Thai Journal of Mathematics : 117–125
Special Issue: Annual Meeting in Mathematics 2017

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Green's Relations and Regularity for the Self-E-Preserving Transformation Semigroups

Chaiwat Namnak

Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
e-mail : chaiwatn@nu.ac.th

Abstract : Let T_X be the full transformation semigroup on a set X and E an arbitrary equivalence relation on X. We define a subsemigroup of T_X as follows:

$$T_{SE}(X) = \{\alpha \in T_X : \forall x \in X, (x, x\alpha) \in E\}$$

which is called the *self-E-preserving transformation semigroup* on X. Then $T_{SE}(X)$ becomes a regular semigroup. The purpose of this paper is to investigate Green's relations for $T_{SE}(X)$. Moreover, we characterize when certain elements of $T_{SE}(X)$ are left regular, right regular and completely regular.

Keywords : transformation semigroup; Green's relations; regular; left (right) regular; completely regular.
2010 Mathematics Subject Classification : 20M20.

1 Introduction

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$, *left regular* if $a = xa^2$ for some $x \in S$, *right regular* if $a = a^2x$ for some $x \in S$ and *completely regular* if $a = axa$ and $ax = xa$ for some $x \in S$. Evidently every completely regular element is regular, left regular and right regular. If all its elements of S are regular we called S a *regular semigroup*.

Let T_X be the full transformation semigroup on a set X under usual composition of mappings. It is well known that T_X is a regular semigroup. Over the last decades, notions of regularity and Green's relations of subsemigroups of T_X have been widely considered see [1–6]. In [1] has introduced a family of subsemigroups of T_X defined by

$$T_E(X) = \{\alpha \in T_X : \forall a, b \in X, (a, b) \in E \Rightarrow (a\alpha, b\alpha) \in E\}$$

where E is an arbitrary equivalence relation on X. [1] has investigated regularity and Green's relations for $T_E(X)$.

In the rest of the paper, let E be an arbitrary equivalence relation on X. The following a subsemigroup of T_X is considered:

$$T_{SE}(X) = \{\alpha \in T_X : \forall x \in X, (x, x\alpha) \in E\}.$$

In [7], $T_{SE}(X)$ is said to be the *self-E-preserving transformation semigroup* on X and $T_{SE}(X) \subseteq T_E(X)$.

The paper is organized as follows. In section 2, we investigate Green's relations of $T_{SE}(X)$. In section 3, we show that $T_{SE}(X)$ is a regular semigroup and give necessary and sufficient conditions for each element of $T_{SE}(X)$ when it is left regular, right regular and completely regular.

In this introductory section, we present a number of notations and propositions most of which will be indispensable for our research. For a set X and $\alpha \in T_X$, we denote by $\pi(\alpha)$ the partition of X induced by α, namely,

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\},$$

and α_* the natural bijection corresponding to α from $\pi(\alpha)$ onto $X\alpha$ defined by

$$P\alpha_* = x\alpha \quad \text{for all } P \in \pi(\alpha) \text{ and all } x \in P.$$

For collections of subsets $\mathcal{A}$ and $\mathcal{B}$ of X, we say that $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ or $\mathcal{B}$ *refines* $\mathcal{A}$ if $\cup\mathcal{A} = \cup\mathcal{B}$ and for every $B \in \mathcal{B}$, there exists an element $A \in \mathcal{A}$ such that $B \subseteq A$.

Proposition 1.1. *Let $\alpha \in T_{SE}(X)$. If $y \in X\alpha$, then there exists a unique $A \in X/E$ such that $y\alpha^{-1} \subseteq A$. Hence $\pi(\alpha)$ refines X/E.*

Proof. Let $y \in X\alpha$. Then $y = x\alpha$ for some $x \in X$. By X/E is a partition of X, there exists a unique $A \in X/E$ such that $x \in A$. Let $z \in y\alpha^{-1}$. Then $z\alpha = y$. Since $\alpha \in T_{SE}(X)$, we have that $(x, y) = (x, x\alpha) \in E$ and $(z, y) = (z, z\alpha) \in E$. By transitive of E, we deduce that $(x, z) \in E$, so $z \in A$. Hence $y\alpha^{-1} \subseteq A$ for some $A \in X/E$. □

