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Bi-Bases of Γ-Semigroups

Pisit Kummoon[†] **and Thawhat Changphas**[†,‡,1]

[†]Department of Mathematics, Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
e-mail : pisit_2534@hotmail.com (P. Kummoon)
[‡]Centre of Excellence in Mathematics, CHE, Si Ayuttaya Rd.,
Bangkok 10400, Thailand
e-mail : thacha@kku.ac.th (T. Changphas)

Abstract : Based on the results of bi-ideals generated by a non-empty subset of a Γ-semigroup S, we introduce in this paper the concept of bi-bases of S. Using the quasi-order on S defined by the principal bi-ideals of S we characterize when a non-empty subset of S is a bi-base of S.

Keywords : Γ-semigroup; bi-ideal; two-sided base; bi-base; quasi-order.
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1 Introduction and Preliminaries

Let S be a semigroup. A subset A of S is called a *two-sided base* or simply *base* of S if it satisfies the following conditions:

(i) $S = A \cup SA \cup AS \cup SAS$;

(ii) if B is a subset of A such that $S = B \cup SB \cup BS \cup SBS$, then $B = A$.

This notion was introduced and studied by Fabrici [1]. Indeed, the author described the structure of semigroups containing two-sided bases.

This is an algebraic structure, generalized the concept of semigroups, called a Γ-semigroup introduced by Sen [2]. This notion has been widely studied, see [3–18]. Let S and Γ be the set of all functions (or mappings) from $\{1, 2, 3, 4, 5\}$ into $\{6, 7, 8\}$, and from $\{6, 7, 8\}$ into $\{1, 2, 3, 4, 5\}$, respectively. It is observed that S is

[1] Corresponding author.

not a semigroup under the composition of functions. Consider the operation, for $a, b \in S$ and $\alpha \in \Gamma$, by

$$(a\alpha b)(x) = a(\alpha(b(x))) \text{ for all } x \in \{1, 2, 3, 4, 5\}$$

we have that

(i) $a\alpha b \in S$ for all $a, b \in S$ and $\alpha \in \Gamma$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Formally, let S and Γ be any two non-empty sets. Then S is called a Γ-*semigroup* [15] if, for any $a, b \in S$ and $\alpha \in \Gamma$, $a\alpha b$ is defined, and the following hold:

(i) $a\alpha b \in S$ for all $a, b \in S$ and $\alpha \in \Gamma$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Example 1.1. [19] Let $S := [0, 1]$ be a unit interval and $\Gamma := \left\{\frac{1}{n} \mid n \text{ is a positive interger}\right\}$. Then S is a Γ-semigroup under the usual multiplication.

The purpose of this paper is to introduce the concept of bi-bases of a Γ-semigroup, and extend some of Fabrici's results.

Let S be a Γ-semigroup, and A, B non-empty subsets of S. The set product $A\Gamma B$ is defined by:

$$A\Gamma B := \{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}.$$

For $a \in S$, we write $B\Gamma a$ for $B\Gamma\{a\}$, and similarly for $a\Gamma B$.

A non-empty subset A of a Γ-semigroup S is called a Γ-*subsemigroup* [2] of S if

$$A\Gamma A \subseteq A.$$

That is, $a\alpha a' \in A$ for all $a, a' \in A$ and $\alpha \in \Gamma$.

A Γ-subsemigroup B of a Γ-semigroup S is called a *bi-Γ-ideal* [19] of S if

$$B\Gamma S\Gamma B \subseteq B.$$

This notion generalizes the notion of one-sided and two-sided Γ-ideals of S.

Let S be a Γ-semigroup, and B_i a bi-Γ-ideal of S for all $i \in I$. It is known that if $\bigcap_{i\in I} B_i \neq \emptyset$, then $\bigcap_{i\in I} B_i$ is a bi-Γ-ideal of S (see, [19]). Moreover, for a non-empty subset A of S, the intersection of all bi-Γ-ideals of S, denoted by $(A)_b$, is the smallest bi-Γ-ideal of S containing A. And it is of the form

$$(A)_b = A \cup A\Gamma A \cup A\Gamma S\Gamma A$$

(see, [19]). In particular, for $A = \{a\}$, we write $(\{a\})_b$ by $(a)_b$.

Example 1.2. [19] Let $\mathbb{N}$ be the set of all positive integers and $\Gamma = \{5\}$. Then $\mathbb{N}$ is a Γ-semigroup under usual addition. We have:

(1) For $A = \{2\}$, $(A)_b = \{2\} \cup \{9\} \cup \{15, 16, 17, \ldots\}$.

