of S if $I^2 \subseteq I$ and $S \cdot I \cdot S \subseteq I$;

(*v*) I is called an (m,n)-*ideal* of S if $I^2 \subseteq I$ and $I^m \cdot S \cdot I^n \subseteq I$ where m, n are non-negative integers where $I^0 \cdot S = S$ and $S \cdot I^0 = S$.

A *left ordered ideal* (resp. *right ordered ideal*, *two-sided ordered ideal* or *ordered ideal*, *ordered bi-ideal*, *ordered interior ideal*, *ordered* (m,n)-*ideal*) I of an ordered semiring S is a left ideal (resp. right ideal, two-sided ideal or ideal, bi-ideal, interior ideal, (m,n)-ideal) of S satisfying the condition if $x \in S$ such that $x \leq a$ for some $a \in I$ then $x \in I$ (i.e., $(I] = I$).

Lemma 2.2. *Let $(S,+,\cdot,\leq)$ be an ordered semiring and A be a nonempty subset of S. If A is an ideal of S, then $[A)$ is also an ideal of S.*

Proof. Assume that A is an ideal of S. Let $x, y \in [A)$ and $s \in S$. There exist $a, b \in A$ such that $a \leq x$ and $b \leq y$. So, $a+b \leq x+y$, that is, $x+y \in [a+b) \subseteq [A)$. Since $a \leq x$, we have $s \cdot a \leq s \cdot x$. Thus, $s \cdot x \in [s \cdot a) \subseteq [A)$. Also, $S \cdot [A) \subseteq [A)$. Similarly, we can show that $[A) \cdot S \subseteq [A)$. Hence, $[A)$ is an ideal of S. □

Let H be a nonempty set. A mapping $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H, is called a *hyperoperation* on H (see, e.g., [4–7]). The structure $(H, \circ)$ is said to be a *hypergroupoid*. If A and B are two nonempty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for any $x, y, z \in H$, we have $(x \circ y) \circ z = x \circ (y \circ z)$. A semihypergroup $(H, \circ)$ is called a *hypergroup* if $a \circ H = H = H \circ a$ for all $a \in S$.

Vougiouklis [8] introduced the notion of a semihyperring which both the sum and the product are hyperoperations as follows.

A structure $(S,+,\cdot)$ is called a *semihyperring* if it satisfies the following axioms:

(*i*) $(S,+)$ is a semihypergroup;

(*ii*) $(S,\cdot)$ is a semihypergroup;

(*iii*) for all $x, y, z \in S, x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$.

A nonempty subset T of a semihyperring $(S,+,\cdot)$ is said to be a *subsemihyperring* of S if for all $x, y \in T$, $x+y \subseteq T$ and $x \cdot y \subseteq T$. A nonempty subset I of a semihyperring $(S,+,\cdot)$ is called a *left* (resp. *right*) *hyperideal* of S if for every $x, y \in I, x+y \subseteq I$ and for any $s \in S, s \cdot x \subseteq I$ (resp. $x \cdot s \subseteq I$). We call I a *two-sided hyperideal* or a *hyperideal* of S if it is both a left and a right hyperideal of S. An element e of a semihyperring $(S,+,\cdot)$ is said to be an *identity* if $x \in e \cdot x$ and $x \in x \cdot e$ for all $x \in S$.

The *EL-heperstructures* are hyperstructures constructed from quasi/partially-ordered (semi)groups using the "Ends lemma". The concept of Ends lemma was introduced by Chvalina [9] as the following lemma.

Lemma 2.3. [9] *Let* $(S, \cdot, \leq)$ *be a partially-ordered semigroup. Hyperoperation* $\circ : S \times S \to \mathcal{P}^*(S)$ *defined by* $a \circ b = [a \cdot b) = \{x \in S \mid a \cdot b \leq x\}$ *is associative. The semihypergroup* $(S, \circ)$ *is commutative if and only if the semigroup* $(S, \cdot)$ *is commutative.*

3 Hyperideals of *EL*-Semihyperrings

We apply the concept of "Ends lemma" to a construction of the EL - semihyperrings from ordered semirings and study the connection between many types of ideals in ordered semirings and hyperideals of its associated EL-semihyperrings.