Proposition 1.2. *Let $\alpha \in T_{SE}(X)$. Then for every $A \in X/E$, $A\alpha \subseteq A$.*

Proof. Let $A \in X/E$ and $x \in A$. By $\alpha \in T_{SE}(X)$, we have that $(x, x\alpha) \in E$. This means that $x\alpha \in A$. □

Let $\alpha \in T_{SE}(X)$ and $A \in X/E$. We denote

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

Proposition 1.3. *Let $\alpha \in T_{SE}(X)$ and $A \in X/E$. Then $A = \bigcup \pi_A(\alpha)$.*

Proof. Let $x \in \bigcup \pi_A(\alpha)$. Then $x \in P$ for some $P \in \pi(\alpha)$ such that $P \cap A \neq \emptyset$. By Proposition 1.1, we deduce that $P \subseteq A$. Thereby $\bigcup \pi_A(\alpha) \subseteq A$. For the reverse inclusion, let $x \in A$. By $\pi(\alpha)$ is a partition of X, we have $x \in P$ for some $P \in \pi(\alpha)$. This implies that $P \in \pi_A(\alpha)$, hence $x \in \bigcup \pi_A(\alpha)$. Therefore $A = \bigcup \pi_A(\alpha)$. □

2 Green's Relations for the Self-E-Preserving Transformation Semigroups

We refer to [8, Chapter 2] for the definitions and notations of Green's relations. In this section, we discuss Green's relations of $T_{SE}(X)$.

Theorem 2.1. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in T_{SE}(X)\beta$ if and only if for every $A \in X/E, A\alpha \subseteq A\beta$.*

Proof. Suppose that $\alpha \in T_{SE}(X)\beta$. Then $\alpha = \delta\beta$ for some $\delta \in T_{SE}(X)$. Let $A \in X/E$. By Proposition 1.2, we then have $A\delta \subseteq A$. Hence $A\alpha = A\delta\beta \subseteq A\beta$.

Conversely, assume that $A\alpha \subseteq A\beta$ for all $A \in X/E$. For each $x \in X$, there exists a unique $A \in X/E$ such that $x \in A$. By assumption, we choose and fix an element $x' \in A$ such that $x\alpha = x'\beta$ for all $x \in X$. Define $\delta : X \to X$ by $x\delta = x'$ for all $x \in X$. Let $x \in X$. Since $x, x' \in A$, $(x, x\delta) = (x, x') \in E$ and $x\delta\beta = (x\delta)\beta = x'\beta = x\alpha$. These verify that $\delta \in T_{SE}(X)$ and $\alpha = \delta\beta$. Hence $\alpha \in T_{SE}(X)\beta$, as required. □

Corollary 2.2. *Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{L}$ if and only if $A\alpha = A\beta$ for all $A \in X/E$.*

Theorem 2.3. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in \beta T_{SE}(X)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$.*

Proof. Assume that $\alpha \in \beta T_{SE}(X)$. Then $\alpha = \beta\delta$ for some $\delta \in T_{SE}(X)$. Using the fact that $\pi(\alpha)$ is a partition of X, we have $\cup\pi(\alpha) = \cup\pi(\beta)$. Let $P \in \pi(\beta)$. Hence $P\beta_* = y$ for some $y \in X\beta$. Thus $P\alpha = P\beta\delta = \{y\delta\}$ which implies that $P \subseteq y\delta\alpha^{-1} \in \pi(\alpha)$. We conclude that $\pi(\beta)$ refines $\pi(\alpha)$.

Conversely, assume that $\pi(\beta)$ refines $\pi(\alpha)$. For each $x \in X\beta$, there exists a unique $P_x \in \pi(\beta)$ such that $P_x\beta_* = x$. By assumption, there exists a unique $Q_x \in \pi(\alpha)$ such that $P_x \subseteq Q_x$. Define $\delta : X \to X$ by

$$x\delta = \begin{cases} Q_x\alpha_* & \text{if } x \in X\beta; \\ x & \text{otherwise.} \end{cases}$$

