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E-Inversive Elements in Some Semigroups of Transformations that Preserve Equivalence

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Abstract : Let X be a nonempty set and $T(X)$ the full transformation semigroup on a set X. For an equivalence relation E on X and a cross-section R of the partition X/E induced by E, let

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\} \text{ and}$$

$$T_E(X, R) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_{E^*}(X)$ and $T_E(X, R)$ are subsemigroups of $T(X)$. In this paper, we describe the E-inversive elements of $T_{E^*}(X)$ and $T_E(X, R)$. We also show that $T_{E^*}(X)$ and $T_E(X, R)$ are E-inversive semigroups in terms of the cardinality of X/E and R, respectively.

Keywords : transformation semigroup; equivalence relation; E-inversive element; E-inversive semigroup.
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1 Introduction

An element a of a semigroup S is called *E-inversive* if there exists x in S such that ax is idempotent of S. A semigroup S is called an *E-inversive semigroup* if

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every element of S is E-inversive. Clearly, regular semigroups, finite semigroups and eventually regular semigroups are E-inversives. Basic properties of E-inversive semigroups were given by Catino and Miccoli [1], Mitsch [2] and Mitsch and Petrich [3,4] and Gigoń [5].

The full transformation semigroup on a nonempty set X is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \to X$ under composition. The semigroup $T(X)$ is known to be regular [6]. Hence $T(X)$ is an E-inversive semigroup.

Let E be an equivalence relation on X. Pei [7] has introduced a family of subsemigroups of $T(X)$ defined by

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}$$

and call it the *semigroup of transformations that preserve an equivalence* on X. He has studied Green's relations and regularity on $T_E(X)$. Recently, Deng, Zeng and Xu [8] introduced the subsemigroup of $T_E(X)$ as follows:

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

The authors considered regularity of elements and Green's relations for $T_{E^*}(X)$.

Let R be a cross-section of the partition X/E induced by E. In [9], Araújo and Konieczny defined a subsemigroup of $T(X)$ as follows:

$$T(X, E, R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Clearly, $T(X, E, R) \subseteq T_E(X)$. They have been proved that the semigroups $T(X, E, R)$ are precisely the centralizers of idempotents of $T(X)$. After year, they discussed regularity of elements and Green's relations for $T(X, E, R)$ in [10]. Now, we consider the following subset of $T_E(X)$:

$$T_E(X, R) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_E(X, R)$ is a subsemigroup of $T(X, E, R)$.

The aim of this paper is to give necessary and sufficient condition for the elements of $T_{E^*}(X)$ and $T_E(X, R)$ are E-inversives. Moreover, a necessary and sufficient condition for $T_{E^*}(X)$ and $T_E(X, R)$ to be an E-inversive semigroup is given in terms of $|X/E|$ and $|R|$, respectively.

In the remainder, let E be an equivalence relation on a set X and R a cross-section of the partition X/E. Denote by X/E the quotient set.

2 Main Results

We denote composition of two mappings obtained by performing first α and then β. We first provide that $T_E(X)$ and $T(X, E, R)$ are E-inversive semigroups. We remark that in view of this fact, if S is any one of $T_E(X)$ and $T(X, E, R)$, then S contains a constant mapping. It thus follows that every $\alpha \in S$ and a constant mapping β of S, $\alpha\beta$ is also constant and hence $\alpha\beta$ is an idempotent element of S. We immediately obtain:

Proposition 2.1. *The semigroups $T_E(X)$ and $T(X,E,R)$ are E-inversive semigroups.*

We have mentioned that every regular element is E-inversive. But there exists an E-inversive element of a semigroup S which is not regular as shown in the following example.

Example 2.2. Let $X = \{1,2,3,4,5,6,7\}$ and $X/E = \{\{1,2,3\},\{4,5\},\{6,7\}\}$. Define $\alpha \in T(X)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 6 & 3 & 3 & 2 & 1 \end{pmatrix}.$$

Then $\alpha \in T_E(X)$, hence α is E-inversive. Suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_E(X)$. Since $1 = 7\alpha = 7\alpha\beta\alpha = 1\beta\alpha$ and $3 = 4\alpha = 4\alpha\beta\alpha = 3\beta\alpha$, we obtain that $1\beta = 7$ and $3\beta \in \{4,5\}$. Since $(1,3) \in E$ and $\beta \in T_E(X)$, $(1\beta, 3\beta) \in E$ which is a contradiction. Hence α is not a regular element of $T_E(X)$.

To prove the main theorem, the following lemma is needed.