Theorem 3.1. *Let* $(S, +, \cdot, \leq)$ *be an ordered semiring. Then* $(S, \oplus, \odot)$ *is a semihyperring where the hyperoperations* $\oplus$ *and* $\odot$ *are defined as follows:*

$$a \oplus b = [a + b) = \{x \in S \mid a + b \leq x\} \text{ and}$$
$$a \odot b = [a \cdot b) = \{x \in S \mid a \cdot b \leq x\}$$

for all $a, b \in S$. *Moreover, the semihyperring* $(S, \oplus, \odot)$ *is commutative if and only if the ordered semiring* $(S, +, \cdot, \leq)$ *is commutative.*

Proof. By Lemma 2.3, $(S, \oplus)$ and $(S, \odot)$ are semihypergroups. Now, we show that the hyperoperation $\odot$ is distributive over the hyperoperation $\oplus$ on S. Let $a, b, c \in S$. We claim that $a \odot (b \oplus c) = [a \cdot (b + c))$. Let $t \in a \odot (b \oplus c)$. Then $t \in a \odot x$ for some $x \in b \oplus c$. Thus, $a \cdot (b + c) \leq a \cdot x \leq t$, that is, $t \in [a \cdot (b + c))$. Hence, $a \odot (b \oplus c) \subseteq [a \cdot (b + c))$. Let $s \in [a \cdot (b + c))$. Then $a \cdot (b + c) \leq s$. So, $s \in a \odot (b + c) \subseteq \bigcup_{x \in b \oplus c} a \odot x = a \odot (b \oplus c)$. Thus, $[a \cdot (b + c)) \subseteq a \odot (b \oplus c)$. It turns out $a \odot (b \oplus c) = [a \cdot (b + c))$. Next, we show that $(a \odot b) \oplus (a \odot c) = [a \cdot b + a \cdot c)$. Let $t \in (a \odot b) \oplus (a \odot c)$. Then $t \in x \oplus y$ for some $x \in a \odot b$ and $y \in a \odot c$. We obtain that $a \cdot b + a \cdot c \leq x + y \leq t$. So $t \in [a \cdot b + a \cdot c)$. Hence, $(a \odot b) \oplus (a \odot c) \subseteq [a \cdot b + a \cdot c)$. Let $s \in [a \cdot b + a \cdot c)$. Then $a \cdot b + a \cdot c \leq s$. Thus, $s \in a \cdot b \oplus a \cdot c \subseteq \bigcup_{x \in a \odot b, y \in a \odot c} x \oplus y = (a \odot b) \oplus (a \odot c)$. Hence, $[a \cdot b + a \cdot c) \subseteq (a \odot b) \oplus (a \odot c)$. Therefore, $(a \odot b) \oplus (a \odot c) = [a \cdot b + a \cdot c)$. Since $[a \cdot (b + c)) = [a \cdot b + a \cdot c)$, we have $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. Similarly, we can show that $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)$. Consequently, $(S, \oplus, \odot)$ forms a semihyperring. The rest of theorem is easy to verify. □

The semihyperring $(S, \oplus, \odot)$ defined in Theorem 3.1 is called the *associated* EL*-semihyperring* of $(S, +, \cdot, \leq)$.

Theorem 3.2. *Let* $(S, +, \cdot, \leq)$ *be an ordered semiring and* $(S, \oplus, \odot)$ *be its associated* EL*-semihyperring. If* A *is a left (ordered) ideal of* S*, which is an upper end of* S*, then* A *is a left hyperideal of* $(S, \oplus, \odot)$*. It is also true for right (ordered) ideals.*

Proof. Let A be a left (ordered) ideal of S, $x, y \in A$ and $s \in S$. Since A is an upper end of S, we have $x \oplus y = [x + y) \subseteq A$ and $s \odot x = [s \cdot x) \subseteq A$. Hence, A is a left hyperideal of S. □

We apply Example 3.4 in [13] to construct an example of a semihyperring.

Example 3.3. Let $S = \{a, b, c, d, e\}$ and two binary operations $+$ and $\cdot$ on S be defined as follows:

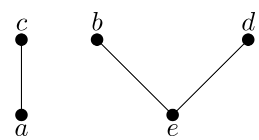
$+$	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	d	e
c	a	b	c	d	e
d	a	b	c	d	e
e	a	b	c	d	e

$\cdot$	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

We define an order relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (e, b), (e, d)\}.$$

The figure of $\leq$ on S is given by



Then $(S, +, \cdot, \leq)$ is an ordered semiring. We obtain its associated semihyperring $(S, \oplus, \odot)$ where $\oplus$ and $\odot$ are shown as follows:

$\oplus$	a	b	c	d	e
a	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{b, d, e\}$
b	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{b, d, e\}$
c	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{b, d, e\}$
d	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{b, d, e\}$
e	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{b, d, e\}$

$\odot$	a	b	c	d	e
a	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{b\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{b\}$	$\{b\}$
d	$\{d\}$	$\{b\}$	$\{d\}$	$\{b\}$	$\{b\}$
e	$\{b, d, e\}$	$\{b, d, e\}$	$\{b, d, e\}$	$\{b, d, e\}$	$\{b, d, e\}$

Now, we can see that $A = \{b, d, e\}$ is a right ordered ideal and $B = \{b, c, d, e\}$ is a left (not ordered) ideal of $(S, +, \cdot, \leq)$. Also, both of them are the upper ends of S. It turns out that A is a right hyperideal and B is a left hyperideal of $(S, \oplus, \odot)$.

Corollary 3.4. *Let $(S, +, \cdot, \leq)$ be an ordered semiring and $(S, \oplus, \odot)$ be its associated EL-semihyperring. If A is an (ordered) ideal of S, which is an upper end of S, then A is a hyperideal of $(S, \oplus, \odot)$.*

We apply Example 3.7 in [13] to construct an example of an ordered semiring to show that the condition "upper end" is necessary.

Example 3.5. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $S = M_2(\mathbb{N}_0)$ denote the set of all 2×2 matrices over $\mathbb{N}_0$. Then S with usual the matrix addition and matrix multiplication is a semiring. Define a binary relation $\leq$ on S by letting $A, B \in S$, $A \leq B$ iff $a_{ij} \leq b_{ij}$ where $i, j \in \{1, 2\}$. Then $(S, +, \cdot, \leq)$ is an ordered semiring. Let $I = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{N}_0 \right\} \subseteq S$. Thus, I is a left ordered ideal of $(S, +, \cdot, \leq)$, which is not an upper end of S. Now, I can not be a left hyperideal of $(S, \oplus, \odot)$, since

$$S \odot I = \bigcup_{A \in S, X \in I} A \odot X = \bigcup_{a,b,c,d,x,y \in \mathbb{N}_0} \left[\begin{bmatrix} ax + by & 0 + 0 \\ cx + dy & 0 + 0 \end{bmatrix} \right) = S \nsubseteq I.$$

The following theorem shows that if the ordered semiring $(S, +, \cdot, \leq)$ contains an identity element, then the converse of Theorem 3.2 is true.

Theorem 3.6. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then A is a left (resp. right, two-sided) ideal of S, which is an upper end of S if and only if A is a left (resp. right, two-sided) hyperideal of $(S, \oplus, \odot)$.*

Proof. The proof of the necessity part follows by Theorem 3.2. For the sufficiency part, we assume that A is a left hyperideal of S. Let $x, y \in A$ and $s \in S$. We have $x + y \in [x + y) = x \oplus y \subseteq A$ and $s \cdot x \in [s \cdot x) = s \odot x \subseteq A$. Hence, A is a left ideal of S. Next, suppose that A is not an upper end of S. There exists $a \in A$ such that $[a) \nsubseteq A$. Since S contains the identity element e, we have $e \odot a = [e \cdot a) = [a) \nsubseteq A$. This is a contradiction. □

Example 3.7. Let $\mathbb{N}$ be the set of natural numbers together with usual addition, multiplication and ordering of numbers. Then $(\mathbb{N}, +, \cdot, \leq)$ is an ordered semiring with identity 1. Consider $I = \{2, 3, 4, 5, \ldots\}$. We can see that I is a hyperideal of $(\mathbb{N}, \oplus, \odot)$ which is also an upper end of $\mathbb{N}$. But, I is not an ordered ideal of $(\mathbb{N}, +, \cdot, \leq)$, since $1 < 2$ but $1 \notin I$.

Theorem 3.8. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity e and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then u is an identity of $(S, \oplus, \odot)$ if and only if $u \leq e$.*

Proof. Assume that u is an identity of $(S, \oplus, \odot)$. We have $e \in u \odot e = [u \cdot e) = [u)$, implies $u \leq e$. Conversely, suppose that $u \leq e$. Let $a \in S$. Since e is an identity of $(S, +, \cdot, \leq)$, we have $a \cdot u \leq a \cdot e = a$ and $u \cdot a \leq e \cdot a = a$. This implies that $a \in a \odot u$ and $a \in u \odot a$. Hence, u is an identity of $(S, \oplus, \odot)$. □

Definition 3.9. A semihyperring S is called *simple* if it does not contain proper hyperideals.