Clearly, δ is well-defined. To show that $\delta \in T_{SE}(X)$, let $x \in X$. If $x \notin X\beta$, then $(x, x\delta) = (x, x) \in E$. If $x \in X\beta$, then by the definition of δ, $x\delta = Q_x\alpha_*$ where $P_x\beta_* = x$ and $P_x \subseteq Q_x$ for some $P_x \in \pi(\beta)$ and $Q_x \in \pi(\alpha)$. By Proposition 1.1, $\pi(\alpha)$ and $\pi(\beta)$ refine X/E, we then have $P_x \subseteq A$ and $Q_x \subseteq B$ for some $A, B \in X/E$. Since $\beta \in T_{SE}(X)$, it follows that $x \in A$. Since $P_x \subseteq Q_x$ and X/E is a partition of X, we have that $A = B$, so $Q_x \subseteq A$. It follows from Proposition 1.2 that $Q_x\alpha_* \in A\alpha \subseteq A$. We deduce that $(x, x\delta) = (x, Q_x\alpha_*) \in E$. Therefore $\delta \in T_{SE}(X)$. Moreover, for $x \in X$,

$$x\beta\delta = (x\beta)\delta = Q_{x\beta}\alpha_* = x\alpha$$

since $x \in P_{x\beta} \subseteq Q_{x\beta}$ where $P_{x\beta} \in \pi(\beta)$ and $Q_{x\beta} \in \pi(\alpha)$. Therefore $\alpha = \beta\delta$, hence the theorem is thereby proved. □

Corollary 2.4. *Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Since $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, the following corollary follows immediately from Corollary 2.2 and Corollary 2.4 .

Corollary 2.5. *Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{H}$ if and only if $\pi(\alpha) = \pi(\beta)$ and $A\alpha = A\beta$ for all $A \in X/E$.*

The next lemma is verified to consider the relation $\mathcal{J}$.

Lemma 2.6. *Let $\alpha, \beta, \delta, \gamma \in T_X$. If $\alpha = \delta\beta\gamma$, then the set $\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta)$ and $Q \cap X\delta \neq \emptyset\}$ is a refinement of $\pi(\alpha)$ where $\mathcal{A}_Q = \{P \in \pi(\delta) : P\delta_* \in Q\}$.*

Proof. Assume that $\alpha = \delta\beta\gamma$. By the first part of the proof Theorem 2.3, we have $\pi(\delta)$ refines $\pi(\alpha)$. Claim that $\cup\mathcal{A} = X$. Let $x \in X$. Then $x \in P$ for some $P \in \pi(\delta)$. We note that $x\delta\beta \in X\beta$, so $x\delta\beta = Q\beta_*$ for some $Q \in \pi(\beta)$. Then $P\delta_* = x\delta \in Q$ and hence $Q \cap X\delta \neq \emptyset$. Thus $P \in \mathcal{A}_Q$ and $x \in P \subseteq \cup\mathcal{A}_Q \subseteq \cup\mathcal{A}$. Hence we have the claim. This means that $\cup\mathcal{A} = \cup\pi(\alpha)$. Let $Q \in \pi(\beta)$ be such that $Q \cap X\delta \neq \emptyset$. To show that there exists $\tilde{P} \in \pi(\alpha)$ such that $\cup\mathcal{A}_Q \subseteq \tilde{P}$, let $x \in Q \cap X\delta$. Then there exists an element $x' \in X$ such that $x'\delta = x$. Since $\pi(\delta)$ is a partition of X, $x' \in P$ for some $P \in \pi(\delta)$ and $P\delta_* = x'\delta$. Since $\pi(\delta)$ refines $\pi(\alpha)$, $P \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\alpha)$. Let $y \in \cup\mathcal{A}_Q$. Then $y \in P'$ for some $P' \in \mathcal{A}_Q$. By the definition of $\mathcal{A}_Q$, $P'\delta_* \in Q$. Hence $y\delta\beta = P'\delta_*\beta = Q\beta_* = x\beta = x'\delta\beta$. Since $x' \in P \subseteq \tilde{P}$, $x'\alpha = \tilde{P}\alpha_*$. Thus

$$y\alpha = y\delta\beta\gamma = x'\delta\beta\gamma = x'\alpha = \tilde{P}\alpha_*$$

which implies that $y \in \tilde{P}$, hence $\cup\mathcal{A}_Q \subseteq \tilde{P}$. This proves that $\mathcal{A}$ refines $\pi(\alpha)$, as required. □

