Thai **J**ournal of **M**athematics : 97–108
Special Issue: Annual Meeting in Mathematics 2017

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Natural Partial Order on the Semigroups of Partial Isometries of a Finite Chain

Chaiwat Namnak, Ekkachai Laysirikul[1] and Nares Sawatraksa

Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
e-mail : chaiwatn@nu.ac.th (C. Namnak)
ekkachail@nu.ac.th (E. Laysirikul)
naress58@nu.ac.th (N. Sawatraksa)

Abstract : Let I_n denote the $1-1$ partial transformation semigroup on a set $\{1, 2, \ldots, n\}$ and let $DP_n = \{\alpha \in I_n : \forall x, y \in Dom\,\alpha, |x\alpha - y\alpha| = |x - y|\}$ and $ODP_n = \{\alpha \in DP_n : \forall x, y \in Dom\,\alpha, x \leq y \Rightarrow x\alpha \leq y\alpha\}$. Then DP_n and ODP_n are subsemigroups of I_n. The purpose of this research, we study the natural partial orders on DP_n and ODP_n and characterize when two elements of DP_n and ODP_n are related under this partial order. Moreover, we give a necessary and sufficient conditions for elements in DP_n and ODP_n to be maximal or minimal elements.

Keywords : natural partial order; partial isometry; transformation semigroup.
2010 Mathematics Subject Classification : 20M20.

1 Introduction

Let X be a set. A partial transformation on X is a map from a subset of X into X. The empty transformation is the partial transformation 0 with empty domain. Let I_X be the set of all $1-1$ partial transformations on X. For $\alpha \in I_X$, let $Dom\,\alpha$ and $Im\,\alpha$ denote respectively the domain and the image of α. Then I_X becomes a semigroup under the composition of maps, that is, for α, $\beta \in I_X$,

$$Dom\,\alpha\beta = \{x \in Dom\,\alpha : x\alpha \in Dom\,\beta\} \text{ and}$$

[1] Corresponding author.

$$x(\alpha\beta) = (x\alpha)\beta \text{ for all } x \in Dom\,\alpha\beta.$$

We call I_X the $1-1$ partial transformation semigroup on X. This was introduced in [1] that I_X is an inverse semigroup (that is, for every $\alpha \in I_X$ there exists a unique $\beta \in I_X$ such that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$). The study of inverse semigroups has many features in common with the theory of groups, and one of the earliest results was a representation theorem to effect that every inverse semigroup has a faithful representation as an inverse semigroup of $1-1$ partial mappings as the Vagner-Preston Theorem in [2].

Given a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way. Denote I_n the $1-1$ partial transformation semigroup on $[n]$. We call $\alpha \in I_n$ is order-preserving (order-reversing) if for every x, $y \in Dom\,\alpha$, $x \leq y$ implies $x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$) and α is isometry if for every $x, y \in Dom\,\alpha$, $|x\alpha - y\alpha| = |x-y|$. Kehinde and Umar [3] have introduced families of subsemigroups of I_n defined as follow:

$$DP_n = \{\alpha \in I_n : \forall x, y \in Dom\,\alpha, |x\alpha - y\alpha| = |x-y|\}$$

and

$$ODP_n = \{\alpha \in DP_n : \forall x, y \in Dom\,\alpha, x \leq y \text{ implies } x\alpha \leq y\alpha\}.$$

Then DP_n and ODP_n are subsemigroups of I_n which are called the semigroup of partial isometries of an n-chain and the semigroup of partial order-preserving isometries of an n-chain, respectively. Green's relations on DP_n and ODP_n have been investigated by Kehinde and Umar [3] and the order of the set of idempotent elements of DP_n has been discussed by Kehinde and Adeshola [4].

As one can easily see, the following lemma holds:

Lemma 1.1. [3] *DP_n and ODP_n are inverse semigroups.*

In 1952, Vagner [2] defined a partial order relation on an inverse semigroup S in a natural way as follows: for a, $b \in S$,

$$a \leq b \text{ if and only if } a = be \text{ for some } e \in E(S)$$

where $E(S) = \{x \in S : x^2 = x\}$. Of course, this relation is indeed a partial order which is called the natural partial order on an inverse semigroup S. Also, the order relation $\leq$ is in fact compatible with multiplication of S, in the sense that

$$a \leq b \text{ and } c \in S \text{ imply that } ac \leq bc \text{ and } ca \leq cb.$$

It was proved very useful in the theory of inverse semigroups.

