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All Maximal Submonoids of Special Regular Classes of $Hyp_G(2)$

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Abstract : The set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ forms a monoid. The purposes of this paper are to determine all maximal submonoids of special regular classes of the set of generalized hypersubstitutions of type $\tau = (2)$.

Keywords : generalized hypersubstitution; maximal submoniod.
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1 Introduction

In theoretical computer science, automata and languages theory are the very important role in this field. The study of mathematical properties of such automata is automata theory. Tree transducers are generalization of automata and tree transformations defined by hypersubstitutions can be realized by tree transducers. The composition of tree transformations is used in computer science to translate a formal language into another one, step by step, with some language in between. Languages are sets of words. What is the word? Let $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-elements set of *letters*. We think of X_n as an *alphabet*. Then a *word* over the alphabet X_n is any letter or any finite string of letters. We can write this definition inductively:

(i) Each letter $x_i \in X_n$ is a word over X_n.

(ii) If t is a word over X_n and x_j is in X_n, then both x_jt and tx_j are words over X_n.

The next, we will give this language in the general setting. Let $n \in \mathbb{N}$ and $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-elements set. The set X_n is called an *alphabet* and its elements are called *variables*. We also need a set $\{f_i \mid i \in I\}$ of operation symbols of type τ, indexed by the set I. The set X_n and $\{f_i \mid i \in I\}$ have to be disjoint. An *n-ary term* of type τ is defined inductively by

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term of type τ.

We denote the smallest set of which contains $x_1, \ldots, x_n$ and is closed under finite number of applications of (ii) by $W_\tau(X_n)$ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ. The useful of terms not only allows us to use concepts and results from semigroup theory to study algebraic structures properties of hypersubstitutions but also use to study especially automata and languages in computer science theory. Sequences of tree transformations offer a convenient method to describe various manipilations that are commonly performed by compilers and language-based editors. If the considered tree transformations are described by certain mappings defined on the set of all terms, then the sequences of tree transformations can be described by products of such mappings. We can consider tree transformation by using of hypersubstitutions and generalized hypersubstitutions. This allows us to describe algebraic properties of set of tree transformations by algebraic properties of the set of all generalized hypersubstitutions.

In 2000, S. Leeratanavalee and K. Denecke [1] generalized the concepts of hypersubstitutions, hyperidentities to generalized hypersubstitutions, strong hyperidentities and studied its algebraic properties. The set of all generalized hypersubstitutions of type τ forms a monoid. In 2014, W. Wonpinit and S. Leeratanavalee [2] determined all maximal idempotent submonoids of $Hyp_G(2)$. In this work, we determine all maximal submonoids of special regular classes of $Hyp_G(2)$.

2 Preliminaries

Our basic concept is the concept of a generalized hypersubstitution. A *generalized hypersubstitution of type τ* is a mapping σ from the set $\{f_i \mid i \in I\}$ into the set $W_\tau(X)$ which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we need the concept of a generalized superposition of terms which is a mapping $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ defined by the following steps:

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Then the generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma}: W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then, the binary operation of two generalized hypersubstitutions σ_1, σ_2 is defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. It turns out that $Hyp_G(\tau)$ is a monoid under $\circ_G$ and the identity element σ_{id} which $\sigma_{id}(f_i) = f_i(x_1, \dots, x_{n_i})$, see [1].

Proposition 2.1. [1] *For arbitrary $t, t_1, t_2, \dots, t_n \in W_\tau(X)$ and for any generalized hypersubstitution $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

3 Main Results

Let S be any semigroup. Recall that an element a in a semigroup S is called *regular* if there exists $b \in S$ such that $a = aba$. A semigroup S is called *regular* if every its element is regular. An element $a \in S$ is called *idempotent* if $aa = a$. We denote the set of all idempotent elements of a semigroup S by $E(S)$. It's easy to see that all idempotent element is regular element. We will introduce definition of some special regular classes of regular semigroup. A semigroup S is called *coregular* if for each $a \in S, a = aba = bab$ for some $b \in S$; S is *anti-regular* if $aba = b$ and $bab = a$ for some $b \in S$; S is *completely-regular* if $a = aba$ and $ab = ba$ for some $b \in S$; S is *left (right) regular* if $ba^2 = a(a^2b = a)$ for some $b \in S$; and S is *intra-regular* if $a \in Sa^2S$. Throughout this paper, let f be a binary operation symbol of type $\tau = (2)$. By σ_t we denote a generalized hypersubstitution which maps f to the term $t \in W_{(2)}(X)$. For $t \in W_{(2)}(X)$ we introduce the following notation :

(i) $leftmost(t)$:= the first variable (from the left) occurring in t,

(ii) $rightmost(t)$:= the last variable occurring in t,

(iii) $var(t)$:= the set of all variables occurring in t.