(2) For $B = \{3, 4\}$, $(B)_b = \{3, 4\} \cup \{11, 12, 13\} \cup \{17, 18, 19, \ldots\}$.

2 Main Results

We begin this section with the following definition of bi-bases of a Γ-semigroup.

Definition 2.1. Let S be a Γ-semigroup. A subset B of S is called a *bi-base* of S if it satisfies the following two conditions:

(i) $S = (B)_b$;

(ii) if A is a subset of B such that $S = (A)_b$, then $A = B$.

Example 2.2. Consider the Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha\}$ and

α	a	b	c	d	e
a	b	a	d	c	a
b	a	b	c	d	b
c	d	c	d	c	c
d	c	d	c	d	d
e	a	b	c	d	e

Then $B = \{e\}$ is a bi-base of S. But $B' = \{b\}$ is not a bi-base of S.

Example 2.3. Consider the Γ-semigroup $S = \{a, b, c, d\}$ with $\Gamma = \{\gamma, \delta\}$ and

γ	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	c	d
d	d	c	d	c

δ	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	d	c
d	c	d	c	d

Then $B_1 = \{a\}$ and $B_2 = \{b\}$ are bi-bases of S. But $B_2' = \{a, b\}$ is not a bi-base of S.

Lemma 2.4. *Let B be a bi-base of a Γ-semigroup S. Let $a, b \in B$. If $a \in b\Gamma b \cup b\Gamma S\Gamma b$, then $a = b$.*

Proof. Assume that $a \in b\Gamma b \cup b\Gamma S\Gamma b$, and suppose that $a \neq b$. Let

$$A := B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$, $b \in A$. We will show that $(A)_b = S$. Clearly, $(A)_b \subseteq S$. We have

$$(B)_b = S.$$

Let $x \in S$. Then

$$x \in B \cup B\Gamma B \cup B\Gamma S\Gamma B.$$

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. By assumption, we have

$$x = a \in b\Gamma b \cup b\Gamma S\Gamma b \subseteq A\Gamma A \cup A\Gamma S\Gamma A \subseteq (A)_b.$$

Case 2: $x \in B\Gamma B$. Then $x = b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (b\Gamma b \cup b\Gamma S\Gamma b)\Gamma(b\Gamma b \cup b\Gamma S\Gamma b) \\ &= b\Gamma b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma S\Gamma b \cup b\Gamma S\Gamma b\Gamma b\Gamma b \cup b\Gamma S\Gamma b\Gamma b\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma A\Gamma A \\ &\quad \cup A\Gamma S\Gamma A\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma S\Gamma b) \\ &= (B \setminus \{a\})\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma b\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (b\Gamma b \cup b\Gamma S\Gamma b)\Gamma(B \setminus \{a\}) \\ &= b\Gamma b\Gamma(B \setminus \{a\}) \cup b\Gamma S\Gamma b\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma S\Gamma A\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_1\gamma b_2 \in (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $x \in B\Gamma S\Gamma B$. Then $x = b_3\gamma_1 s\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $s \in S$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_4 &\in (b\Gamma b \cup b\Gamma S\Gamma b)\Gamma S\Gamma(b\Gamma b \cup b\Gamma S\Gamma b) \\ &= b\Gamma b\Gamma S\Gamma b\Gamma b \cup b\Gamma b\Gamma S\Gamma b\Gamma S\Gamma b \cup b\Gamma S\Gamma b\Gamma S\Gamma b\Gamma b \\ &\quad \cup b\Gamma S\Gamma b\Gamma S\Gamma b\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma S\Gamma A\Gamma A \cup A\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma A \\ &\quad \cup A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_3 &\in (B \setminus \{a\})\Gamma S\Gamma(b\Gamma b \cup b\Gamma S\Gamma b) \\ &= (B \setminus \{a\})\Gamma S\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma S\Gamma b\Gamma S\Gamma b \\ &\subseteq A\Gamma S\Gamma A\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_4 &\in (b\Gamma b \cup b\Gamma S\Gamma b)\Gamma S\Gamma(B \setminus \{a\}) \\ &= b\Gamma b\Gamma S\Gamma(B \setminus \{a\}) \cup b\Gamma S\Gamma b\Gamma S\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_3\gamma_1 s\gamma_2 b_4 \in (B \setminus \{a\})\Gamma S\Gamma(B \setminus \{a\}) = A\Gamma S\Gamma A \subseteq (A)_b.$$

This implies $(A)_b = S$. This is a contradiction. Therefore, $a = b$. □

Lemma 2.5. *Let B be a bi-base of a Γ-semigroup S. Let $a, b, c \in B$. If $a \in c\Gamma b \cup c\Gamma S\Gamma b$, then $a = b$ or $a = c$.*