The purpose of this paper, we discuss the natural partial orders on both inverse semigroups, namely, DP_n and ODP_n and characterize when two elements of DP_n and ODP_n are related under this partial order. Also, their maximal and minimal elements of each semigroup are described.

Throughout of this paper, let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way and for any x, $y \in [n]$ with $x \leq y$, the set $[x, y] = \{z \in [n] : x \leq z \leq y\}$ is called a *closed interval* of $[n]$.

2 Main Results

In this section, we present the characterization of the natural partial orders on DP_n and ODP_n, respectively. Let S be any one of I_n, DP_n and ODP_n. Since S is an inverse semigroup, the natural partial order on S are defined by

$$\alpha \leq \beta \text{ if and only if } \alpha = \beta\mu \text{ for some } \mu \in E(S)$$

for all α, $\beta \in S$.

We first need the following result is quoted.

Theorem 2.1. [5] *Let $\alpha \in I_n$. Then α is an idempotent if and only if $Im\,\alpha \subseteq Dom\,\alpha$ and the restriction $\alpha|_{Im\,\alpha}$ is the identity transformation on $Im\,\alpha$.*

The following theorem investigates the condition when $\alpha \leq \beta$ for α, $\beta \in S$ where S is any one of I_n, DP_n and ODP_n.

Theorem 2.2. *Let S be any one of I_n, DP_n and ODP_n. Then $\alpha \leq \beta$ on S if and only if $\alpha \subseteq \beta$.*

Proof. Suppose that $\alpha \leq \beta$ on S. Then there exists $\mu \in E(S)$ such that $\alpha = \beta\mu$. Thus $Im\,\alpha \subseteq Im\,\mu$. To verify that $\alpha \subseteq \beta$, let $(x, y) \in \alpha$. Then $y = x\alpha \in Im\,\alpha \subseteq Im\,\mu$. Since $\mu \in E(S)$, we deduce that $y\mu = y$ by Theorem 2.1. Hence $y\mu = y = x\alpha = x\beta\mu$. Since μ is $1-1$, it follows that $x\beta = y$ and hence $(x, y) \in \beta$ as required.

Conversely, assume that $\alpha \subseteq \beta$. Then $x\alpha = x\beta$ for all $x \in Dom\,\alpha$. Define $\mu : Im\,\alpha \to [n]$ by $x\mu = x$ for all $x \in Im\,\alpha$. Obviously, $\mu \in S$ and μ is an idempotent by Theorem 2.1. By assumption and β is $1-1$, we obtain that

$$Dom\,(\beta\mu) = (Im\,\beta \cap Dom\,\mu)\beta^{-1} = (Im\,\beta \cap Im\,\alpha)\beta^{-1} = Dom\,\alpha.$$

Let $x \in Dom\,\alpha$. Then $x\alpha \in Im\,\alpha = Dom\,\mu$ and $x\alpha = x\beta$. Thus $x\alpha = x\alpha\mu = x\beta\mu$. This proves that $\alpha = \beta\mu$ and consequently $\alpha \leq \beta$. □

Next, we investigate the condition under which an element of DP_n and ODP_n to be maximal and minimal with respect to the natural partial order. For convenience, we quote the following result.

Lemma 2.3. [2] *Let $\alpha \in DP_n$. Then α is either order-preserving or order-reversing.*

By the above lemma, we get the following results immediately.

Lemma 2.4. *Let $\alpha \in DP_n$ and x, $y \in Im\,\alpha$.*

1. *If α is order-preserving and $x < y$, then $x\alpha^{-1} < y\alpha^{-1}$.*
2. *If α is order-reversing and $x < y$, then $x\alpha^{-1} > y\alpha^{-1}$.*

Lemma 2.5. *Let* $\alpha \in DP_n$*. Then either*

$$(\min(Dom\,\alpha))\alpha = \min(Im\,\alpha) \text{ and } (\max(Dom\,\alpha))\alpha = \max(Im\,\alpha) \text{ or}$$

$$(\min(Dom\,\alpha))\alpha = \max(Im\,\alpha) \text{ and } (\max(Dom\,\alpha))\alpha = \min(Im\,\alpha).$$

As above result motivates the following theorem.

