Thai Journal of Mathematics : 59–74
Special Issue: Annual Meeting in Mathematics 2017
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Idempotent of Weak Projection Cohypersubstitutions

Nareupanat Lekkoksung[†,1], **Prakit Jampachon**[†]
and Somsak Lekkoksung[‡]

[†]Department of Mathematics, Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
e-mail : n.lekkoksung@kkumail.com (N. Lekkoksung)
prajam@kku.ac.th (P. Jampachon)
[‡]Division of Mathematics, Faculty of Engineering, Rajamangala University of Technology Isaan, Khon Kaen Campus, Khon Kaen 40000, Thailand
e-mail : lekkoksung_somsak@hotmail.com (S. Lekkoksung)

Abstract : K. Denecke and K. Saengsura established a Galois-connection between monoids of cohypersubstitutuions of a given type τ and varieties of the same type, showing that for any monoid M of cohypersubstitutions of type τ, the collection of all M-solid varieties of type τ forms a complete sublattice of the lattice of all varieties ([1]). It is of interest to know which semigroup properties of cohypersubstitutions can be transferred by this Galois connection. In this paper, we study the semigroup properties of weak projection cohypersubstitutions of all mappings from the set of cooperation symbols to the set of coterms which preserve arities type $(2, 2)$. In particular, we characterize all idempotent elements of such cohypersubstitutions.

Keywords : idempotent element; coterm; cohypersubstitution; weak projection cohypersubstitution.

2010 Mathematics Subject Classification : 16W30; 20M12.

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[1] Corresponding author.

1 Introduction

Let A be a nonempty set and n a natural number. The set $A^{\sqcup n} = \underline{n} \times A$ where $\underline{n} := \{1, \dots, n\}$ is called the n-th copower. Dualizing the concept of n-ary operation we obtain that of an n-ary cooperation on A is a mapping $f^A : A \to A^{\sqcup n}$ and the number n is called the arity of the cooperation f^A. Each n-ary cooperation f^A is uniquely determined by the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$, $f_2^A : A \to A$ and $f^A(a) = (f_1^A(a), f_2^A(a))$. The mappings f_1^A and f_2^A is called the labelling and the mapping of f^A, respectively, (see, [2]). An indexed coalgebra is a pair $(A; (f_i^A)_{i\in I})$, where f_i^A is an n_i-ary cooperation defined on A, and $\tau = (n_i)_{i\in I}$ is called the *type of the coalgebra*, (see, [1, 3, 4]). This particular structure was introduced by Drbohlav and the Birkhoff's variety theorem for coalgebra was proven [5].

Let $cO_A^{(n)}$ be the set of all n-ary cooperations defined on A. In [2], Csákány introduced the notion of superposition as follows. If $f^A \in cO_A^{(n)}$ and $g_1^A, \dots, g_n^A \in cO_A^{(k)}$, then define a k-ary cooperation $f^A[g_1^A, \dots, g_n^A] : A \to A^{\sqcup k}$ by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. We call the cooperation $f^A[g_1^A, \dots, g_n^A]$ a superposition of f^A and $g_1^A, \dots, g_n^A$. Instead of $f^A[g_1^A, \dots, g_n^A]$ we also write $\mathrm{comp}_k^n(f^A, g_1^A, \dots, g_n^A)$. For example, let $A = \{a_1, a_2, a_3\}$ and $f^A, g_1^A, g_2^A, g_3^A : A \to A^{\sqcup 3}$ by

$$\begin{array}{c|c|c|c} f^A(a_1) = (2, a_2) & g_1^A(a_1) = (2, a_2) & g_2^A(a_1) = (2, a_2) & g_3^A(a_1) = (3, a_1) \\ f^A(a_2) = (3, a_1) & g_1^A(a_2) = (3, a_1) & g_2^A(a_2) = (3, a_1) & g_3^A(a_2) = (3, a_2) \\ f^A(a_3) = (1, a_3) & g_1^A(a_3) = (1, a_3) & g_2^A(a_1) = (2, a_3) & g_3^A(a_3) = (3, a_2). \end{array}$$