Theorem 2.7. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ if and only if there exists a refinement $\mathcal{A}$ of $\pi(\alpha)$ and $\varphi : \mathcal{A} \to \pi(\beta)$ such that φ is an injection and for every $P \in \mathcal{A}$, $P, P\varphi \subseteq A$ for some $A \in X/E$.*

Proof. Assume that $\alpha \in T_{SE}(X)\beta T_{SE}(X)$. Then $\alpha = \delta\beta\gamma$ for some $\delta, \gamma \in T_{SE}(X)$. Let $\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta) \text{ and } Q \cap X\delta \neq \emptyset\}$ where $\mathcal{A}_Q = \{P \in \pi(\alpha) : P\delta_* \in Q\}$. Then by Lemma 2.6, $\mathcal{A}$ is a refinement of $\pi(\alpha)$. Define $\varphi : \mathcal{A} \to \pi(\beta)$ by $(\cup\mathcal{A}_Q)\varphi = Q$. It is clear that φ is well-defined. Suppose that $(\cup\mathcal{A}_Q)\varphi = (\cup\mathcal{A}_{Q'})\varphi$. By the definition of φ, $Q = Q'$. Thus $\mathcal{A}_Q = \mathcal{A}_{Q'}$ and so φ is an injection. Let $\cup\mathcal{A}_Q \in \mathcal{A}$ where $Q \in \pi(\beta)$. Since $\pi(\beta)$ refines X/E, $Q \subseteq A$ for some $A \in X/E$. For each $P \in \mathcal{A}_Q$, we have $P\delta_* \in Q$ and so $P\delta_* \in A$. Hence $P \subseteq A$ because $\delta \in T_{SE}(X)$. Thus $\cup\mathcal{A}_Q \subseteq A$, hence $(\cup\mathcal{A}_Q)\varphi = Q \subseteq A$.

Conversely, suppose that $\varphi : \mathcal{A} \to \pi(\beta)$ is an injection where $\mathcal{A}$ is a refinement of $\pi(\alpha)$ and for every $P \in \mathcal{A}$, $P, P\varphi \subseteq A$ for some $A \in X/E$. Let $x \in X$. Then $x \in P$ for some $P \in \mathcal{A}$, we choose and fix an element $\tilde{x} \in P\varphi$. We define $\delta : X \to X$ by $x\delta = \tilde{x}$ for all $x \in X$. By assumption, there exists $A \in X/E$ such that $P, P\varphi \subseteq A$ which implies that $\delta \in T_{SE}(X)$. Since $\beta_* : \pi(\beta) \to X\beta$ is an injection and by assumption, $\varphi\beta_* : \mathcal{A} \to X\beta$ is an injection. For each $x \in \mathcal{A}\varphi\beta_*$, there exists a unique $P_x \in \mathcal{A}$ such that $x = P_x\varphi\beta_*$. We fix $x' \in P_x$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x'\alpha & \text{if } x \in \mathcal{A}\varphi\beta_*; \\ x & \text{otherwise.} \end{cases}$$

Let $x \in X$. If $x \notin \mathcal{A}\varphi\beta_*$, then $(x, x\gamma) = (x, x) \in E$. If $x \in \mathcal{A}\varphi\beta_*$, then $x = P_x\varphi\beta_*$ for some $P_x \in \mathcal{A}$. By Proposition 1.1, there is $A \in X/E$ such that $P_x\varphi \subseteq A$. It follows from assumption that $P_x \subseteq A$. We conclude that $x = P_x\varphi\beta_* \in A\beta \subseteq A$ by Proposition 1.2. From $(x', x'\alpha) \in E$ and $x' \in A$, we then have $x'\alpha \in A$. Thus $(x, x\gamma) = (x, x'\alpha) \in E$. This shows that $\gamma \in T_{SE}(X)$. We show that $\alpha = \delta\beta\gamma$. Let $x \in X$. Then $x \in P$ for some $P \in \mathcal{A}$. Since $\tilde{x}\beta = P\varphi\beta_* \subseteq \mathcal{A}\varphi\beta_*$, we conclude that $P\varphi\beta_* = \tilde{x}\beta = P_{\tilde{x}\beta}\varphi\beta_*$. It follows from $\varphi\beta_*$ is injective that $P = P_{\tilde{x}\beta}$. Hence $x, (\tilde{x}\beta)' \in P$. Since $\mathcal{A}$ is a refinement of $\pi(\alpha)$, there exists $P' \in \pi(\alpha)$ such that $P \subseteq P'$. Thus $x, (\tilde{x}\beta)' \in P'$ which implies that $x\alpha = (\tilde{x}\beta)'\alpha$. It would follow that

$$x\delta\beta\gamma = \tilde{x}\beta\gamma = (\tilde{x}\beta)'\alpha = x\alpha.$$