Proposition 2.6. *Let* $\alpha \in DP_n$*. If* $|Dom\,\alpha| = n$*, then* α *is either the identity mapping or* $x\alpha = n + 1 - x$ *for all* $x \in Dom\,\alpha$*.*

Proof. Suppose that $|Dom\,\alpha| = n$. Then $Im\,\alpha = Dom\,\alpha = [n]$. By Lemma 2.5, we deduce that either $n\alpha = n$ or $n\alpha = 1$.

Case 1. $n\alpha = 1$. Let $x \in [n]$. Since α is isometry, we have that

$$x\alpha - n\alpha = |x\alpha - n\alpha| = |x - n| = n - x.$$

Therefore $x\alpha = n - x + n\alpha = n - x + 1$ for all $x \in [n]$.

Case 2. $n\alpha = n$. Let $x \in [n]$. Since α is isometry, we obtain that

$$n\alpha - x\alpha = |n\alpha - x\alpha| = |n - x| = n - x = n\alpha - x.$$

Therefore $x\alpha = x$ for all $x \in [n]$. □

Before we state our main result, it pays to prove the following lemma.

Lemma 2.7. *Let* $\alpha \in DP_n$*. If* $Dom\,\alpha$ *is a closed interval, then* $Im\,\alpha$ *is also closed.*

Proof. Suppose that $Dom\,\alpha$ is a closed interval of $[n]$. Then $Dom\,\alpha = [k, k+t]$ for some $k \in [n]$ and $0 \leq t \leq n - k$. If $t = 0$, then $Im\,\alpha$ is closed. Suppose that $t > 0$. Since α is $1-1$, we have $|Im\,\alpha| = |Dom\,\alpha| = t + 1$. By Lemma 2.3, α is order-preserving or α is order-reversing. If α is order-preserving, then $k\alpha < (k+1)\alpha < \ldots < (k+t)\alpha$. Since $\alpha \in DP_n$, for each $i \in [1, t]$,

$$(k+i)\alpha - k\alpha = |(k+i)\alpha - k\alpha| = |(k+i) - k| = i.$$

Thus $(k+i)\alpha = k\alpha + i$ for all $i \in [1, t]$. Therefore, $Im\,\alpha = [k\alpha, k\alpha + t]$ as required. Similarly, we can show that $Im\,\alpha = [(k+t)\alpha, (k+t)\alpha + t]$ by using the fact that α is order-reversing. □

Now coming back to our purpose.

Theorem 2.8. *Let* $\alpha \in DP_n$*. If* $|Dom\,\alpha| = n$*, then* α *is maximal.*

Proof. Suppose that $|Dom\,\alpha| = n$. Then $Dom\,\alpha = [n]$. Let $\beta \in DP_n$ be such that $\alpha \leq \beta$. By Theorem 2.2, we have that $\alpha \subseteq \beta$ and hence $Dom\,\beta = [n]$. By assumption and α is $1-1$, we deduce that $\alpha = \beta$. Hence α is a maximal element of DP_n as desired. □

Theorem 2.9. *Let $\alpha \in DP_n$ and $|Dom\,\alpha| < n$. Then α is maximal if and only if $Dom\,\alpha$ is closed and one of the following occurs:*

1. $1 \in Dom\,\alpha$, $1 \in Im\,\alpha$ *and* $1\alpha \neq 1$.
2. $1 \in Dom\,\alpha$, $n \in Im\,\alpha$ *and* $1\alpha \neq n$.
3. $n \in Dom\,\alpha$, $1 \in Im\,\alpha$ *and* $n\alpha \neq 1$.
4. $n \in Dom\,\alpha$, $n \in Im\,\alpha$ *and* $n\alpha \neq n$.

Proof. Assume that α is a maximal element in DP_n. Suppose that $Dom\,\alpha$ is not closed. Then there exists $a \in [\min(Dom\,\alpha), \max(Dom\,\alpha)] \setminus Dom\,\alpha$. Without loss of generality, we may assume that α is order-preserving.