We can see that $f_1^A(a_i)$ and $f_2^A(a_i)$, $1 \leq i \leq 3$, are a natural number in the first and an element of A in the second component of $f^A(a_i)$, respectively. The labelling and the mapping of g_1^A, g_2^A, g_3^A can be considered similarly. Thus,

$$f^A[g_1^A, g_2^A, g_3^A](a_1) = ((g_2^A)_1(a_2), (g_2^A)_2(a_2)) = (3, a_1),$$

$$f^A[g_1^A, g_2^A, g_3^A](a_2) = ((g_3^A)_1(a_1), (g_3^A)_2(a_1)) = (3, a_1),$$

and

$$f^A[g_1^A, g_2^A, g_3^A](a_3) = ((g_1^A)_1(a_3), (g_1^A)_2(a_3)) = (1, a_3).$$

The injection $\iota_i^{n,A}$ are special cooperations which are defined by $\iota_i^{n,A} : A \to A^{\sqcup n}$ with $a \mapsto (i, a)$ for $1 \leq i \leq n$. Then we obtain a multi-based algebra

$$((cO_A^{(n)})_{n\geq 1}; (\mathrm{comp}_k^n)_{k,n\geq 1}, (\iota_i^{n,A})_{1\leq i\leq n}).$$

In [2], Csákány mentioned that it is a clone.

Coalgebras are pairs consisting of a nonempty set and a set of cooperations defined on this set. In [1], K. Denecke and K. Saengsura defined terms for coalgebras, coidentities and cohyperidentities. These concepts can be applied to give a new solution of the completeness problem for clones of cooperations defined on a two-element set and to separate clones of cooperations by coidentities. The concepts of coidentities and cohyperidentities, help to solve the functional completeness problem, are defined by coterm.

Let $\tau = (n_i)_{i\in I}$ be an indexed family of natural numbers and let $(f_i)_{i\in I}$ be an indexed set of cooperation symbols. To each cooperation symbol we assign n_i as its arity. Let $\{e^n_j : n \in \mathbb{N}, 1 \leq j \leq n\}$ be a set of symbols which is disjoint from the set $\{f_i : i \in I\}$. To each e^n_j we assign the positive integer n as its arity. *Coterm* of type τ are defined by the following recursion:

(i) For every $i \in I$, the cooperation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \in \mathbb{N}$ and $1 \leq j \leq n$, the symbol e^n_j is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are m-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an m-ary coterm of type τ and if $t_1, \ldots, t_n$ are m-ary coterms of type τ, then $e^n_j[t_1, \ldots, t_n]$ is an m-ary coterm of type τ where $1 \leq j \leq n$.

Let $\mathrm{cT}^{(n)}_\tau$ be the set of all n-ary coterms of type τ and let $\mathrm{cT}_\tau := \bigcup_{n\in\mathbb{N}} \mathrm{cT}^{(n)}_\tau$ be the set of all coterms of type τ. For simply, we write the set $\{e^n_j : n \in \mathbb{N}, 1 \leq j \leq n\}$ by E. Let $n \in \mathbb{N}$, we denote the set $\{e^n_j : 1 \leq j \leq n\}$ by E_n.

Definition 1.1. For each $m, n \in \mathbb{N}$. A *superposition* of coterms $S^n_m : \mathrm{cT}^{(n)}_\tau \times (\mathrm{cT}^{(m)}_\tau)^n \to \mathrm{cT}^{(m)}_\tau$ defined inductively by the following steps;

(i) if $t = e^n_i$, $1 \leq i \leq n$, then $S^n_m(t, t_1, \ldots, t_n) := t_i$ where $t_1, \ldots, t_n \in \mathrm{cT}^{(m)}_\tau$,

(ii) if $t = f_i$ is an n_i-ary cooperation symbol, then $S^{n_i}_{n_i}(t, e^{n_i}_1, \ldots, e^{n_i}_{n_i}) := f_i$,

(iii) if $t = g_j$ is an n_j-ary cooperation symbol, then
$S^{n_j}_m(t, t_1, \ldots, t_{n_j}) := g_j[t_1, \ldots, t_{n_j}]$ where $t_1, \ldots, t_{n_j} \in \mathrm{cT}^{(m)}_\tau$,