Therefore, the theorem is completely proved. □

Corollary 2.8. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ if and only if there exists an injection $\varphi' : \pi(\alpha) \to \pi(\beta)$ such that for every $P \in \pi(\alpha)$, $P, P\varphi' \subseteq A$ for some $A \in X/E$.*

Proof. Assume that $\alpha \in T_{SE}(X)\beta T_{SE}(X)$. By Theorem 2.7 that there exist a refinement $\mathcal{A}$ of $\pi(\alpha)$ and $\varphi : \mathcal{A} \to \pi(\beta)$ such that φ is an injection and for every $P \in \mathcal{A}$, $P, P\varphi \subseteq A$ for some $A \in X/E$. For each $P \in \pi(\alpha)$, we choose and fix $P' \in \mathcal{A}$ such that $P' \subseteq P$. Define $\varphi' : \pi(\alpha) \to \pi(\beta)$ by

$$P\varphi' = P'\varphi \text{ for all } P \in \pi(\alpha).$$

It is easy to see that φ is well-defined. Let $P, Q \in \pi(\alpha)$ be such that $P\varphi' = Q\varphi'$. Hence $P'\varphi = Q'\varphi$. By φ is injective, $P' = Q'$. Since $\pi(\alpha)$ is a partition of X, $P = Q$. Thus φ' is injective. Let $P \in \pi(\alpha)$. We then have $P', P'\varphi \subseteq A$ for some

$A \in X/E$ by Theorem 2.7, whence $P \cap A \neq \emptyset$. From Proposition 1.1, we deduce that $P, P\varphi' \subseteq A$, as required. □

Corollary 2.9. *Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{J}$ if and only if there exist injections $\psi : \pi(\alpha) \to \pi(\beta)$ and $\psi' : \pi(\beta) \to \pi(\alpha)$ such that for every $P \in \pi(\alpha)$ and $Q \in \pi(\beta)$, $P, P\psi \subseteq A$ and $Q, Q\psi' \subseteq A'$ for some $A, A' \in X/E$.*

The remaining matter is only to consider the relation $\mathcal{D}$.

Lemma 2.10. *Let $\alpha, \beta \in T_{SE}(X)$ and $A \in X/E$. If $\varphi : \pi_A(\beta) \to \pi_A(\alpha)$ is a bijection, then there exists $\delta_A : A \to X$ satisfies $\pi(\delta_A) = \pi_A(\beta)$ and $A\delta_A = A\alpha$.*

Proof. Assume that $\varphi : \pi_A(\beta) \to \pi_A(\alpha)$ is bijective. Let $x \in A$. By Proposition 1.3, there exists $P_x \in \pi_A(\beta)$ such that $x \in P_x$. We define $\delta_A : A \to X$ by

$$x\delta_A = (P_x\varphi)\alpha_* \text{ for all } x \in A.$$

It is clear that δ_A is well-defined. We then have $\cup\pi_A(\beta) = A = \cup\pi(\delta_A)$ by Proposition 1.3. Claim that $\pi_A(\beta) = \pi(\delta_A)$. From the definition of δ_A, we note that for every $P \in \pi_A(\beta)$, $P\delta_A = \{P\varphi\alpha_*\}$, hence $P \subseteq Q$ for some $Q \in \pi(\delta_A)$. For each $Q \in \pi(\delta_A)$, $Q\delta_{A*} = P\varphi\alpha_*$ for some $P \in \pi_A(\beta)$. Next, to show $Q \subseteq P$, let $x \in Q$. Then there exists $P_x \in \pi_A(\beta)$ such that $x \in P_x$. Hence $Q\delta_{A*} = x\delta_A = P_x\varphi\alpha_*$, we deduce that $P\varphi\alpha_* = P_x\varphi\alpha_*$. Since $\varphi\alpha_*$ is a bijection, $P = P_x$. Hence $x \in P$, so $Q \subseteq P$. Therefore we conclude that $\pi_A(\beta) = \pi(\delta_A)$. To show that $A\delta_A = A\alpha$, let $x \in A$. Then $x \in P_x$ for some $P_x \in \pi_A(\beta)$, whence $x\delta_A = P_x\varphi\alpha_* \in A\alpha$. Thus $A\delta_A \subseteq A\alpha$. For the reverse inclusion, let $y \in A$. Since $y\alpha \in A\alpha$, $y\alpha = P\alpha_*$ for some $P \in \pi_A(\alpha)$. Since φ is bijective, there exists a unique $P' \in \pi_A(\beta)$ such that $P'\varphi = P$. Choose $z \in P'$, we have that $z\delta_A = P'\varphi\alpha_* = P\alpha_* = y\alpha$ which implies that $y\alpha \in A\delta_A$. Hence $A\alpha \subseteq A\delta_A$ as we wished to show. □