Let $k = \max(Dom\,\alpha) - a$. By Lemma 2.5, we have that

$$\begin{aligned} k &< \max(Dom\,\alpha) - \min(Dom\,\alpha) \\ &= |\max(Dom\,\alpha) - \min(Dom\,\alpha)| \\ &= |(\max(Dom\,\alpha))\alpha - (\min(Dom\,\alpha))\alpha| \\ &= (\max(Dom\,\alpha))\alpha - (\min(Dom\,\alpha))\alpha \\ &= (\max(Dom\,\alpha))\alpha - \min(Im\,\alpha). \end{aligned}$$

Hence $\min(Im\,\alpha) < (\max(Dom\,\alpha))\alpha - k < \max(Im\,\alpha)$. Let $b = (\max(Dom\,\alpha))\alpha - k$. Then $b \in [\min(Im\,\alpha), \max(Im\,\alpha)]$ and

$$|\max(Dom\,\alpha) - a| = |(\max(Dom\,\alpha))\alpha - b|.$$

To show that $b \notin Im\,\alpha$, suppose not. Then there exists $x \in Dom\,\alpha$ such that $x\alpha = b$. Since $\alpha \in DP_n$, we have that

$$\begin{aligned} \max(Dom\,\alpha) - x &= |\max(Dom\,\alpha) - x| \\ &= |(\max(Dom\,\alpha))\alpha - b| \\ &= |\max(Dom\,\alpha) - a| = \max(Dom\,\alpha) - a. \end{aligned}$$

We infer that $x = a$, a contradiction. Hence $b \notin Im\,\alpha$.

Define $\beta : Dom\,\alpha \cup \{a\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ b & \text{if } x = a. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$. This is a contradiction since α is maximal. Hence $Dom\,\alpha$ is a closed subset of $[n]$. Similarly, we can prove that the domain of an order reversing mapping is a closed inteval of $[n]$.

If $1, n \in Dom\,\alpha$, then $Dom\,\alpha = [n]$ which is a contradiction. Hence $1 \notin Dom\,\alpha$ or $n \notin Dom\,\alpha$. We will verify that either $1 \in Dom\,\alpha$ or $n \in Dom\,\alpha$.

Suppose that $1, n \notin Dom\,\alpha$. Since $\alpha \in DP_n$, we have that 1 and n are not all elements in $Im\,\alpha$.

Case 1. $1 \notin Im\,\alpha$ and $n \in Im\,\alpha$.

Subcase 1.1. $(\max(Dom\,\alpha))\alpha = n$. Define $\beta : Dom\,\alpha \cup \{1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ n - (\max(Dom\,\alpha) - 1) & \text{if } x = 1. \end{cases}$$

Since $1 \notin Dom\,\alpha$, we obtain that $n - (\max(Dom\,\alpha) - 1) \notin Im\,\alpha$. For each $x \in Dom\,\alpha$, we deduce $x\alpha - n = x - \max(Dom\,\alpha)$. Hence $|x\beta - 1\beta| = |x - 1|$ for all $x \in Dom\,\alpha$. Then $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Subcase 1.2. $(\min(Dom\,\alpha))\alpha = n$. Define $\beta : Dom\,\alpha \cup \{n\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ \min(Dom\,\alpha) & \text{if } x = n. \end{cases}$$

Since $n \notin Dom\,\alpha$, we then have $\min(Dom\,\alpha) \notin Im\,\alpha$. Each $x \in Dom\,\alpha$, we get $n - x\alpha = x - \min(Dom\,\alpha)$. Then we can show that $|x\beta - n\beta| = |x - n|$ for all $x \in Dom\,\alpha$, hence $\beta \in DP_n$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Case 2. $1 \in Im\,\alpha$ and $n \notin Im\,\alpha$.

Subcase 2.1. $(\max(Dom\,\alpha))\alpha = 1$. Define $\beta : Dom\,\alpha \cup \{1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ \max(Dom\,\alpha) & \text{if } x = 1. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Subcase 2.2. $(\min(Dom\,\alpha))\alpha = 1$. Define $\beta : Dom\,\alpha \cup \{n\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ n - \min(Dom\,\alpha) + 1 & \text{if } x = n. \end{cases}$$

Similarly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Case 3. $1 \notin Im\,\alpha$ and $n \notin Im\,\alpha$. Then $\min(Im\,\alpha) - 1 \geq 1$ and $\max(Im\,\alpha) + 1 \leq n$.

