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On Left and Right A-Ideals of a Γ-Semigroup

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Abstract : In this paper, we define the concepts of left A-ideals and right A-ideals of a Γ-semigroup. The main result is a characterization of a Γ-semigroup contains no proper left (respectively, right) A-ideals. Indeed, it is proved that a Γ-semigroup S contains no proper left (respectively, right) A-ideals if and only if (1) S is a left (respectively, right) zero Γ-semigroup; or (2) (S, γ) is a group for all $\gamma \in \Gamma$ and the cardinality $|S| = 2$. It is observed that the result obtained by Grosek and Satko [1] become then a special case.

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1 Introduction and Preliminaries

Let S be a semigroup. O. Grosek and L. Satko [1] defined the following interesting notions: a non-empty subset G_L $[G_R]$ of S is called a *left* [*right*] *A-ideal* of S if

$$sG_L \cap G_L \neq \emptyset \ [G_R s \cap G_R \neq \emptyset] \text{ for all } s \in S.$$

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It is observed that a left [right] ideal of S is a left [right] A-ideal of S. We recall that S is said to be a *left [right] zero semigroup* if

$$ab = a\ [ab = b] \text{ for all } a, b \in S.$$

For the cardinality $|S| > 1$, the authors characterized when S contains no proper left [right] A-ideals. In fact, it is proved that S contains no proper left [right] A-ideal if and only if

(1) S is left [right] zero; or

(2) S is a group of order 2.

The purpose of this paper is to do in this line on an algebraic structure generalized semigroups called Γ-*semigroups*.

For positive integers $m < n$, let us consider two non-empty sets $A = \{1, 2, ..., n\}$ and $B = \{n+1, n+2, ..., n+m\}$. Let S and Γ be the set of all mappings from A into B, and from B into A, respectively. For $f, g \in S$ and $\alpha \in \Gamma$, define

$$(f\alpha g)(k) = f(\alpha(g(k))) \text{ for all } k \in A.$$

It is observed that

(i) $\forall (f, g, \alpha) \in S^2 \times \Gamma,\ f\alpha g \in S$;

(ii) $\forall (f, g, h, \alpha, \beta) \in S^3 \times \Gamma^2,\ f\alpha(g\beta h) = (f\alpha g)\beta h$.

This structure was abstracted, credited to M. K. Sen [2] (see also in [3–6]), as follows:

Let S and Γ be two non-empty sets. Then S is called a Γ-*semigroup* if there is an operation such that

(i) $\forall (a, b, \alpha) \in S^2 \times \Gamma,\ a\alpha b \in S$;

(ii) $\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2,\ a\alpha(b\beta c) = (a\alpha b)\beta c$.

The concept of Γ-semigroups has been widely studied, see, for examples, [7–13]. Other that the example mentioned above, we present here one more an example of a Γ-semigroup: Let $m < n$ be non-negative integers. Let us consider the following two non-empty sets: S denotes the set of all $m \times n$ matrics over the field of real numbers, and Γ denotes the set of all $n \times m$ matrices over the same field. It is observed that S is not a semigroup under the usual multiplication of matrices. Define, for $a, b \in S$ and $\alpha \in \Gamma$ by:

$$(a, \alpha, b) \mapsto a\alpha b$$

Here, on the right hand side, $a\alpha b$, is the usual multiplication of matices. It is clear that S is a Γ-semigroup.

Let S be a Γ-semigroup. An element 0 in S is called a *right [left] zero element* of S if

$$a\alpha 0 = 0\ [0\alpha a = 0] \text{ for all } a \in S \text{ and } \alpha \in \Gamma.$$

For non-empty subsets A, B of a Γ-semigroup S, the set product $A\Gamma B$ is defined by:

$$A\Gamma B := \{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}.$$

We write $A\Gamma b$ for $A\Gamma\{b\}$, and similarly for $b\Gamma A$.