Lemma 2.3. *Let $\alpha \in T_{E^*}(X)$. If α is idempotent, then $A\alpha \subseteq A$ for all $A \in X/E$.*

Proof. Suppose that α is idempotent. Then $\alpha^2 = \alpha$. Let $A \in X/E$ and $a \in A$. Then $a\alpha^2 = a\alpha$ and hence $(a\alpha, (a\alpha)\alpha) \in E$. Since $\alpha \in T_{E^*}(X)$, it follows that $(a, a\alpha) \in E$. From $a \in A$, we deduce that $a\alpha \in A$. Therefore, $A\alpha \subseteq A$. □

The nature of regular elements in $T_{E^*}(X)$ and condition under which $T_{E^*}(X)$ is regular were considered in [8].

Theorem 2.4. [8, Theorem 3.1] *Let $\alpha \in T_{E^*}(X)$. Then α is regular if and only if $A \cap X\alpha \neq \emptyset$ for all $A \in X/E$.*

Theorem 2.5. [8, Theorem 3.2] *$T_{E^*}(X)$ is a regular semigroup if and only if $|X/E|$ is finite.*

Theorem 2.6. *Let $\alpha \in T_{E^*}(X)$. Then α is E-inversive if and only if $A \cap X\alpha \neq \emptyset$ for all $A \in X/E$.*

Proof. Suppose that α is E-inversive. Then there exists $\beta \in T_{E^*}(X)$ such that $\alpha\beta$ is idempotent. Let $A \in X/E$. Then $A\beta \subseteq B$ for some $B \in X/E$. By Lemma 2.3, we deduce that $B\alpha\beta \subseteq B$. Let $b \in B$. Then $b\alpha\beta \in B$. If $a \in A$, then $a\beta \in B$ and so $(b\alpha\beta, a\beta) \in E$. Since $\beta \in T_{E^*}(X)$, it follows that $(b\alpha, a) \in E$. Thus $b\alpha \in A$. Hence $B\alpha \subseteq A$. Consequently, $A \cap X\alpha \neq \emptyset$.

Conversely, it follows from Theorem 2.4 and the fact that every regular element is E-inversive. □

The next result follows immediately from Theorem 2.4 and Theorem 2.6.

Corollary 2.7. *Let $\alpha \in T_{E^*}(X)$. Then the following statements are equivalent.*

(1) *α is a regular element.*

(2) *α is an E-inversive element.*

(3) *$A \cap X\alpha \neq \emptyset$ for all $A \in X/E$.*

Corollary 2.7 and Theorem 2.5 can be summarized as follows:

Corollary 2.8. *The following statements are equivalent.*

(1) *$T_{E^*}(X)$ is a regular semigroup.*

(2) *$T_{E^*}(X)$ is an E-inversive semigroup.*

(3) *$|X/E|$ is finite.*

The following theorem characterizes the regular elements of $T_E(X,R)$. Denote E_r the $E-$class containing r for all $r \in R$.

Theorem 2.9. *Let $\alpha \in T_E(X,R)$. Then α is regular if and only if $\alpha|_R$ is an injection.*

Proof. Suppose that α is regular. Then there exists $\beta \in T_E(X,R)$ such that $\alpha = \alpha\beta\alpha$. Let $r,s \in R$ be such that $r\alpha = s\alpha$. Since $\beta \in T_E(X,R)$, $R\beta = R$ and hence $r = r'\beta$ and $s = s'\beta$ for some $r', s' \in R$. Since $R\alpha = R$, there exist $r'', s'' \in R$ such that $r' = r''\alpha$ and $s' = s''\alpha$. We have that

$$r' = r''\alpha = r''\alpha\beta\alpha = r'\beta\alpha = r\alpha = s\alpha = s'\beta\alpha = s''\alpha\beta\alpha = s''\alpha = s'.$$

This implies that $r = r'\beta = s'\beta = s$. Hence $\alpha|_R$ is an injection, as required.

Conversely, assume that $\alpha|_R$ is an injection. Claim that for every $r \in R$, there exists $r' \in R$ such that $E_r \cap X\alpha = E_{r'}\alpha$. Let $r \in R$. Since $R\alpha = R$, there is $r' \in R$ such that $r = r'\alpha$. Since $\alpha \in T_E(X)$, it then follows that $E_{r'}\alpha \subseteq E_r \cap X\alpha$. For the reverse inclusion, if $y \in E_r \cap X\alpha$, then $y = x\alpha$ for some $x \in X$. This implies that $x \in E_s$ for some $s \in R$ and so $s\alpha = r$. By assumption and $s\alpha = r'\alpha$, we have $s = r'$. Hence $y \in E_{r'}\alpha$. This shows that $E_r \cap X\alpha = E_{r'}\alpha$. So we have the claim.