Theorem 2.11. *Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if for every $A \in X/E$, there exists a bijection $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$.*

Proof. Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then there exists $\delta \in T_{SE}(X)$ such that $(\alpha, \delta) \in \mathcal{L}$ and $(\delta, \beta) \in \mathcal{R}$. Let $A \in X/E$. For each $P \in \pi_A(\beta)$, we then have $P \in \pi(\beta)$ and $P \cap A \neq \emptyset$. By Corollary 2.4, we have $\pi(\delta) = \pi(\beta)$, so $P \in \pi(\delta)$. Since $\pi(\delta)$ refines X/E and $P \cap A \neq \emptyset$, we deduce that $P \subseteq A$, hence $P\delta_* \in A\delta$. From Corollary 2.2, we obtain that $A\alpha = A\delta$, that is $P\delta_* \in A\alpha \subseteq X\alpha$. Then there exists $Q_P \in \pi(\alpha)$ such that $Q_P\alpha_* = P\delta_*$. Since $Q_P\alpha_* \in A$ and $\alpha \in T_{SE}(X)$, $Q_P \subseteq A$. Then $Q_P \in \pi_A(\alpha)$. Define $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$ by

$$P\varphi_A = Q_P \qquad \text{for all } P \in \pi_A(\beta).$$

If $Q' \in \pi_A(\alpha)$ is such that $Q'\alpha_* = P\delta_*$, then $Q_P = Q'$ because $Q_P\alpha_* = Q'\alpha_*$. This shows that φ_A is well-defined. Suppose that $P\varphi_A = P'\varphi_A$. Then $Q_P = Q_{P'}$, hence $P\delta_* = Q_P\alpha_* = Q_{P'}\alpha_* = P'\delta_*$ which implies that $P = P'$. Therefore φ is an injection. Claim that φ_A is onto, let $Q \in \pi_A(\alpha)$. Then $Q\alpha_* \in A\alpha = A\delta$.

Thus there exists $P \in \pi(\delta)$ such that $P\delta_* = Q\alpha_*$. Since $\delta \in T_{SE}(X)$, $P \subseteq A$. By $\pi(\delta) = \pi(\beta)$, we have $P \in \pi(\beta)$ and so $P \in \pi_A(\beta)$. Thus $P\varphi_A = Q$, so we have the claim. Therefore φ_A is bijective, as required.

Conversely, for every $A \in X/E$, there exists a bijection $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$. It follows from Lemma 2.10 that there exists $\delta_A : A \to X$ corresponding to $A \in X/E$ such that $\pi_A(\beta) = \pi(\delta_A)$ and $A\delta_A = A\alpha$. Thus we define $\delta : X \to X$ by $\delta|_A = \delta_A$ for all $A \in X/E$. Since X/E is a partition of X, δ is well-defined. We note that for each $A \in X/E$, $A\delta = A\delta_A = A\alpha \subseteq A$ by $\alpha \in T_{SE}(X)$, hence $\delta \in T_{SE}(X)$. Finally, we can see that

$$\pi(\beta) = \bigcup_{A \in X/E} \pi_A(\beta) = \bigcup_{A \in X/E} \pi(\delta_A) = \bigcup_{A \in X/E} \pi_A(\delta) = \pi(\delta),$$

it follows by Corollary 2.4 that $(\delta, \beta) \in \mathcal{R}$. Since $A\delta = A\delta|_A = A\delta_A = A\alpha$, we deduce that $(\alpha, \delta) \in \mathcal{L}$ by Corollary 2.2. Therefore $(\alpha, \beta) \in \mathcal{L} \circ \mathcal{R} = \mathcal{D}$.