Theorem 3.10. *Let $(S, +, \cdot, \leq)$ be an ordered semiring which $(S, +, \cdot)$ is simple. Then the associated EL-semihyperring $(S, \oplus, \odot)$ is simple.*

Proof. Let I be an arbitrary hyperideal of $(S, \oplus, \odot)$. We can show that I is an ideal of S. Since $(S, +, \cdot)$ is simple, we have $I = S$. □

The converse of Theorem 3.10 is not true, as the following example.

Example 3.11. Let $S = \{a, b, c\}$ and two operations $+$ and $\cdot$ on S be defined as follows:

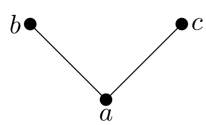
$+$	a	b	c
a	a	a	a
b	a	b	a
c	a	c	a

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	a

We define an order relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c)\}.$$

The figure of $\leq$ on S is given by



Then $(S, +, \cdot, \leq)$ is an ordered semiring. Clearly, $(S, +, \cdot)$ is not simple, Since $\{a\}$ and $\{a, b\}$ are proper ideals of S. By Theorem 3.1, the hyperoperations $\oplus$ and $\odot$ of the associated EL-semihyperring $(S, \oplus, \odot)$ of $(S, +, \cdot, \leq)$ are shown as the following tables:

$\oplus$	a	b	c
a	S	S	S
b	S	$\{b\}$	S
c	S	$\{c\}$	S

$\odot$	a	b	c
a	S	S	S
b	S	$\{b\}$	S
c	S	S	S

It is easy to check that $(S, \oplus, \odot)$ has no proper hyperideal.

A left (resp. right, two-sided) hyperideal I of a semihyperring S is called a *maximal left* (resp. *right*, *two-sided*) *hyperideal* of S if there is no proper left (resp. right, two-sided) hyperideal J of S such that $I \subset J \subset S$. A left (resp. right, two-sided) hyperideal I of a semihyperring S is called a *minimal left* (resp. *right*, *two-sided*) *hyperideal* of S if there is no proper left (resp. right, two-sided) hyperideal K of S such that $K \subset I \subset S$.

Theorem 3.12. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is a maximal left (resp. right, two-sided) ideal of the set of all left (resp. right, two-sided) ideals of S, which are also upper ends of S if and only if I is a maximal left (resp. right, two-sided) hyperideal of the set of all left (resp. right, two-sided) hyperideals of $(S, \oplus, \odot)$.*

Proof. Assume that I is maximal left ideal of the set of all left ideals of S, which are also upper ends of S. Let J be a left hyperideal of S such that $I \subset J \subseteq S$. By Theorem 3.6, J is a left ideal of S, which is also an upper end of S. By maximality of I, we obtain that $J = S$. The proof of the sufficiency part is similar. □

The proof of the following theorem is similar to the proof of Theorem 3.12.

Theorem 3.13. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is a minimal left (resp. right, two-sided) ideal of the set of all left (resp. right, two-sided) ideals of S, which are also upper ends of S if and only if I is a minimal left (resp. right, two-sided) hyperideal of $(S, \oplus, \odot)$.*

The statements of Theorem 3.6, Theorem 3.12 and Theorem 3.13 are not true for the ordered ideals of $(S, +, \cdot, \leq)$. However, if we add some condition, then these theorems are also true for ordered ideals.

Theorem 3.14. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is a left (resp. right, two-sided) ordered ideal of S, which is also an upper end of S if and only if I is a left (resp. right, two-sided) hyperideal of $(S, \oplus, \odot)$ with $(I] = I$.*

Theorem 3.15. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is a maximal left (resp. right, two-sided) ordered ideal of the set of all left (resp. right, two-sided) ordered ideals of S, which are also upper ends of S if and only if I is a maximal left (resp. right, two-sided) hyperideal of the set of all left (resp. right, two-sided) hyperideals of $(S, \oplus, \odot)$ with $(I] = I$.*

Theorem 3.16. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is a minimal left (resp. right, two-sided) ordered ideal of the set of all left (resp. right, two-sided) ordered ideals of S, which are also upper ends of S if and only if I is a minimal left (resp. right, two-sided) hyperideal of the set of all left (resp. right, two-sided) hyperideals of $(S, \oplus, \odot)$ with $(I] = I$.*

Definition 3.17. A nonempty subset B of a semihyperring $(S, \oplus, \odot)$ is said to be a *bi-hyperideal* of S if B is a subsemihyperring of S and $B \odot S \odot B \subseteq B$.