Let S be a Γ-semigroup. A non-empty subset S_1 of S is called a Γ-*subsemigroup* of S if $S_1\Gamma S_1 \subseteq S_1$, that is if $a\alpha b \in S_1$ for all $a, b \in S_1$ and $\alpha \in \Gamma$. A non-empty subset L of S is said to be a *left ideal* of S if $S\Gamma L \subseteq L$. For a *right ideal* of S can be defined dually. And S is said to be *left simple* (respectively, *right simple*) if S is its only left (respectively, right) ideal.

Let S be a Γ-semigroup, and $e \in S$. Then e is called an *idempotent* of S if

$$e\alpha e = e \text{ for all } \alpha \in \Gamma.$$

Finally, let S be a Γ-semigroup. Then S is called a *left group* if

(i) S is left simple;

(ii) S is right cancellative, i.e.,

$$\forall (a, b, c, \alpha) \in S^3 \times \Gamma,\ a\alpha c = b\alpha c \Rightarrow a = b.$$

Lemma 1.1. Let S be a Γ-semigroup. If S is left simple, and if $e\alpha e = e$ for some $(e, \alpha) \in S \times \Gamma$, then $x\alpha e = x$ for all $x \in S$.

Proof. Assume that S is left simple, and if $e\alpha e = e$ for some $(e, \alpha) \in S \times \Gamma$. Let $x \in S$. Since $S\alpha e$ is a left ideal of S, it follows by assumption that $S\alpha e = S$, and so $y\alpha e = x$ for some $y \in S$. We have

$$x\alpha e = (y\alpha e)\alpha e = y\alpha(e\alpha e) = y\alpha e = x$$

as required. □

Theorem 1.2. *Let S be a Γ-semigroup. If S is left simple, and if S contains an idempotent, then S is a left group.*

Proof. Assume that S is left simple, and it contains an idempotent element e. Let $w, y, z \in S$ and $\alpha \in \Gamma$ such that $z\alpha w = y\alpha w$. Since $S\alpha w = S$, we have $u\alpha w = e$ for some $u \in S$. By $e\alpha e = e$ and Lemma 1.1,

$$(w\alpha u)\alpha(w\alpha u) = w\alpha(u\alpha w)\alpha u = w\alpha e\alpha u = w\alpha u.$$

Since $z\alpha w = y\alpha w$, we have

$$z\alpha w\alpha u = y\alpha w\alpha u.$$

By Lemma 1.1, we have $z = y$. □

2 Main Results

We begin this section with the following definition of a left and a right A-ideal of a Γ-semigroup.

Definition 2.1. Let S be a Γ-semigroup, and $\emptyset \neq G_L[G_R] \subseteq S$. Then G_L $[G_R]$ is called a *left* [*right*] *A-ideal* of S if

$$s\Gamma G_L \cap G_L \neq \emptyset \ [G_R\Gamma s \cap G_R \neq \emptyset] \text{ for all } s \in S.$$

A left or right A-ideal of S is said to be *proper* if it is a proper subset of S.

We give here some examples:

Example 2.2. Consider a Γ-semigroup $S = \{a, b, c, d\}$ with $\Gamma = \{\alpha, \beta\}$ and

α	a	b	c	d
a	b	a	b	b
b	a	b	a	a
c	b	a	b	b
d	b	a	b	b

β	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	d	c
d	a	b	c	d

The left A-ideals of S are: $\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}$.

And the right A-ideals of S are: $\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}$.

Example 2.3. Consider a Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha, \beta, \gamma\}$ and

α	a	b	c	d	e
a	a	d	a	d	e
b	e	e	e	e	e
c	c	b	c	b	e
d	e	e	e	e	e
e	e	e	e	e	e

β	a	b	c	d	e
a	a	d	a	d	e
b	c	b	c	b	e
c	c	b	c	b	e
d	a	d	a	d	e
e	e	e	e	e	e

γ	a	b	c	d	e
a	e	e	e	e	e
b	e	e	e	e	e
c	e	e	e	e	e
d	e	e	e	e	e
e	e	e	e	e	e

The left A-ideals of S are: $\{e\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}, \{a, b, e\}, \{a, c, e\}, \{a, d, e\}, \{b, c, e\}, \{c, d, e\}, \{b, d, e\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}$.