Hence the theorem is completely proved. □

3 Regularity for the Self-E-Order Preserving Transformation Semigroups

Proposition 3.1. *The semigroup $T_{SE}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in T_{SE}(X)$. Then for each $x \in X\alpha$, there exists $P_x \in \pi(\alpha)$ such that $P_x\alpha_* = x$. We choose an element $x' \in P_x$, whence $x'\alpha = x$ for all $x \in X\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha; \\ x & \text{otherwise.} \end{cases}$$

Let $x \in X$. If $x \notin X\alpha$, then $(x, x\beta) = (x, x) \in E$. If $x \in X\alpha$, then $(x, x\beta) = (x, x') = (x'\alpha, x') \in E$. We also have that

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha.$$

These show that $\beta \in T_{SE}(X)$ and $\alpha = \alpha\beta\alpha$, respectively. Therefore α is a regular element of $T_{SE}(X)$. □

Theorem 3.2. *Let $\alpha \in T_{SE}(X)$. Then α is left regular if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.*

Proof. Assume that α is a left regular element of $T_{SE}(X)$. Then $\alpha = \beta\alpha^2$ for some $\beta \in T_{SE}(X)$. Let $P \in \pi(\alpha)$. Then $P\alpha_* = y$ for some $y \in X\alpha$. For each $x \in P$, we have that $x\alpha = y$. Hence

$$y = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha.$$

This implies that $x\beta\alpha \in y\alpha^{-1} = P$ and $x\beta\alpha \in X\alpha$. Therefore $P \cap X\alpha \neq \emptyset$.

Conversely, suppose that $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. For each $x \in X$, there exists $P \in \pi(\alpha)$ such that $x \in P$. By assumption, we choose and fix $y_x \in P \cap X\alpha$. So $x\alpha = y_x\alpha$. Since $y_x \in X\alpha$, we can fix $y'_x \in X$ such that $y'_x\alpha = y_x$. Define $\beta : X \to X$ by

$$x\beta = y'_x \quad \text{for all } x \in X.$$

Since $\pi(\alpha)$ is a partition of X, β is well-defined. To show that $\beta \in T_{SE}(X)$, let $x \in X$. Since $(x, x\alpha), (y_x, y_x\alpha) \in E$ and E is transitive, $(x, y_x) \in E$. Since $(y'_x, y_x) = (y'_x, y'_x\alpha) \in E$, it then follows by transitive of E that, $(x, x\beta) = (x, y'_x) \in E$. Hence $\beta \in T_{SE}(X)$. And

$$x\beta\alpha^2 = y'_x\alpha^2 = (y'_x\alpha)\alpha = y_x\alpha = x\alpha.$$

Thus α is a left regular element of $T_{SE}(X)$. □

Theorem 3.3. *Let $\alpha \in T_{SE}(X)$. Then α is right regular if and only if $\alpha|_{X\alpha}$ is an injection.*

Proof. Assume that α is a right regular element of $T_{SE}(X)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T_{SE}(X)$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Since $x, y \in X\alpha$, $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Hence

$$x = x'\alpha = x'\alpha^2\beta = (x'\alpha)\alpha\beta = (x\alpha)\beta = (y\alpha)\beta = (y'\alpha)\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

This proves that $\alpha|_{X\alpha}$ is injective.

For the converse, assume that $\alpha|_{X\alpha}$ is an injection. We construct $\beta \in T_{SE}(X)$ such that $\alpha = \alpha^2\beta$. For any $x \in X\alpha^2$, we choose $x' \in X\alpha$ such that $x = x'\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha^2; \\ x & \text{otherwise.} \end{cases}$$

It is easy to verify that $\beta \in T_{SE}(X)$. Let $x \in X$. Then $x\alpha = y$ for some $y \in X\alpha$ and $y\alpha \in X\alpha^2$. Then there exists $(y\alpha)' \in X\alpha$ such that $(y\alpha)'\alpha = y\alpha$. It follows by assumption that $(y\alpha)' = y$. Thus $x\alpha^2\beta = (x\alpha)\alpha\beta = (y\alpha)\beta = (y\alpha)' = y = x\alpha$. Hence α is a right regular element of $T_{SE}(X)$. □

Theorem 3.4. *Let $\alpha \in T_{SE}(X)$. Then α is completely regular if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$.*

Proof. Suppose that α is a completely regular element of $T_{SE}(X)$. Then α is left regular and right regular. By Theorem 3.2, $P \cap X\alpha \neq \emptyset$. It follows from Theorem 3.3, $|P \cap X\alpha| = 1$.