Proof. Assume that $a \in c\Gamma b \cup c\Gamma S\Gamma b$, and suppose that $a \neq b$ and $a \neq c$. Let

$$A := B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(A)_b = S$. Clearly, $(A)_b \subseteq S$. We have

$$(B)_b = S.$$

Let $x \in S$. Then

$$x \in B \cup B\Gamma B \cup B\Gamma S\Gamma B.$$

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. By assumption, we have

$$x = a \in c\Gamma b \cup c\Gamma S\Gamma b \subseteq A\Gamma A \cup A\Gamma S\Gamma A \subseteq (A)_b.$$

Case 2: $x \in B\Gamma B$. Then $x = b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (c\Gamma b \cup c\Gamma S\Gamma b)\Gamma(c\Gamma b \cup c\Gamma S\Gamma b) \\ &= c\Gamma b\Gamma c\Gamma b \cup c\Gamma b\Gamma c\Gamma S\Gamma b \cup c\Gamma S\Gamma b\Gamma c\Gamma b \cup c\Gamma S\Gamma b\Gamma c\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma A\Gamma A \\ &\quad \cup A\Gamma S\Gamma A\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma S\Gamma b) \\ &= (B \setminus \{a\})\Gamma c\Gamma b \cup (B \setminus \{a\})\Gamma c\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_1\gamma b_2 &\in (c\Gamma b \cup c\Gamma S\Gamma b)\Gamma(B \setminus \{a\}) \\ &= c\Gamma b\Gamma(B \setminus \{a\}) \cup c\Gamma S\Gamma b\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma S\Gamma A\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_1\gamma b_2 \in (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $x \in B\Gamma S\Gamma B$. Then $x = b_3\gamma_1 s\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $s \in S$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_4 &\in (c\Gamma b \cup c\Gamma S\Gamma b)\Gamma S\Gamma(c\Gamma b \cup c\Gamma S\Gamma b) \\ &= c\Gamma b\Gamma S\Gamma c\Gamma b \cup c\Gamma b\Gamma S\Gamma c\Gamma S\Gamma b \cup c\Gamma S\Gamma b\Gamma S\Gamma c\Gamma b \\ &\quad \cup c\Gamma S\Gamma b\Gamma S\Gamma c\Gamma S\Gamma b \\ &\subseteq A\Gamma A\Gamma S\Gamma A\Gamma A \cup A\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma A \\ &\quad \cup A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_3 &\in (B \setminus \{a\})\Gamma S\Gamma(c\Gamma b \cup c\Gamma S\Gamma b) \\ &= (B \setminus \{a\})\Gamma S\Gamma c\Gamma b \cup (B \setminus \{a\})\Gamma S\Gamma c\Gamma S\Gamma b \\ &\subseteq A\Gamma S\Gamma A\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} x = b_3\gamma_1 s\gamma_2 b_4 &\in (c\Gamma b \cup c\Gamma S\Gamma b)\Gamma S\Gamma(B \setminus \{a\}) \\ &= c\Gamma b\Gamma S\Gamma(B \setminus \{a\}) \cup c\Gamma S\Gamma b\Gamma S\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$x = b_3\gamma_1 s\gamma_2 b_4 \in (B \setminus \{a\})\Gamma S\Gamma(B \setminus \{a\}) = A\Gamma S\Gamma A \subseteq (A)_b.$$

This implies $(A)_b = S$. This is a contradiction. Therefore, $a = b$. □

To characterised when a non-empty subset of a Γ-semigroup is a bi-base of the Γ-semigroup we need the quasi-order defined as follows:

Definition 2.6. Let S be a Γ-semigroup. Define a *quasi-order* on S by, for any $a, b \in S$,

$$a \leqslant_b b :\Leftrightarrow (a)_b \subseteq (b)_b.$$

The following examples show that the order $\leqslant_b$ defined above is not, in general, a partial order.

Example 2.7. From Example 2.3, we have that $(a)_b \subseteq (b)_b$ (i.e., $a \leqslant_b b$) and $(b)_b \subseteq (a)_b$ (i.e., $b \leqslant_b a$), but $a \neq b$. Thus, $\leqslant_b$ is not a partial order on S.

Example 2.8. Consider the Γ-semigroup $S = \{u, v, x, y, z\}$ with $\Gamma = \{\alpha, \beta\}$ and

α	u	v	x	y	z
u	u	u	u	u	u
v	u	z	y	x	v
x	u	y	v	z	x
y	u	x	z	v	y
z	u	v	x	y	z

β	u	v	x	y	z
u	u	u	u	u	u
v	u	y	v	z	x
x	u	v	x	y	z
y	u	z	y	x	v
z	u	x	z	v	y

We have that $(v)_b \subseteq (x)_b$ (i.e., $v \leqslant_b x$) and $(x)_b \subseteq (v)_b$ (i.e., $x \leqslant_b v$). But $v \neq x$. Thus, $\leqslant_b$ is not a partial order on S.