(iv) if $t = e^p_j[s_1, \ldots, s_p]$ where $s_1, \ldots, s_p$ are n-ary coterms and assume that $S^n_m(s_k, t_1, \ldots, t_n)$ are already defined for $t_1, \ldots, t_n \in \mathrm{cT}^{(m)}_\tau$, $1 \leq k \leq p$, then

$$S^n_m(t, t_1, \ldots, t_n) := e^p_j[S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_p, t_1, \ldots, t_n)],$$

(v) if $t = f_i[s_1, \ldots, s_{n_i}]$ where f_i is an n_i-ary cooperation symbol, $s_1, \ldots, s_{n_i}$ are n-ary coterms and assume that $S^n_m(s_k, t_1, \ldots, t_n)$ are already defined for $t_1, \ldots, t_n \in \mathrm{cT}^{(m)}_\tau$, $1 \leq k \leq n_i$, then

$$S^n_m(t, t_1, \ldots, t_n) := f_i[S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n)].$$

The above definition is defined slightly different from [1,3]. Indeed, the property (iv) is added since an n-ary coterm of type τ can start with symbol e^p_j for $p \in \mathbb{N}$, $1 \leq j \leq p$. For instance, the binary coterm t can be written by $e^3_2[e^2_2, e^2_1, e^2_2]$. In [1], the authors proved that the multi-based algebra

$$((\mathrm{c}T^{(n)}_\tau)_{n\geq 1}; (S^n_m)_{m,n\geq 1}, (e^n_j)_{1\leq j\leq n})$$

is a clone. That is, it satisfied the conditions

(C1) $\hat{S}^p_m(z, \hat{S}^n_m(y_1, x_1, \ldots, x_n), \ldots, \hat{S}^n_m(y_p, x_1, \ldots, x_n))$
$\approx \hat{S}^n_m(\hat{S}^p_m(z, y_1, \ldots, y_p), x_1, \ldots, x_n),$
$(m, n, p \in \mathbb{N}),$

(C2) $\hat{S}^n_m(e^n_i, x_1, \ldots, x_n) \approx x_i,\ m \in \mathbb{N}, 1 \leq i \leq n,$

(C3) $\hat{S}^n_n(y, e^n_1, \ldots, e^n_n) \approx y,\ (n \in \mathbb{N}).$

Here $\hat{S}^n_m, \hat{S}^p_m, \hat{S}^n_n$ and e^n_i are operation symbols corresponding to the clone type.

The concept of cohypersubstitution was introduced in [1] as making precise the concept of cohyperidentities.

Definition 1.2. A *cohypersubstitution* of type τ is a mapping $\sigma : \{f_i : i \in I\} \cup E \to \mathrm{cT}_\tau$ which maps each n_i-ary cooperation symbols of type τ to an n_i-ary coterm of this type and $\sigma(e) = e$ if $e \in E$. Any cohypersubstitution σ can be extended to a mapping $\hat{\sigma} : \mathrm{cT}_\tau \to \mathrm{cT}_\tau$ on the set of all coterms of type τ inductively defined as follows:

(i) $\hat{\sigma}[f_i] := f_i$ for all $i \in I$,

(ii) $\hat{\sigma}[e^n_i] := e^n_i$ for each $n \in \mathbb{N}$ and $1 \leq i \leq n$,

(iii) $\hat{\sigma}[e^n_i[t_1, \ldots, t_n]] := \hat{\sigma}[t_i]$ for each $n \in \mathbb{N}$ and $1 \leq i \leq n$,

(iv) $\hat{\sigma}[f_i[t_1, \ldots, t_{n_i}]] := S^{n_i}_n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

This definition is also slightly different from [1,3] by the same reason as defining superposition of coterms. Moreover, we set $\sigma(e) = e$ for all $e \in E$. We denote by $\mathrm{cHyp}(\tau)$ the set of all cohypersubstitutions of type τ. In [1], the authors defined a binary operation $\circ_{\mathrm{C}}$ on cT_τ by $\sigma_1 \circ_{\mathrm{C}} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in \mathrm{cT}_\tau$ where $\circ$ is a usual composition of mapping. They showed that the structure $\mathbf{cHyp}(\tau) := (\mathrm{cHyp}(\tau); \circ_{\mathrm{C}}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity cohypersubstitution defined by $\sigma_{\mathrm{id}}(f_i) = f_i$ for all $i \in I$.