For each $r \in R$, we choose $a_r \in R$ such that $E_r \cap X\alpha = E_{a_r}\alpha$. Thus $r = a_r\alpha$. For each $y \in (E_r \cap X\alpha) \setminus \{r\}$, we choose $a_y \in E_{a_r}$ such that $a_y\alpha = y$. Define $\beta_r : E_r \to E_{a_r}$ by

$$x\beta_r = \begin{cases} a_x & \text{if } x \in X\alpha, \\ a_r & \text{otherwise.} \end{cases}$$

Then β_r is well-defined, $E_r\beta_r \subseteq E_{a_r}$ and $r\beta_r = a_r \in R$. Let $\beta : X \to X$ be defined by $\beta|_{E_r} = \beta_r$ for all $r \in R$. Since R is a cross-section of the partition X/E induced by E, β is well-defined. Obviously, $\beta \in T_E(X)$ and $R\beta \subseteq R$. Let $r \in R$. Then $r\alpha = s$ for some $s \in R$. Thus $s\beta_s = a_s$ for some $a_s \in R$ with $a_s\alpha = s$. Therefore, $a_s\alpha = r\alpha$. By assumption, we have that $a_s = r$ and thus

$s\beta = s\beta|_{E_s} = s\beta_s = a_s = r$. It follows that $R\beta = R$ and therefore $\beta \in T_E(X,R)$. Let $x \in X$. Then $x\alpha \in E_r$ for some $r \in R$. Thus

$$x\alpha\beta\alpha = (x\alpha)\beta|_{E_r}\alpha = (x\alpha)\beta_r\alpha = a_{x\alpha}\alpha = x\alpha$$

and therefore $\alpha = \alpha\beta\alpha$. Hence α is regular. □

We also have the following theorem for which characterizes when $T_E(X,R)$ is a regular semigroup.

Theorem 2.10. *$T_E(X,R)$ is a regular semigroup if and only if $|R|$ is finite.*

Proof. Suppose that R is an infinite set. Let $r \in R$. Then $R \setminus \{r\}$ is also infinite and $|R \setminus \{r\}| = |R|$. Thus there exists a bijection $\varphi : R \setminus \{r\} \to R$. Choose and fix $r' \in R \setminus \{r\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} r' & \text{if } x \in E_r, \\ s\varphi & \text{if } x \in E_s. \end{cases}$$

Then $\alpha \in T_E(X)$. Since $r\alpha = r'$ and $\varphi : R \setminus \{r\} \to R$, we get that $R\alpha \subseteq R$. Let $s \in R$. Since φ is surjective, $s = t\varphi$ for some $t \in R\setminus\{r\}$. Since $t \neq r$, it follows that $t\alpha = t\varphi = s$. Therefore $R \subseteq R\alpha$. Hence $\alpha \in T_E(X,R)$. Since $r' \in R$, $r' = r''\varphi$ for some $r'' \in R \setminus \{r\}$. This implies that $r'' \neq r$ and $r''\alpha = r''\varphi = r' = r\alpha$. Consequently, $\alpha|_R$ is not injective. By Theorem 2.9, we have that α is not regular. Hence $T_E(X)$ is not a regular semigroup.

Conversely, suppose that R is finite. Let $\alpha \in T_E(X,R)$. Then $R\alpha = R$ and so $\alpha|_R : R \to R$ is a surjection. By the finiteness of R, $\alpha|_R$ is injective. From Theorem 2.9, α is regular. We conclude that $T_E(X,R)$ is a regular semigroup. □

The next thorem use [6, page 4] that if $\alpha \in T(X)$ and $\alpha^2 = \alpha$, then $x\alpha = x$ for all $x \in X\alpha$.

Theorem 2.11. *Let $\alpha \in T_E(X,R)$. Then α is E-inversive if and only if $\alpha|_R$ is an injection.*

Proof. Suppose that α is E-inversive. Then there exists $\beta \in T_E(X,R)$ such that $\alpha\beta$ is idempotent. Let $r,s \in R$ be such that $r\alpha = s\alpha$. Since $\alpha\beta \in T_E(X,R)$, we have $R\alpha\beta = R$. Thus $r,s \in X\alpha\beta$. Since $\alpha\beta$ is idempotent and $r\alpha = s\alpha$, we deduce that $r = r\alpha\beta = s\alpha\beta = s$. Thereby $\alpha|_R$ is an injection.

Conversely, if $\alpha|_R$ is injective, then α is regular by Theorem 2.9. Therefore α is E-inversive. □

As a consequence of Theorems 2.9 and 2.11 are useful to obtain this result.

Corollary 2.12. *Let $\alpha \in T_E(X,R)$. Then the following statements are equivalent.*

(1) *α is a regular element.*

(2) *α is an E-inversive element.*

(3) $\alpha|_R$ *is an injection.*

As a consequence of Corollary 2.12 and Theorem 2.10, the following result follows readily.

Corollary 2.13. *The following statements are equivalent.*

(1) $T_E(X, R)$ *is a regular semigroup.*

(2) $T_E(X, R)$ *is an E-inversive semigroup.*

(3) $|R|$ *is finite.*

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