Subcase 3.1. $(\max(Dom\,\alpha))\alpha = \max(Im\,\alpha)$. Since $n \notin Dom\,\alpha$, we have $\max(Dom\,\alpha) + 1 \leq n$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha) + 1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha + 1 & \text{if } x = \max(Dom\,\alpha) + 1. \end{cases}$$

Similarly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Subcase 3.2. $(\max(Dom\,\alpha))\alpha = \min(Im\,\alpha)$. We note that $\max(Dom\,\alpha)+1 \leq n$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha)+1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha - 1 & \text{if } x = \max(Dom\,\alpha)+1. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Hence, either $1 \in Dom\,\alpha$ or $n \in Dom\,\alpha$.

Assume that $1 \in Dom\,\alpha$ but $n \notin Dom\,\alpha$. Since $\alpha \in DP_n$, we obtain that $1 \notin Im\,\alpha$ or $n \notin Im\,\alpha$.

Suppose that $1, n \notin Im\,\alpha$. Since $n \notin Dom\,\alpha$, it follows that $\max(Dom\,\alpha)+1 \leq n$. By Lemma 2.5, we obtain that either $1\alpha = \min(Im\,\alpha)$ or $1\alpha = \max(Im\,\alpha)$.

Case 1. $1\alpha = \min(Im\,\alpha)$. Since $n \notin Im\,\alpha$, we have that $(\max(Dom\,\alpha))\alpha+1 \leq n$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha)+1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha + 1 & \text{if } x = \max(Dom\,\alpha)+1. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Case 2. $1\alpha = \max(Im\,\alpha)$. Since $1 \notin Im\,\alpha$, we have that $(\max(Dom\,\alpha))\alpha - 1 \geq 1$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha)+1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha - 1 & \text{if } x = \max(Dom\,\alpha)+1. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Hence, either $1 \in Im\,\alpha$ or $n \in Im\,\alpha$.

By using the similar proof as above for $n \in Dom\,\alpha$ but $1 \notin Dom\,\alpha$, we obtain that either $1 \in Im\,\alpha$ or $n \in Im\,\alpha$. Next, the proof falls into four cases as follow.

Case 1. $1 \in Dom\,\alpha$ and $1 \in Im\,\alpha$. Suppose that $1\alpha = 1$. It follows that $Dom\,\alpha = Im\,\alpha$. Since $|Dom\,\alpha| < n$, there exists $a \in [n]$ such that $a \notin Dom\,\alpha = Im\,\alpha$.

Define $\beta : Dom\,\alpha \cup \{a\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ a & \text{if } x = a. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $1\alpha \neq 1$.

Case 2. $1 \in Dom\,\alpha$ and $n \in Im\,\alpha$. Suppose that $1\alpha = n$. Since $|Dom\,\alpha| < n$, there exists $a \in [n]$ and $a \notin Dom\,\alpha$. Let $b = n - a + 1$. Then $b \geq 1$ and $|1\alpha - b| = |n - (n - a + 1)| = |1 - a|$. To show that $b \notin Im\,\alpha$, suppose that

$b \in Im\,\alpha$. Then there exists $x \in Dom\,\alpha$ such that $x\alpha = b$. Since $\alpha \in DP_n$, we have that

$$x - 1 = |x - 1| = |x\alpha - 1\alpha| = |b - 1\alpha| = |1 - a| = a - 1.$$

Therefore $x = a$ which is a contradiction. Hence $b \notin Im\,\alpha$.

Define $\beta : Dom\,\alpha \cup \{a\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ b & \text{if } x = a. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $1\alpha \neq n$.

Case 3. $n \in Dom\,\alpha$ and $1 \in Im\,\alpha$. Suppose that $n\alpha = 1$. Since $|Dom\,\alpha| < n$, there exists $a \in [n]$ and $a \notin Dom\,\alpha$. Let $b = n - a + 1$. Then $b \geq 1$ and $|n\alpha - b| = |1 - (n - a + 1)| = |a - n|$. To show that $b \notin Im\,\alpha$, suppose that $b \in Im\,\alpha$. Then there exists $x \in Dom\,\alpha$ such that $x\alpha = b$. Since $\alpha \in DP_n$, we have

$$n - x = |n - x| = |n\alpha - x\alpha| = |n\alpha - b| = |a - n| = n - a.$$

Therefore $x = a$ which is a contradiction. Hence $b \notin Im\,\alpha$.

Define $\beta : Dom\,\alpha \cup \{a\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ b & \text{if } x = a. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $n\alpha \neq 1$.