In semigroup theory, it is of interest to consider various type of its elements, including regular, idempotent, completely regular, etc. In [6], the authors characterized idempotent and regular elements of $\mathbf{cHyp}(2)$. The characterizations of idempotent and regular elements of cohypersubstitutions of type (3) and type (n) was given in [7] and [8], respectively, by D. Boonchari and K. Saengsura.

In this paper, we continue in this vein, by consider the submonoid of cohypersubstitutions of type $(2, 2)$, so-called *weak projection cohypersubstitutions* and characterize its idempotent elements.

2 Some Submonoids

In this section, we present two submonoids based on various properties of cohypersubstitutions.

Definition 2.1. A cohypersubstitution σ of type τ is called a *projection cohypersubstitution* of type τ if $\sigma(f_i) \in E_{n_i}$ for all $i \in I$.

Denoted by $\mathrm{P}(\tau)$ the set of all projection cohypersubstitutions of type τ.

Proposition 2.2. *An algebra* $(P(\tau)\cup\{\sigma_{id}\};\circ_C,\sigma_{id})$ *is a submonoid of* $(cHyp(\tau);\circ_C,\sigma_{id})$.

Proof. Let $\sigma_1,\sigma_2\in \mathrm{P}(\tau)\cup\{\sigma_{\mathrm{id}}\}$.

- If $\sigma_1\in\mathrm{P}(\tau)$ and $\sigma_2=\sigma_{\mathrm{id}}$, then

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(f_i)=f_i=\sigma_{\mathrm{id}}(f_i).$$

- If $\sigma_1=\sigma_{\mathrm{id}}$ and $\sigma_2\in\mathrm{P}(\tau)$, then $\sigma_2(f_i)=e^n_j$ for some $n\in\mathbb{N}$ and $1\leq j\leq n$. Thus,

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(e^n_j)=e^n_j.$$

- If $\sigma_1,\sigma_2\in\mathrm{P}(\tau)$, then $\sigma_2(f_i)=e^n_j$ for some $n\in\mathbb{N}$ and $1\leq j\leq n$. Thus,

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(e^n_j)=e^n_j.$$

- If $\sigma_1=\sigma_{\mathrm{id}}=\sigma_2$, then

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(f_i)=f_i=\sigma_{\mathrm{id}}(f_i).$$

Therefore, $\sigma_1\circ_{\mathrm{C}}\sigma_2\in\mathrm{P}(\tau)\cup\{\sigma_{\mathrm{id}}\}$. Altogether, we have that $(\mathrm{P}(\tau)\cup\{\sigma_{\mathrm{id}}\};\circ_{\mathrm{C}},\sigma_{\mathrm{id}})$ is a submonoid of $(\mathrm{cHyp}(\tau);\circ_{\mathrm{C}},\sigma_{\mathrm{id}})$. □

Definition 2.3. A cohypersubstitution σ of type τ is called a *weak projection cohypersubstitution* of type τ if there is $i\in I$ such that $\sigma(f_i)\in E_{n_i}$.

Denoted by $\mathrm{WP}(\tau)$ the set of all weak projection cohypersubstitutions of type τ.

Proposition 2.4. *An algebra* $(WP(\tau)\cup\{\sigma_{id}\};\circ_C,\sigma_{id})$ *is a submonoid of* $(cHyp(\tau);\circ_C,\sigma_{id})$.

Proof. Let $\sigma_1,\sigma_2\in\mathrm{WP}(\tau)\cup\{\sigma_{\mathrm{id}}\}$.