Theorem 3.18. *Let $(S, \oplus, \odot)$ be the associated EL-semihyperring of an ordered semiring $(S, +, \cdot, \leq)$. If B is a (an ordered) bi-ideal of S, which is also an upper end of S, then B is a bi-hyperideal of $(S, \oplus, \odot)$.*

Proof. Since $B + B \subseteq B$ and $B \cdot B \subseteq B$, we have $B \oplus B \subseteq B$ and $B \odot B \subseteq B$. Let $a, b \in B$ and $s \in S$. So $a \odot s \odot b = \bigcup_{t \in a \odot s} t \odot b = \bigcup_{t \in [a \cdot s)} [t \cdot b)$. Then $a \cdot s \leq t$ implies $a \cdot s \cdot b \leq t \cdot b$. Since B is a bi-ideal of S, $a \cdot s \cdot b \in B$. Since B is an upper end of S, $t \cdot b \in B$. We obtain that $[t \cdot b) \subseteq B$. So $a \odot s \odot b \subseteq B$. Hence, $B \odot S \odot B \subseteq B$. □

Theorem 3.19. *Let $(S, +, \cdot, \leq)$ be an ordered semiring, $(S, \oplus, \odot)$ be its associated EL-semihyperring and B be an upper end of S. Then B is a (an ordered) bi-ideal of S if and only if B is a bi-hyperideal of $(S, \oplus, \odot)$ (with $(B] = B$).*

Definition 3.20. A subsemihyperring I of $(S, \oplus, \odot)$ is called an *interior hyperideal* of S if $S \odot I \odot S \subseteq I$.

The proof of the following theorem is similar to the proof of Theorem 3.18.

Theorem 3.21. *Let $(S, +, \cdot, \leq)$ be an ordered semiring and $(S, \oplus, \odot)$ be its associated EL-semihyperring. If I is an (ordered) interior ideal of S, which is also an upper end of S, then I is an interior hyperideal of $(S, \oplus, \odot)$.*

Because each interior hyperideal of $(S, \oplus, \odot)$ is an interior ideal of $(S, +, \cdot, \leq)$ but it is not necessarily an upper end of S. Then every interior hyperideal of S need not to be an ordered interior ideal of S. But, if an ordered semiring $(S, +, \cdot, \leq)$ contains an identity element, then the converse of Theorem 3.21 is true.

Theorem 3.22. *Let $(S, +, \cdot, \leq)$ be an ordered semiring with identity and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then I is an (ordered) interior ideal of S, which is also an upper end of S if and only if I is an interior hyperideal of $(S, \oplus, \odot)$ (with $(I] = I$).*

Definition 3.23. Let $(S, \oplus, \odot)$ be a semihyperring. A nonempty subset A of S is called an *(m, n)-hyperideal* of S if A is a subsemihyperring of S and $A^m \odot S \odot A^n \subseteq A$, where m, n are non-negative integers, $A^0 \odot S = S$ and $S \odot A^0 = S$.

Theorem 3.24. *Let $(S, \oplus, \odot)$ be the associated EL-semihyperring of an ordered semiring $(S, +, \cdot, \leq)$. If A is an (m, n)-ideal (ordered (m, n)-ideal) of S, which is also an upper end of S, then A is an (m, n)-hyperideal of $(S, \oplus, \odot)$.*