Conversely, assume that $|P \cap X\alpha| = 1$ for all $P \in \pi(\alpha)$. For each $P \in \pi(\alpha)$, by assumption, there exists a unique $x_P \in P \cap X\alpha$. Since $x_P \in X\alpha$, $P'\alpha_* = x_P$ for some $P' \in \pi(\alpha)$. Similarly, there is a unique $x_{P'} \in P' \cap X\alpha$ and hence $x_{P'}\alpha = x_P$. Define $\beta : X \to X$ by

$$x\beta = x_{P'} \quad \text{for all } x \in P \text{ and for each } P \in \pi(\alpha).$$

Then $\beta \in T_X$. To show that $\beta \in T_{SE}(X)$, let $x \in X$. Thus $x \in P$ for some $P \in \pi(\alpha)$. By assumption $x\alpha = x_P\alpha$ where $x_P \in P \cap X\alpha$, so $(x, x_P\alpha) = (x, x\alpha) \in E$. Since $(x, x_P\alpha), (x_P, x_P\alpha) \in E$, $(x, x_P) \in E$ by transitive of E. Then we obtain that $(x, x_P) = (x, x_{P'}\alpha)$. Since $(x_{P'}, x_{P'}\alpha) \in E$, it follows that $(x, x_{P'}) \in E$. We deduce that $(x, x\beta) = (x, x_{P'}) \in E$. Thus $\beta \in T_{SE}(X)$. Finally, to show that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, let $x \in X$. Hence $x\alpha = x_P$ for some $x_P \in P \cap X\alpha$ where $P \in \pi(\alpha)$. Then $x\alpha\beta\alpha = x_P\beta\alpha = x_{P'}\alpha = x_P = x\alpha$. Moreover, we get $x \in P'$ where $P' \in \pi(\alpha)$ and $P'\alpha_* = x_P$. Then there exists a unique $x_{P''} \in P'' \cap X\alpha$ where $P'' \in \pi(\alpha)$ and $x_{P''}\alpha = x_{P'}$. By the definition of β, we have $x\beta = x_{P''}$. Also, we have that $x\alpha\beta = x_P\beta = x_{P'} == x_{P''}\alpha = x\beta\alpha$. These mean that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$. Therefore α is completely regular of $T_{SE}(X)$. □

Acknowledgement : The author would like to thank the referees for valuable comments and suggestions which improved the paper.

References

[1] H. Pei, Regularity and Green's relations for semigroups of transformations that preserve an equivalence, Communications in Algebra 33 (2005) 109-118.

[2] H. Pei, D. Zou, Green's equivalences on semigroups of transformations preserving order and an equivalence relation, Semigroup Forum 71 (2005) 241-251.

[3] M. Ma, T. You, Sh. Luo, Y. Yang, L. Wang, Regularity and Green's relations for finite E-order-preserving transformations semigroups, Semigroup Forum 80 (2010) 164-173.

[4] W. Mora, Y. Kemprasit, Regular elements of some order-preserving partial transformations, International Mathematical Forum 48 (2010) 2381-2385.

[5] W. Mora, Y. Kemprasit, Regular transformation semigroups on some dictionary chains, Thai Journal of Mathematics (2007) 87-93.

[6] N. Sirasuntorn, Y. Kemprasit, Left regular and right regular elements of semigroups of 1-1 transformations and 1-1 linear transfromations, International Journal of Algebra 28 (2010) 1399-1406.

[7] C. Namnak, E. Laysirikul, A note on the regularity for semigroups of self-E-preserving transformations, AGRC Conference Proceedings, 2012, Thailand, March 1-2, 2012, pp. ST164-167.

[8] J.M. Howie, Fundamentals of semigroup theory, Oxford University Press Inc., New York, 1995.

(Received 2 April 2017)
(Accepted 8 July 2017)