Let $\sigma_t \in Hyp_G(2)$, we denote

$R_1 := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ and $x_2 \notin var(t')\}$, $R_2 := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $x_1 \notin var(t')\}$, $R_3 := \{\sigma_t | t \in \{x_1, x_2, f(x_1, x_2)\}\}$ and $R_4 := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2010, W. Puninagool and S. Leeratanavalee [3] proved that : $\bigcup_{i=1}^{4} R_i = E(Hyp_G(2))$.

Let $\sigma_t \in Hyp_G(2)$, we denote
$R_1' := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$, $x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$ and $R_2' := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$, $x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$.

And denote $(MI)_{Hyp_G(2)} = R_1' \cup R_2' \cup R_3 \cup R_4$, $(MI_1)_{Hyp_G(2)} = R_1 \cup R_3 \cup R_4$, $(MI_2)_{Hyp_G(2)} = R_2 \cup R_3 \cup R_4$ and $(MSR)_{Hyp_G(2)} = \{\sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}, \sigma_{f(x_2,x_1)}\} \cup R_3 \cup R_4$.

In 2014, W. Wonpinit and S. Leeratanavalee [2] determined that the set $(MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are all of maximal idempotent submonoids of $Hyp_G(2)$. In 2014, W. Wongpinit and S. Leeratanavalee [4] proved that $E(Hyp_G(2)) \cup \{\sigma_{f(x_2,x_1)}\}$ is the set of all coregular elements, anti-regular elements, completely-regular elements, left regular elements, right regular elements, and intra-regular elements in $Hyp_G(2)$. Let S be any semigroup. A nonempty subset T of S is called a *subsemigroup* of S if $T^2 \subseteq T$. A subsemigroup T of S is called a *regular subsemigroup* if, for any element $a \in T$, there exists $b \in T$ such that $a = aba$. The next results describe the great important relationship between special regular subsemigroups of $Hyp_G(2)$.

Lemma 3.1. [4] *Let R be a subsemigroups of $Hyp_G(2)$. Then the following conditions are equivalent:*

(a) R is coregular,

(b) R is anti-regular,

(c) R is completely-regular,

(d) R is left regular,

(e) R is right regular,

(f) R is intra-regular.

Using these facts and for convenient, the classes coregular, anti-regular, completely regular, left regular, right regular, and intra-regular are called the *special regular classes of* $Hyp_G(2)$. So we are able to prove the following propositions of special regular on the monoid $Hyp_G(2)$.

Proposition 3.2. *$(MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are the maximal submonoids of the special regular classes of $Hyp_G(2)$.*

Proof. Since the set $(MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are the set of idempotent submonoids of $Hyp_G(2)$, we obtain that they are the submonoids of the special regular classes of $Hyp_G(2)$. We will show that $(MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are the maximal submonoids.

Case $(MI)_{Hyp_G(2)}$: Let K be a proper submonoid of $Hyp_G(2)$ such that $(MI)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$. Suppose that $\sigma_{f(x_2,x_1)} \in K$, choose $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_1, x_2 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_{f(x_2,x_1)})(f) &= \hat{\sigma}_t[f(x_2,x_1)] \\ &= S^2(f(x_1,t'), x_2, x_1) \\ &= f(x_2,t') \qquad \text{where } x_1, x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \notin K$ which is a contradiction. So $\sigma_{f(x_2,x_1)} \notin K$. Therefore $(MI)_{Hyp_G(2)} = K$ is a maximal submonoids of coregular, anti-regular, completely-regular, left regular, right regular, and intra-regular of $Hyp_G(2)$.

Case $(MI_1)_{Hyp_G(2)}$: Let K be a proper submonoid of $Hyp_G(2)$ such that $(MI_1)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$. Suppose that $\sigma_{f(x_2,x_1)} \in K$, choose $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_1, x_2 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_{f(x_2,x_1)})(f) &= \hat{\sigma}_t[f(x_2,x_1)] \\ &= S^2(f(x_1,t'), x_2, x_1) \\ &= f(x_2,t') \qquad \text{where } x_1, x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \notin K$ which is a contradiction. So $\sigma_{f(x_2,x_1)} \notin K$. Therefore $(MI_1)_{Hyp_G(2)} = K$ is a maximal submonoids of coregular, anti-regular, completely-regular, left regular, right regular, and intra-regular of $Hyp_G(2)$.

Case $(MI_2)_{Hyp_G(2)}$ can be proved similarly as Case $(MI_1)_{Hyp_G(2)}$. □

Corollary 3.3. *Every maximal idempotent submonoids of $Hyp_G(2)$ is the maximal submonoids of the special regular classes of $Hyp_G(2)$.*

Next, we will show that the converse of Corollary 3.3 is not true in general.