Lemma 2.9. *Let B be a bi-base of a Γ-semigroup S. If $a, b \in B$ such that $a \neq b$, then neither $a \leqslant_b b$, nor $b \leqslant_b a$.*

Proof. Assume that $a, b \in B$ such that $a \neq b$. Suppose that $a \leqslant_b b$; then

$$a \in (a)_b \subseteq (b)_b.$$

By assumption, we have $a \neq b$, so

$$a \in b\Gamma b \cup b\Gamma S\Gamma b.$$

By Lamma 2.4, $a = b$. This is a contradiction. The case $b \leqslant_b a$ can be proved similarly. □

Lemma 2.10. *Let B be a bi-base of a Γ-semigroup S. Let $a, b, c \in B$ and $\gamma_1, \gamma_2 \in \Gamma$ and $s \in S$:*

(1) If $a \in \{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma S\Gamma\{b\gamma_1 c\}$, then $a = b$ or $a = c$.

(2) If $a \in \{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma\{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma S\Gamma\{b\gamma_1 s\gamma_2 c\}$, then $a = b$ or $a = c$.

Proof. (1) Assume that $a \in \{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma S\Gamma\{b\gamma_1 c\}$, and suppose that $a \neq b$ and $a \neq c$. Let

$$A := B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, if suffices to show that $B \subseteq (A)_b$. Let $x \in B$. If $x \neq a$, then $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$\begin{aligned} x = a &\in \{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma S\Gamma\{b\gamma_1 c\} \\ &\subseteq A\Gamma A \cup A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma S\Gamma A\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of S,

$$S = (B)_b \subseteq (A)_b \subseteq S.$$

Therefore, $S = (A)_b$. This is a contradiction.
(2) Assume that $a \in \{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma\{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma S\Gamma\{b\gamma_1 s\gamma_2 c\}$, and suppose that $a \neq b$ and $a \neq c$. Let

$$A := B \setminus \{a\}.$$

Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, if suffices to show that $B \subseteq (A)_b$. Let $x \in B$. If $x \neq a$, then $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$\begin{aligned} x = a &\in \{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma\{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma S\Gamma\{b\gamma_1 s\gamma_2 c\} \\ &\subseteq A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma A\Gamma S\Gamma A \cup A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \\ &\subseteq A\Gamma S\Gamma A \\ &\subseteq (A)_b. \end{aligned}$$

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of S,

$$S = (B)_b \subseteq (A)_b \subseteq S.$$

Therefore, $S = (A)_b$. This is a contradiction.

□

Lemma 2.11. *Let B be a bi-base of a Γ-semigroup S.*

(1) For any $a, b, c \in B$, $\gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b b\gamma_1 c$.

(2) For any $a, b, c \in B$, $\gamma_2, \gamma_3 \in \Gamma$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleqslant_b b\gamma_2 s\gamma_3 c$.

Proof. (1) For any $a, b, c \in B$, $\gamma_1 \in \Gamma$, let $a \neq b$ and $a \neq c$. Suppose that

$$a \leqslant_b b\gamma_1 c,$$

we have

$$a \in (a)_b \subseteq (b\gamma_1 c)_b = \{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma S\Gamma\{b\gamma_1 c\}.$$

By Lamma 2.10 (1), it follows that $a = b$ or $a = c$. This contradicts to assumption.

(2) For any $a, b, c \in B$, $\gamma_2, \gamma_3 \in \Gamma$ and $s \in S$, let $a \neq b$ and $a \neq c$. Suppose that

$$a \leqslant_b b\gamma_1 s\gamma_2 c,$$

we have

$$\begin{aligned} a \in (a)_b &\subseteq (b\gamma_1 s\gamma_2 c)_b \\ &= \{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma\{b\gamma_1 s\gamma_2 c\} \cup \{b\gamma_1 s\gamma_2 c\}\Gamma S\Gamma\{b\gamma_1 s\gamma_2 c\}. \end{aligned}$$

By Lamma 2.10 (2), it follows that $a = b$ or $a = c$. This contradicts to assumption.

□

The following theorem characterizes when a non-empty subset of a Γ-semigroup S is a bi-base of S.