Proof. It is easy to show that A is a subsemihyperring of S. Let $x \in A^m \odot S \odot A^n$. There exist $a \in A^m, s \in S$ and $b \in A^n$ such that $x \in a \odot s \odot b = \bigcup_{t \in s \odot b} a \odot t = \bigcup_{t \in [s \cdot b)} [a \cdot t)$. Then $x \in [a \cdot t_1)$ for some $t_1 \in [s \cdot b)$, that is, $a \cdot t_1 \leq x$ and $s \cdot b \leq t_1$. This implies that $a \cdot s \cdot b \leq a \cdot t_1 \leq x$. Since $a \in A^m$ and $b \in A^n$, we have $a \in x_1 \odot x_2 \odot \cdots \odot x_m$ and $b \in y_1 \odot y_2 \odot \cdots \odot y_n$. Thus, $a \in [x_1 \cdot x_2 \cdot \cdots \cdot x_m)$ and $b \in [y_1 \cdot y_2 \cdot \cdots \cdot y_n)$. Also, $x_1 \cdot x_2 \cdot \cdots \cdot x_m \cdot s \cdot y_1 \cdot y_2 \cdot \cdots \cdot y_n \leq a \cdot s \cdot b \leq x$. So, $x \in [x_1 \cdot x_2 \cdot \cdots \cdot x_m \cdot s \cdot y_1 \cdot y_2 \cdot \cdots \cdot y_n)$, which $x_1 \cdot x_2 \cdot \cdots \cdot x_m \cdot s \cdot y_1 \cdot y_2 \cdot \cdots \cdot y_n \in A^m \cdot S \cdot A^n \subseteq A$. Since A is an upper end of S, $[x_1 \cdot x_2 \cdot \cdots \cdot x_m \cdot s \cdot y_1 \cdot y_2 \cdot \cdots \cdot y_n) \subseteq A$. Therefore, $x \in A$. □

A hyperideal P of a semihyperring $(S, \oplus, \odot)$ is called *semiprime* if for any hyperideal A of S, $A \odot A \subseteq P$ implies $A \subseteq P$, and is called *prime* if for any hyperideals A, B of S, $A \odot B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Theorem 3.25. *Let $(S, \oplus, \odot)$ be the associated EL-semihyperring of an ordered semihyperring $(S, +, \cdot, \leq)$ and a nonempty subset P of S be an upper end. Then P is a prime (ordered) ideal of S if and only if P is a prime hyperideal of $(S, \oplus, \odot)$ (with $(P] = P$).*

Proof. Assume that P is a prime ideal of S. By Corollary 3.4, P is a hyperideal of S. Let A and B be hyperideals of S such that $A \odot B \subseteq P$. We can show that A and B are ideals of S. Now, $A \cdot B \subseteq A \odot B \subseteq P$. This implies that $A \subseteq P$ or $B \subseteq P$. Hence, P is a prime hyperideal of S. Conversely, suppose that P is a prime hyperideal of S. Then P is an ideal of S. Let I and J be ideals of S such that $I \cdot J \subseteq P$. By Lemma 2.2 and Remark 2.1, we have that $[I)$ and $[J)$ are ideals of S which are upper ends. By Corollary 3.4, $[I)$ and $[J)$ are hyperideals of S. Let $x \in [I)$ and $y \in [J)$. There exist $u \in I$ and $v \in J$ such that $u \leq x$ and $v \leq y$. So, $u \cdot v \leq x \cdot y$, that is, $x \cdot y \in [u \cdot v)$. Since P is an upper end of S, $x \odot y = [x \cdot y) \subseteq [u \cdot v) \subseteq P$. Hence, $[I) \odot [J) \subseteq P$. By assumption, $[I) \subseteq P$ or $[J) \subseteq P$ and then $I \subseteq P$ or $J \subseteq P$. Therefore, P is a prime ideal of S. □

The proof of the following theorem is similar to the proof of Theorem 3.25.

Theorem 3.26. *Let $(S, \oplus, \odot)$ be the associated EL-semihyperring of an ordered semihyperring $(S, +, \cdot, \leq)$ and a nonempty subset P of S be an upper end. Then P is a semiprime (ordered) ideal of S if and only if P is a semiprime hyperideal of $(S, \oplus, \odot)$ (with $(P] = P$).*

Definition 3.27. A semihyperring $(S, \oplus, \odot)$ is called an *idempotent semihyperring* if $a \in a \oplus a$ and $a \in a \odot a$ for all $a \in S$.

Theorem 3.28. *Let $(S, +, \cdot, \leq)$ be an ordered semiring which $(S, +, \cdot)$ is idempotent and $(S, \oplus, \odot)$ be its associated EL-semihyperring. Then $(S, \oplus, \odot)$ is an idempotent semihyperring.*

Proof. Let $x \in S$. Then $x \in [x) = [x+x) = x \oplus x$ and $x \in [x) = [x \cdot x) = x \odot x$. □

The converse of Theorem 3.28 is not true as the following example.

Example 3.29. Let $(S, +\cdot, \leq)$ be the ordered semiring defined in Example 3.11. Since $c + c \neq c$, $(S, +, \cdot)$ is not idempotent, while its associated EL-semihyperring $(S, \oplus, \odot)$ is idempotent.

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