Case 4. $n \in Dom\,\alpha$ and $n \in Im\,\alpha$. Suppose that $n\alpha = n$. Then $Dom\,\alpha = Im\,\alpha$. Since $|Dom\,\alpha| < n$, there exists $a \in [n]$ such that $a \notin Dom\,\alpha = Im\,\alpha$.

Define $\beta : Dom\,\alpha \cup \{a\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ a & \text{if } x = a. \end{cases}$$

Clearly, $\beta \in DP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $n\alpha \neq n$.

Conversely, suppose that the converse conditions hold and $\alpha \leq \beta$ for some $\beta \in DP_n$. Then $\alpha \subseteq \beta$. Thus $x\alpha = x\beta$ for all $x \in Dom\,\alpha$. To show that $\alpha = \beta$, suppose not. Then there exists $(x, y) \in \beta \setminus \alpha$ and hence $x\beta = y$.

Case 1. Suppose that $1 \in Dom\,\alpha$. Since $Dom\,\alpha$ is closed and $|Dom\,\alpha| < n$, we have $n \notin Dom\,\alpha$. Then there exists $k \in [1, n-1]$ such that $Dom\,\alpha = [1, k]$. This implies that $x \notin [1, k]$ via $\alpha \subseteq \beta$. Then $k < x \leq n$.

Subcase 1.1. $1 \in Im\,\alpha$ and $1\alpha \neq 1$. Since $\alpha \in DP_n$ and $Dom\,\alpha$ is closed, by Lemma 2.7, we get that $Im\,\alpha$ is also closed. Note that $1 \in Im\,\alpha$ and

$Dom\,\alpha = [1,k]$, we obtain that $Im\,\alpha = [1,k]$. By Lemma 2.5, we have that $1\alpha = k$ and $k\alpha = 1$. Since $\beta \in DP_n$, we have

$$x - k = |x - k| = |x\beta - k\beta| = |y - 1| = y - 1.$$

Since β is injective and $\alpha \subseteq \beta$, we obtain that $k < y \leq n$, so that

$$x - 1 = |x - 1| = |x\beta - 1\beta| = |y - k| = y - k.$$

This implies that $k = x - y + 1 = y - x + 1$. Thus $x = y$ and so $y - 1 = y - k$. Therefore $k = 1$ which leads to a contradiction.

Subcase 1.2. $n \in Im\,\alpha$ and $1\alpha \neq n$. Since $\alpha \in DP_n$ and $Dom\,\alpha$ is closed, by Lemma 2.7, we get that $Im\,\alpha$ is also closed. Note that $n \in Im\,\alpha$ and $Dom\,\alpha = [1,k]$, we obtain that $Im\,\alpha = [n-k+1, n]$. By Lemma 2.5, we have that $1\alpha = n-k+1$ or $1\alpha = n$. Thus $1\alpha = n-k+1$ and $k\alpha = n$. Since $\beta \in DP_n$, we have

$$x - k = |x - k| = |x\beta - k\beta| = |y - n| = n - y.$$

Since $\alpha \subseteq \beta$ and β is injective, we get that $y \notin Im\,\alpha$, that is $1 \leq y < n-k+1$. Thus

$$x - 1 = |x - 1| = |x\beta - 1\beta| = |y - (n-k+1)| = n - k + 1 - y.$$

Therefore $k = n-y+1-x+1 = x-k+2-x = 2-k$. Thus $k = 1$, a contradiction.

Case 2. Suppose that $n \in Dom\,\alpha$. Since $Dom\,\alpha$ is closed and $|Dom\,\alpha| < n$, we have $1 \notin Dom\,\alpha$. Then there exists $k \in [2,n]$ such that $Dom\,\alpha = [k,n]$. This implies that $x \notin [k,n]$. Then $1 \leq x < k$.

Subcase 2.1. $1 \in Im\,\alpha$ and $n\alpha \neq 1$. Since $\alpha \in DP_n$ and $Dom\,\alpha$ is closed, by Lemma 2.7, we get that $Im\,\alpha$ is also closed. Note that $1 \in Im\,\alpha$ and $Dom\,\alpha = [k,n]$, we obtain that $Im\,\alpha = [1, n-k+1]$. Since $n\alpha \neq 1$ and Lemma 2.5, we conclude that $n\alpha = n-k+1$ and $k\alpha = 1$. Since $\beta \in DP_n$, we have

$$k - x = |k - x| = |k\beta - x\beta| = |1 - y| = y - 1.$$

Since β is injective and $\alpha \subseteq \beta$, we get that $n-k+1 < y \leq n$, so that

$$n - x = |n - x| = |n\beta - x\beta| = |n - k + 1 - y| = y - (n - k + 1).$$

Therefore $k = y + x - 1 = 2n - k + 1 - 1 = 2n - k$. Thus $k = n$, a contradiction.