- If $\sigma_1=\sigma_{\mathrm{id}}$ and $\sigma_2\in\mathrm{WP}(\tau)$, then there is $i\in I$ such that $\sigma_2(f_i)=e^n_j$ for some $n\in\mathbb{N}$ and $1\leq j\leq n$. Thus,

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(e^n_j)=e^n_j.$$

- If $\sigma_1\in\mathrm{WP}(\tau)$ and $\sigma_2=\sigma_{\mathrm{id}}$, then there is $i\in I$ such that $\sigma_1(f_i)=e^n_j$ for some $n\in\mathbb{N}$ and $1\leq j\leq n$. Thus,

$$(\sigma_1\circ_{\mathrm{C}}\sigma_2)(f_i)=\hat{\sigma}_1(\sigma_2(f_i))=\hat{\sigma}_1(f_i)=f_i=\sigma_{\mathrm{id}}(f_i).$$

- If $\sigma_1, \sigma_2 \in \mathrm{WP}(\tau)$, then there is $i \in I$ such that $\sigma_2(f_i) = e_j^n$ for some $n \in \mathbb{N}$ and $1 \leq j \leq n$. Thus,

$$(\sigma_1 \circ_{\mathrm{C}} \sigma_2)(f_i) = \hat{\sigma}_1(\sigma_2(f_i)) = \hat{\sigma}_1(e_j^n) = e_j^n.$$

- If $\sigma_1 = \sigma_{\mathrm{id}} = \sigma_2$, then

$$(\sigma_1 \circ_{\mathrm{C}} \sigma_2)(f_i) = \hat{\sigma}_1(\sigma_2(f_i)) = \hat{\sigma}_1(f_i) = f_i = \sigma_{\mathrm{id}}(f_i).$$

Therefore, $\sigma_1 \circ_{\mathrm{C}} \sigma_2 \in \mathrm{WP}(\tau) \cup \{\sigma_{\mathrm{id}}\}$. Altogether, we have that $(\mathrm{WP}(\tau) \cup \{\sigma_{\mathrm{id}}\}; \circ_{\mathrm{C}}, \sigma_{\mathrm{id}})$ is a submonoid of $(\mathrm{cHyp}(\tau); \circ_{\mathrm{C}}, \sigma_{\mathrm{id}})$. □

Corollary 2.5. *An algebra $(P(\tau) \cup \{\sigma_{id}\}; \circ_C, \sigma_{id})$ is a submonoid of $(WP(\tau) \cup \{\sigma_{id}\}; \circ_C, \sigma_{id})$.*

3 Idempotent Elements of Weak Projection Cohypersubstitutions

For a semigroup S, an element $e \in S$ is called an *idempotent element* of S if $e = ee$. We consider the idempotent elements of $\mathrm{WP}(2,2)$. It is clear that every element of $\mathrm{P}(2,2)$ is idempotent. Thus, we only consider the idempotent elements of $\mathrm{WP}(2,2) \setminus \mathrm{P}(2,2)$. Let f and g be the binary cooperation symbols. We denote the cohypersubstitution σ with $\sigma(f) = t_1$ and $\sigma(g) = t_2$ by σ_{t_1,t_2}. We start with the following proposition:

Proposition 3.1. *Let $\sigma_{(t_i)_{i\in I}}$ be a cohypersubstitution of type $\tau = (n_i)_{i\in I}$. Then the following statements are equivalent:*

(i) $\sigma_{(t_i)_{i\in I}}$ is idempotent;

(ii) $\hat{\sigma}_{(t_i)_{i\in I}}[t_j] = t_j$ for all $j \in I$.

Proof. (i) ⇒ (ii): Let $j \in I$. Then

$$\hat{\sigma}_{(t_i)_{i\in I}}[t_j] = \hat{\sigma}_{(t_i)_{i\in I}}[\sigma_{(t_i)_{i\in I}}(f_j)] = (\sigma_{(t_i)_{i\in I}} \circ_{\mathrm{C}} \sigma_{(t_i)_{i\in I}})(f_j) = \sigma_{(t_i)_{i\in I}}(f_j) = t_j.$$

(ii) ⇒ (i): For each $j \in I$, we obtain

$$(\sigma_{(t_i)_{i\in I}} \circ_{\mathrm{C}} \sigma_{(t_i)_{i\in I}})(f_j) = \hat{\sigma}_{(t_i)_{i\in I}}[\sigma_{(t_i)_{i\in I}}(f_j)] = \hat{\sigma}_{(t_i)_{i\in I}}[t_j] = t_j = \sigma_{(t_i)_{i\in I}}(f_j).$$