The right A-ideals of S are: $\{e\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}, \{a, b, e\}, \{a, c, e\}, \{a, d, e\}, \{b, c, e\}, \{c, d, e\}, \{b, d, e\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}$.

Hereafter, we deal only with a left A-ideal of a Γ-semigroup because the results on right A-ideal of a Γ-semigroup are left-right dual.

Proposition 2.4. *Let S be a Γ-semigroup. If L is a left ideal of S, then L is a left A-ideal of S.*

Proof. Let L be a left ideal of S. If $s \in S$, then $s\Gamma L \subseteq L$, and so $s\Gamma L \cap L \neq \emptyset$. Thus L is a left A-ideal of S. □

Proposition 2.5. *Let S be a Γ-semigroup. If G_L is a left A-ideal of S and if $H \subseteq S$ with $G_L \subseteq H$, then H is a left A-ideal of S.*

Proof. Assume that G_L is a left A-ideal of S, and let $H \subseteq S$ such that $G_L \subseteq H$. For $s \in S$, we have

$$\emptyset \neq s\Gamma G_L \cap G_L \subseteq s\Gamma H \cap H.$$

It follows that H is a left A-ideal of S. □

Corollary 2.6. *If G_L and G'_L are left A-ideals of a Γ-semigroup S, then $G_L \cup G'_L$ is a left A-ideal of S.*

Proof. This follows by Proposition 2.5. □

Proposition 2.7. *Let S be a Γ-semigroup containing a right zero element 0_R. If $0_R \in G_L \subseteq S$, then G_L is a left A-ideal of S.*

Proof. Assume that $0_R \in G_L \subseteq S$. If $s \in S$, then

$$s\Gamma 0_R \subseteq s\Gamma G_L \cap G_L \neq \emptyset.$$

Thus G_L is a left A-ideal of S. □

Let us consider Example 2.2; we have $G_{L_1} = \{b, c\}$ is not a subsemigroup of S because $b\alpha c = a \notin G_{L_1}$. Thus we obtain:

Remark 2.8. *A left A-ideal of a Γ-semigroup need not be a Γ-subsemigroup.*

In Example 2.2, we have $G_{L_2} = \{b, c, d\}$ and $G_{L_3} = \{a, c, d\}$ are left A-ideals of S, but $G_{L_2} \cap G_{L_3} = \{c, d\}$ is not. Thus we have:

Remark 2.9. *The intersection of two left A-ideals of a Γ-semigroup S need not be a left A-ideal of S.*

Remark 2.10. *There is a Γ-semigroup which contains no proper left A-ideal.*

The following example asserts that the remark 2.10:

Example 2.11. Consider a Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha\}$ and

α	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

It is a routine matter to check that S has no proper left A-ideals.

For convenience we introduce the following: Let

$$\begin{aligned} G_{\mathcal{L}} &= \{S \mid S \text{ is a } \Gamma\text{-semigroup having no proper left } A\text{-ideals}\}, \\ G_{\mathcal{R}} &= \{S \mid S \text{ is a } \Gamma\text{-semigroup having no proper right } A\text{-ideals}\}. \end{aligned}$$

Proposition 2.12. *If $S \in G_{\mathcal{L}}$, then S is left simple.*

Proof. This follows by Proposition 2.4. □

Proposition 2.13. *$S \in G_{\mathcal{L}}$ if and only if for any $a \in S$ there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) = \{a\}$.*

Proof. Assume first that $S \in G_{\mathcal{L}}$. Then S contains no proper left A-ideals of S. Let $a \in S$. Then $S \setminus \{a\}$ is not a left A-ideal of S. Thus there exists $s_a \in S$ such that

$$s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset.$$

This implies

$$s_a\Gamma(S \setminus \{a\}) = \{a\}.$$

Conversely, suppose that $S \notin G_{\mathcal{L}}$. Then S contains a proper left A-ideal, say G_L. Since $G_L \subset S$, there exists $a \in S \setminus G_L$. Since $G_L \subseteq S \setminus \{a\}$, it follows by Proposition 2.5 that $S \setminus \{a\}$ is a left A-ideal of S. That is,

$$s\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\}) \neq \emptyset \text{ for any } s \in S.$$