Subcase 2.2. $n \in Im\,\alpha$ and $n\alpha \neq n$. Since $\alpha \in DP_n$ and $Dom\,\alpha$ is closed, by Lemma 2.7, we get that $Im\,\alpha$ is also closed. Note that $n \in Im\,\alpha$ and $Dom\,\alpha = [k,n]$, we obtain that $Im\,\alpha = [k,n]$. By Lemma 2.5, we have $n\alpha = k$ or $n\alpha = n$. Thus $n\alpha = k$ and $k\alpha = n$. Since $\beta \in DP_n$, we have

$$k - x = |k - x| = |k\beta - x\beta| = |n - y| = n - y.$$

Since β is injective and $\alpha \subseteq \beta$, we deduce that $1 \leq y < k$, so that

$$n - x = |n - x| = |n\beta - x\beta| = |k - y| = k - y.$$

Therefore $y = n - k + x = (k - y + x) - k + x = 2x - y$ and so $x = y$. We infer that $k = n$ which leads to a contradiction.

The proof of the theorem is now complete. □

Now, we aim to prove an analogue results for ODP_n and this result is a similar for DP_n.

Theorem 2.10. *Let α be an element in ODP_n. Then α is maximal if and only if $Dom\,\alpha$ is closed such that $1 \in Dom\,\alpha$ or $n \in Dom\,\alpha$ and*

1. *if $1 \in Dom\,\alpha$, then $n \in Im\,\alpha$ or*
2. *if $n \in Dom\,\alpha$, then $1 \in Im\,\alpha$.*

Proof. Assume that α is a maximal element in ODP_n. The proof is essentially the same as the proof of Theorem 2.9. We then have that $Dom\,\alpha$ is closed.

Suppose that $1, n \notin Dom\,\alpha$. Since $\alpha \in ODP_n$, we get that 1 and n are not all elements in $Im\,\alpha$. Next, we consider three possible cases as following.

Case 1. $1 \notin Im\,\alpha$ and $n \in Im\,\alpha$. Then we have $(\max(Dom\,\alpha))\alpha = n$. Define $\beta : Dom\,\alpha \cup \{1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ n - (\max(Dom\,\alpha) - 1) & \text{if } x = 1. \end{cases}$$

Clearly, $\beta \in ODP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Case 2. $1 \in Im\,\alpha$ and $n \notin Im\,\alpha$. Then we have $(\min(Dom\,\alpha))\alpha = 1$. Define $\beta : Dom\,\alpha \cup \{n\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ n - \min(Dom\,\alpha) + 1 & \text{if } x = n. \end{cases}$$

Since $n \notin Dom\,\alpha$, we obtain that $n - \min(Dom\,\alpha) + 1 \notin Im\,\alpha$. Thus $\beta \in ODP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Case 3. $1 \notin Im\,\alpha$ and $n \notin Im\,\alpha$. Then $\max(Im\,\alpha) + 1 \leq n$. Since $\alpha \in ODP_n$, we get that $(\max(Dom\,\alpha))\alpha = \max(Im\,\alpha)$. Since $n \notin Dom\,\alpha$, we have $\max(Dom\,\alpha) + 1 \leq n$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha) + 1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha + 1 & \text{if } x = \max(Dom\,\alpha) + 1. \end{cases}$$

Clearly, $\beta \in ODP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction.

Therefore, three cases imply that $1 \in Dom\,\alpha$ or $n \in Dom\,\alpha$. We now consider two cases.

Case 1. $1 \in Dom\,\alpha$. Suppose that $n \notin Im\,\alpha$. Then $(\max(Dom\,\alpha))\alpha + 1 \leq n$. Since $\alpha \in ODP_n$, we have that $n \notin Dom\,\alpha$. Thus $\max(Dom\,\alpha) + 1 \leq n$. Define $\beta : Dom\,\alpha \cup \{\max(Dom\,\alpha) + 1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\max(Dom\,\alpha))\alpha + 1 & \text{if } x = \max(Dom\,\alpha) + 1. \end{cases}$$

Clearly, $\beta \in ODP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $n \in Im\,\alpha$.