Thus, we complete the proof. □

For a cohypersubstitution σ_{t_1,t_2} of $\mathrm{WP}(2,2) \setminus \mathrm{P}(2,2)$ we separate our consideration into four cases:

(i) $t_1 \in E_2$, $t_2 \notin E_2$ and $\mathrm{co}(t_2) = 1$,

(ii) $t_2 \in E_2$, $t_1 \notin E_2$ and $\mathrm{co}(t_1) = 1$,

(iii) $t_1 \in E_2$, $t_2 \notin E_2$ and $\text{co}(t_2) > 1$,

(iv) $t_2 \in E_2$, $t_1 \notin E_2$ and $\text{co}(t_1) > 1$,

where $\text{co}(t_1)$ and $\text{co}(t_2)$ denote the number of all cooperation symbols occurring in the coterms t_1 and t_2, respectively. We will start with some notions that used to prove our main results.

For $n \in \mathbb{N}$, $1 \le j \le n$ and F be a variable over the two-elements alphabet $\{f, g\}$ where $\text{ar}(F)$ denotes the arity of F. We define $M^i(t)$, $1 \le i \le \text{ar}(F)$ by

(i) if $t = e_j^n$, then $M^i(t) = t$,

(ii) if $t = F$, then $M^i(t) = F$,

(iii) if $t = F[s_1, \ldots, s_{\text{ar(F)}}]$ and $1 \le i \le \text{ar}(F)$, then $M^i(t) = M^i(s_i)$,

(iv) if $t = e_j^n[s_1, \ldots, s_n]$, then $M^i(t) = M^i(s_j)$.

For example, let f, g be binary cooperation symbols and $t = f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]$. Then

$$\begin{aligned} M^1(t) &= M^1(f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]) \\ &= M^1(g[e_1^2[f, e_2^2], e_1^2]) \\ &= M^1(e_1^2[f, e_2^2]) \\ &= M^1(f) \\ &= f, \end{aligned}$$

$$\begin{aligned} M^2(t) &= M^2(f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]) \\ &= M^2(e_2^2[g, e_1^2[e_2^2, e_1^2]]) \\ &= M^2(e_1^2[e_2^2, e_1^2]) \\ &= M^2(e_2^2) \\ &= e_2^2. \end{aligned}$$

For $n \in \mathbb{N}$, $1 \le j \le n$ and F be a variable over the two-element alphabet $\{f, g\}$ where $\text{ar}(F)$ denotes the arity of F. For a coterm t, we let $\text{inn}(t)$ be the set of inner coterm of the coterm t defined inductively by the following,

(i) if $t = F$, then $\text{inn}(t) = \{F\}$,

(ii) if $t = e_j^n$, then $\text{inn}(t) = \{e_j^n\}$,

(iii) if $t = e_j^n[s_1, \ldots, s_n]$, then $\text{inn}(t) = \bigcup_{i=1}^{n} \text{inn}(s_i)$,

(iv) if $t = F[s_1, \ldots, s_{\text{ar(F)}}]$, then $\text{inn}(t) = \bigcup_{i=1}^{\text{ar(F)}} \text{inn}(s_i)$.

For example, let f, g be binary cooperation symbols and $t = f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]$. Then

$$\begin{aligned}
\mathrm{inn}(t_2) &= \mathrm{inn}(f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]) \\
&= \mathrm{inn}(g[e_1^2[f, e_2^2], e_1^2]) \cup \mathrm{inn}(e_2^2[g, e_1^2[e_2^2, e_1^2]]) \\
&= \mathrm{inn}(e_1^2[f, e_2^2]) \cup \mathrm{inn}(e_1^2) \cup \mathrm{inn}(g) \cup \mathrm{inn}(e_1^2[e_2^2, e_1^2]) \\
&= \mathrm{inn}(f) \cup \mathrm{inn}(e_2^2) \cup \{e_1^2\} \cup \{g\} \cup \mathrm{inn}(e_2^2) \cup \mathrm{inn}(e_1^2) \\
&= \{f\} \cup \{e_2^2\} \cup \{e_1^2\} \cup \{g\} \cup \{e_2^2\} \cup \{e_1^2\} \\
&= \{e_1^2, e_2^2, f, g\}.
\end{aligned}$$