And

$$s\Gamma(S \setminus \{a\}) \neq \{a\} \text{ for all } s \in S.$$

□

Moreover, we need the following lemma:

Lemma 2.14. *If $S \in G_{\mathcal{L}}$, then S contains an idempotent.*

Proof. Assume that $S \in G_{\mathcal{L}}$. Let $a \in S$. By Proposition 2.13, there exists $s_a \in S$ such that

$$s_a\Gamma(S \setminus \{a\}) = \{a\}.$$

There are 2 cases to consider.

Case 1: $s_a = a$. Then $a\Gamma(S \setminus \{a\}) = \{a\}$. Suppose that $a\alpha a \neq a$ for some $\alpha \in \Gamma$. Then $a\alpha a \in S \setminus \{a\}$. Let $\beta \in \Gamma$. Then

$$a\beta(a\alpha a) = a,$$

and

$$(a\alpha a)\beta(a\alpha a) = a\alpha a.$$

Case 2: $s_a \neq a$. Then $s_a \in S \setminus \{a\}$; hence

$$s_a \Gamma s_a = \{a\}.$$

Suppose that $a\alpha a \neq a$ for some $\alpha \in \Gamma$. Then $a\alpha a \in S \setminus \{a\}$, so

$$s_a \Gamma (a\alpha a) = \{a\}.$$

Let $\beta \in \Gamma$. Then

$$s_a \beta (a\alpha a) = \{a\}.$$

Case 2.1: $s_a \beta a = a$. This is a contradiction.

Case 2.2: $s_a \beta a \neq a$. Then $s_a \beta a \in S \setminus \{a\}$. That is, $s_a \Gamma (s_a \beta a) = \{a\}$. So $a\beta a = a$, this is a contradiction. □

By Theorem 1.2, we have:

Corollary 2.15. *If $S \in G_{\mathcal{L}}$, then S is a left group.*

The following result can be found in [7], but we give here a proof.

Theorem 2.16. *Let S be a Γ-semigroup. If (S, α) is a group for some $\alpha \in \Gamma$, then (S, γ) is a group for all $\gamma \in \Gamma$.*

Proof. Assume that (S, α) is a group for some $\alpha \in \Gamma$. Let $\gamma \in \Gamma$. For $a \in S$, by $S\gamma a$ is a left ideal of (S, α) it follows that

$$S = S\gamma a.$$

Similarly,

$$S = a\gamma S.$$

Hence (S, γ) is a group. □

Lemma 2.17. *If $S \in G_{\mathcal{L}}$ and S contains at least two idempotents, then for any $a \in S, a\Gamma(S \setminus \{a\}) = \{a\}$.*

Proof. Assume that $S \in G_{\mathcal{L}}$ and that $e, f \in S$ are idempotents such that $e \neq f$. Let $a \in S$. By Lemma 2.13, there exists $s_a \in S$ such that

$$s_a \Gamma (S \setminus \{a\}) = \{a\}.$$

If $e, f \notin S \setminus \{a\}$, then $e = f = a$, this is a contradiction. Hence $e \in S \setminus \{a\}$ or $f \in S \setminus \{a\}$. If $e \in S \setminus \{a\}$, then $s_a \Gamma e = a$. That is $s_a = a$. For $f \in S \setminus \{a\}$ can be proved similarly. Thus $a\Gamma(S \setminus \{a\}) = \{a\}$. □

We now prove the main result of this paper.

Theorem 2.18. *Let S be a Γ-semigroup with the cardinality $|S| > 1$. Then $S \in G_{\mathcal{L}}$ if and only if*

(1) *S is left zero; or*

(2) *(S,α) is a group for all $\alpha \in \Gamma$ and $|S| = 2$.*

Proof. Assume first that $S \in G_{\mathcal{L}}$. By Proposition 2.12, S is left simple. And by Lemma 2.14, the set $E(S)$ of all idempotents of S is non-empty. There are two cases to consider: $|E(S)| = 1$ or $|E(S)| > 1$.