Theorem 2.12. *A non-empty subset B of a Γ-semigroup S is a bi-base of S if and only if B satisfies the following conditions:*

(1) For any $x \in S$,

(1.a) there exists $b \in B$ *such that* $x \leqslant_b b$*; or*

(1.b) there exist $b_1, b_2 \in B$ *and* $\gamma \in \Gamma$ *such that* $x \leqslant_b b_1\gamma b_2$*; or*

(1.c) there exist $b_3, b_4 \in B, s \in S$ *and* $\gamma_1, \gamma_2 \in \Gamma$ *such that* $x \leqslant_b b_3\gamma_1 s\gamma_2 b_4$.

(2) For any $a, b, c \in B$, $\gamma_1 \in \Gamma$, *if* $a \neq b$ *and* $a \neq c$, *then* $a \nleqslant_b b\gamma_1 c$.

(3) For any $a, b, c \in B$, $\gamma_2, \gamma_3 \in \Gamma$ *and* $s \in S$, *if* $a \neq b$ *and* $a \neq c$, *then* $a \nleqslant_b b\gamma_2 s\gamma_3 c$.

Proof. Assume first that B is a bi-base of S. Then

$$S = (B)_b.$$

To show that (1) holds, let $x \in S$. Then

$$x \in B \cup B\Gamma B \cup B\Gamma S\Gamma B.$$

We consider three cases:

Case 1 : $x \in B$. Then $x = b$ for some $b \in B$. This implies $(x)_b \subseteq (b)_b$. Hence, $x \leqslant_b b$.

Case 2 : $x \in B\Gamma B$. Then $x = b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. This implies $(x)_b \subseteq (b_1\gamma b_2)_b$. Hence, $x \leqslant_b b_1\gamma b_2$.

Case 3 : $x \in B\Gamma S\Gamma B$. Then $x = b_3\gamma_1 s\gamma_2 b_4$ for some $b_3, b_4 \in B, s \in S$ and $\gamma_1, \gamma_2 \in \Gamma$. This implies $(x)_b \subseteq (b_3\gamma_2 s\gamma_3 b_4)_b$. Hence, $x \leqslant_b b_3\gamma_1 s\gamma_2 b_4$.

The validity of (2) and (3) follow, respectively, from Lemma 2.11 (1), and Lemma 2.11 (2).

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that B is a bi-base of S. To show that $S = (B)_b$. Clearly, $(B)_b \subseteq S$. By (1),

$$S \subseteq (B)_b,$$

and

$$S = (B)_b.$$

It remains to show that B is a minimal subset of S with the property: $S = (B)_b$. Suppose that $S = (A)_b$ for some $A \subset B$. Since $A \subset B$, there exists $b \in B \setminus A$. Since $b \in B \subseteq S = (A)_b$ and $b \notin A$, it follows that

$$b \in A\Gamma A \cup A\Gamma S\Gamma A.$$

There are two cases to consider:

Case 1: $b \in A\Gamma A$. Then $b = a_1\gamma_1 a_2$ for some $a_1, a_2 \in A$ and $\gamma_1 \in \Gamma$. We have $a_1, a_2 \in B$. Since $b \notin A$, so $b \neq a_1$ and $b \neq a_2$. Since $b = a_1\gamma_1 a_2$, $(b)_b \subseteq (a_1\gamma_1 a_2)_b$. Hence, $b \leqslant_b a_1\gamma_1 a_2$. This contradicts to (2).

Case 2: $b \in A\Gamma S\Gamma A$. Then $b = a_3\gamma_2 s\gamma_3 a_4$ for some $a_3, a_4 \in A$, $\gamma_2, \gamma_3 \in \Gamma$ and $s \in S$. Since $b \notin A$, we have $b \neq a_3$ and $b \neq a_4$. Since $A \subset B$, $a_3, a_4 \in B$. Since $b = a_3\gamma_2 s\gamma_3 a_4$, so $(b)_b \subseteq (a_3\gamma_2 s\gamma_3 a_4)_b$. Hence, $b \leqslant_b a_3\gamma_2 s\gamma_3 a_4$. This contradicts to (3).

Therefore, B is a bi-base of S as required, and the proof is completed. □

Theorem 2.13. *Let B be a bi-base of a Γ-semigroup S. Then B is a Γ-subsemigroup of S if and only if for any $a, b \in B$ and $\beta \in \Gamma$, $a\beta b = a$ or $a\beta b = b$.*

Proof. Let $a, b \in B$ and $\beta \in \Gamma$. If B is a Γ-subsemigroup of S, then $a\beta b \in B$. Since $a\beta b \in a\Gamma b \cup a\Gamma S\Gamma b$, it follows by Lemma 2.5 that $a\beta b = a$ or $a\beta b = b$. The opposite direction is clear. □

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