Proposition 3.4. *$(MSR)_{Hyp_G(2)}$ is a maximal submonoid of the special regular classes of $Hyp_G(2)$.*

Proof. We will show that the set $(MSR)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$. Since $\sigma_{id} \in (MSR)_{Hyp_G(2)}$, $\{\sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}, \sigma_{f(x_2,x_1)}\}(MSR)_{Hyp_G(2)}$ $\subseteq (MSR)_{Hyp_G(2)}$ and $(R_3 \cup R_4)(MSR)_{Hyp_G(2)} \subseteq (MSR)_{Hyp_G(2)}$, we have that $(MSR)^2_{Hyp_G(2)} \subseteq (MSR)_{Hyp_G(2)}$. So $(MSR)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$. We will show that $(MSR)_{Hyp_G(2)}$ is a maximal submonoid of $Hyp_G(2)$. Let K be a proper submonoid of the special regular classes of $Hyp_G(2)$ such that $(MSR)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$. If $\sigma_t \in R_2$ such that $t = f(t', x_2) \neq f(x_2, x_2)$ where $x_1 \notin var(t')$. Consider $(\sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) = \widehat{\sigma}_{f(x_2,x_1)}[f(t', x_2)] = S^2(f(x_2, x_1), \widehat{\sigma}_{f(x_2,x_1)}[t'], x_2) = f(x_2, \widehat{\sigma}_{f(x_2,x_1)}[t']) \neq f(x_2, x_2)$. So $\sigma_{f(x_2,x_1)} \circ_G \sigma_t$ is not closed which is a contradiction. so $t = f(x_2, x_2)$. If $\sigma_t \in R_1$ such that $t = f(x_1, t') \neq f(x_1, x_1)$ where $x_2 \notin var(t')$. Consider $(\sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) = \widehat{\sigma}_{f(x_2,x_1)}[f(x_1, t')] = S^2(f(x_2, x_1), x_1, \widehat{\sigma}_{f(x_2,x_1)}[t']) = f(\widehat{\sigma}_{f(x_2,x_1)}[t'], x_1) \neq f(x_1, x_1)$. We obtain $\sigma_{f(x_2,x_1)} \circ_G \sigma_t$ is not closed which is a contradiction. Hence $t = f(x_1, x_1)$. Therefore, $K = (MSR)_{Hyp_G(2)}$. □

Theorem 3.5. *The set $(MSR)_{Hyp_G(2)}, (MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are all maximal submonoids of the special regular classes of $Hyp_G(2)$.*

Proof. Let M be any maximal submonoid of the special regular classes of $Hyp_G(2)$. We consider into two cases.

Case 1: $\sigma_{f(x_2,x_1)} \notin M$. Then M is a maximal idempotent submonoid of $Hyp_G(2)$. By using Corollay 3.3 we have $M \in \{(MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}, (MI_2)_{Hyp_G(2)}\}$.

Case 2: $\sigma_{f(x_2,x_1)} \in M$. Let $\sigma_t \in M \setminus \{\sigma_{f(x_2,x_1)}\} \cup R_3 \cup R_4$. If $\sigma_t \in R_2$ such that $t = f(t', x_2)$ where $x_1 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_{f(x_2,x_1)}[f(t', x_2)] \\ &= S^2(f(x_2, x_1), \widehat{\sigma}_{f(x_2,x_1)}[t'], x_2) \\ &= f(x_2, \widehat{\sigma}_{f(x_2,x_1)}[t']) \in M.\end{aligned}$$

So $t = f(x_2, x_2)$. If $\sigma_t \in R_1$ such that $t = f(x_1, t')$ where $x_2 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_{f(x_2,x_1)}[f(x_1, t')] \\ &= S^2(f(x_2, x_1), x_1, \widehat{\sigma}_{f(x_2,x_1)}[t']) \\ &= f(\widehat{\sigma}_{f(x_2,x_1)}[t'], x_1) \in M.\end{aligned}$$

So that $t = f(x_1, x_1)$. Therefore, $\sigma_t \in (MSR)_{Hyp_G(2)}$ and then $M \subseteq (MSR)_{Hyp_G(2)}$. Since M is a maximal submonoid of coregular, anti-regular, completely-regular, left regular, right regular, and intra-regular of $Hyp_G(2)$, we obtain that $M = (MSR)_{Hyp_G(2)}$. By Case 1 and Case 2, we get that

$(MSR)_{Hyp_G(2)}, (MI)_{Hyp_G(2)}, (MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are all maximal submonoids of the special regular classes of $Hyp_G(2)$. □

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