Case 2. $n \in Dom\,\alpha$. Suppose that $1 \notin Im\,\alpha$. Then $(\min(Dom\,\alpha))\alpha - 1 \geq 1$. Since $\alpha \in ODP_n$, we have that $1 \notin Dom\,\alpha$. Thus $\min(Dom\,\alpha) - 1 \geq 1$. Define $\beta : Dom\,\alpha \cup \{\min(Dom\,\alpha) - 1\} \to [n]$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in Dom\,\alpha, \\ (\min(Dom\,\alpha))\alpha - 1 & \text{if } x = \min(Dom\,\alpha) - 1. \end{cases}$$

Clearly, $\beta \in ODP_n$, $\alpha \subseteq \beta$ and $\alpha \neq \beta$. By Theorem 2.2, we have that $\alpha \leq \beta$ which is a contradiction. Hence $1 \in Im\,\alpha$.

Conversely, assume that $Dom\,\alpha$ is closed such that $1 \in Dom\,\alpha$ or $n \in Dom\,\alpha$. Suppose that $\alpha \leq \beta$ for some $\beta \in ODP_n$. Then $\alpha \subseteq \beta$. Thus $x\alpha = x\beta$ for all $x \in Dom\,\alpha$.

Suppose that $1 \in Dom\,\alpha$. Thus $n \in Im\,\alpha$. Let $(x, y) \in \beta$. Then $x\beta = y$. Since $Dom\,\alpha$ is closed and $1 \in Dom\,\alpha$, there exists $k \in [n]$ such that $Dom\,\alpha = [1, k]$ and by Lemma 2.7, $Im\,\alpha$ is also closed. We get that $Im\,\alpha = [n - k + 1, n]$ and so and $k\alpha = n$.

Since $\alpha \subseteq \beta$, we have $k\beta = n$. We obtain from $x\beta \leq n$ that $x \leq k$. This implies that $x \in Dom\,\alpha$ and thus $(x, y) \in \alpha$ via $\alpha \subseteq \beta$. Therefore $\beta \subseteq \alpha$.

Similarly, we can prove that $\beta \subseteq \alpha$ by using the fact that $n \in Dom\,\alpha$. Consequently, $\alpha = \beta$ and our proof is complete. □

Since 0 is the minimum element of DP_n and ODP_n. The following theorems determine minimal elements in $DP_n \setminus \{0\}$ and $ODP_n \setminus \{0\}$ with respect to this order.

Theorem 2.11. *Let α be an element in $DP_n \setminus \{0\}$, then α is minimal if and only if $|Dom\,\alpha| = 1$.*

Proof. Suppose that $|Dom\,\alpha| \geq 2$. Then there are distinct elements $u, v \in Dom\,\alpha$. Define $\beta : \{u\} \to [n]$ by $u\beta = u\alpha$. Thus $\beta \in DP_n$, $\beta \neq \alpha$ and $\beta \subseteq \alpha$. By Theorem 2.2, we have $\beta \leq \alpha$. This shows that α is not minimal.

The sufficiency of the theorem is obvious. □

The next theorem is proved essentially the same as the proof of Theorem 2.11.

Theorem 2.12. *Let α be an element in $ODP_n \setminus \{0\}$, then α is minimal if and only if $|Dom\,\alpha| = 1$.*

Acknowledgement(s) : The authors are grateful to the referees for their careful reading of the manuscript and their useful comments.

References

[1] J.M. Howie, Fundamentals of Semigroup Theory, Oxford University Press Inc., New York, 1995.

[2] V. Vagner, Generalized groups, Doklady Akademiĭ Nauk SSSR 84 (1952) 1119-1122.

[3] R. Kehinde, A. Umar, On the semigroup of partial isometries of a finite chain, http://arXiv:1101.0049v1 [math.GR] 2 (December 30, 2010).

[4] R. Kehinde, A. Adeshola, The order of the set of idempotent elements of semigroup of partial isometries of a finite chain, Int. J. Sci. Research 2 (2013) 291-292.

[5] O. Ganyushkin, V. Mazorchuk, Classical Finite Transformation Semigroups, Springer-Verlag, London, 2009.

(Received 4 March 2017)
(Accepted 17 June 2017)