Case 1: $|E(S)| = 1$. Let $e \in E(S)$. We will show that (S,α) is a group for all $\alpha \in \Gamma$. Let $\alpha \in \Gamma$. Let $x \in S$. Since S is left simple and $S\alpha x$ is a left ideal of S, we have $S\alpha x = S$, and so

$$y\alpha x = e \text{ for some } y \in S.$$

By

$$(x\alpha y)\alpha(x\alpha y) = x\alpha(y\alpha x)\alpha y = x\alpha e\alpha y = x\alpha y,$$

it follows that

$$x\alpha y = e.$$

By Lemma 1.1, (S,α) is a group. By Theorem 2.16, (S,α) is a group for all $\alpha \in \Gamma$.

Suppose that $|S| \neq 2$; then $|S| \geq 3$. For $x \in S$,

$$x\Gamma(S \setminus \{e\}) \cap (S \setminus \{e\}) = S \setminus (x\Gamma e \cup \{e\}) \neq \emptyset.$$

Then $S \setminus \{e\}$ is a proper left A-ideal of S. This contradicts to $S \in G_{\mathcal{L}}$.

Case 2: $|E(S)| > 1$. We will show that $a\Gamma S = \{a\}$ for all $a \in S$. Let $a \in S$. By Lemma 2.17, we have

$$a\Gamma(S \setminus \{a\}) = \{a\}.$$

By the case, there exist $e, f \in E(S)$ such that $e \neq f$. If $e, f \notin S \setminus \{a\}$, then $e = a = f$. This is a contradiction. Hence $e \in S \setminus \{a\}$ or $f \in S \setminus \{a\}$.

Case 2.1: $e \in S \setminus \{a\}$. To show that $a\Gamma a = \{a\}$, we suppose that $a\Gamma a \neq \{a\}$. Then $a\alpha a \neq a$ for some $\alpha \in \Gamma$. Since $a\alpha a \in S \setminus \{a\}$, it follows that

$$(a\alpha a)\alpha(S \setminus a\alpha a) = a\alpha a.$$

And

$$a\alpha a\alpha a = a\alpha a.$$

Hence, $a\alpha a = a$. This is a contradiction. Thus $a\alpha S = \{a\}$ for all $a \in S$.

Case 2.2: $f \in S \setminus a$. That the case holds can be proved similarly.

Conversely, assume that S is left zero, or (S,α) is a group for all $\alpha \in \Gamma$ and $|S| = 2$.

Case 1: S is left zero. Taking $a \in S$. We have

$$a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\}) = \{a\} \cap (S \setminus \{a\}) = \emptyset.$$

We have $a\Gamma(S \setminus \{a\}) = \{a\}$. Then $S \in G_{\mathcal{L}}$.

Case 2: (S,α) is a group for all $\alpha \in \Gamma$ and $|S| = 2$. Let $S = \{a, e\}$ with identity e. Hence $S \in G_{\mathcal{L}}$.

The proof is completed. □

Following is the dual statement of Theorem 2.18, and so the proof is omitted.

Theorem 2.19. *Let S be a Γ-semigroup with $|S| > 1$. Then $S \in G_{\mathcal{R}}$ if and only if*

(1) *S is right zero; or*

(2) *(S, α) is a group for all $\alpha \in \Gamma$ and $|S| = 2$.*

The following two corollaries (see in [1]) are special cases of the two main results presented above:

Corollary 2.20. *Let S be a semigroup with the cardinality $|S| > 1$. Then $S \in G_{\mathcal{L}}$ if and only if*

(1) *S is left zero; or*

(2) *S is a group and $|S| = 2$.*

Corollary 2.21. *Let S be a semigroup with $|S| > 1$. Then $S \in G_{\mathcal{R}}$ if and only if*

(1) *S is right zero; or*

(2) *S is a group and $|S| = 2$.*

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