For $n \in \mathbb{N}$, $1 \leq j \leq n$ and F be variables over the two-elements alphabet $\{f, g\}$ where $\mathrm{ar}(F)$ denotes the arity of F. Let $1 \leq i \leq \mathrm{ar}(F)$. We define $P^i(t)$ by

(i) if $t = e_j^n$, then $P^i(t) = t$,

(ii) if $t = F$, then $P^i(t) = F$,

(iii) if $t = e_j^n[s_1, \ldots, s_n]$,then $P^i(t) = P^i(s_j)$

(iv) if $t = F(t_1, \ldots, t_i, \ldots, t_{\mathrm{ar}(F)})$, then $P^i(t) = FP^i(t_i)$.

For example, let f, g be binary cooperation symbols and $t = f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]$. Then

$$\begin{aligned}
P^1(t) &= P^1(f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]) \\
&= fP^1(g[e_1^2[f, e_2^2], e_1^2]) \\
&= fgP^1(e_1^2[f, e_2^2]) \\
&= fgP^1(f) \\
&= fgf
\end{aligned}$$

and

$$\begin{aligned}
P^2(t) &= P^2(f[g[e_1^2[f, e_2^2], e_1^2], e_2^2[g, e_1^2[e_2^2, e_1^2]]]) \\
&= fP^2(e_2^2[g, e_1^2[e_2^2, e_1^2]]) \\
&= fP^2(e_1^2[e_2^2, e_1^2]) \\
&= fP^2(e_2^2) \\
&= fe_2^2.
\end{aligned}$$

Now, we are ready to prove our results. In the case that $t_1 \in E_2$, $t_2 \notin E_2$ and $\mathrm{co}(t_2) = 1$, we obtain the following propositions.

Proposition 3.2. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_1 \in E_2$, $co(t_2) = 1$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} is idempotent.

(ii) t_2 *is one of the following forms;*

- $t_2 \in \{f, g\}$,
- *in* $P^1(t_2) = F_1 \cdots F_k$ *where* $F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$, *there is the smallest positive integer* $l \in \{1, \ldots, k\}$ *such that* $F_l = g$ *with a subcoterm* t'_2 *of* t_2 *such that*
  - $t'_2 = g$ *or*
  - $t'_2 = g[x_1, x_2]$ *where* $x_1, x_2 \in cT^{(2)}_{(2,2)}$ *and the following conditions hold;*
    - *if* $e^2_1 \in inn(t_2)$, *then* $M^1(x_1) = e^2_1$,
    - *if* $e^2_2 \in inn(t_2)$, *then* $M^2(x_2) = e^2_2$.

Proof. (i) $\Rightarrow$ (ii): Assume that $t_2 \notin \{f, g\}$. Suppose that, in $P^1(t_2) = F_1 \cdots F_k$ where $F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$, there is the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = f$. This implies that $\hat{\sigma}_{t_1,t_2} \in E_2$, which is a contradiction. This implies that in $P^1(t_2) = F_1 \cdots F_k$ where $F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$, there is the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with a subcoterm t'_2 of t_2. Let $t'_2 \neq g$. Assume that $t'_2 = g[x_1, x_2]$ where $x_1, x_2 \in \mathrm{cT}^{(2)}_{(2,2)}$. Since σ_{t_1,t_2} is idempotent,

$$t_2 = \hat{\sigma}_{t_1,t_2}[t_2] = S^2(t_2, \hat{\sigma}_{t_1,t_2}[x_1], \hat{\sigma}_{t_1,t_2}[x_2]).$$

Let $e^2_1 \in \mathrm{inn}(t_2)$. Suppose that $M^1(x_1) = e^2_2$. Then we have to replace e^2_1 in $\mathrm{inn}(t_2)$ of the coterm t_2 by e^2_2. Thus, $S^2(t_2, \hat{\sigma}_{t_1,t_2}[x_1], \hat{\sigma}_{t_1,t_2}[x_2]) \neq t_2$, which is a contradiction. Hence, if $e^2_1 \in \mathrm{inn}(t_2)$, then $M^1(x_1) = e^2_1$. Similarly, if $e^2_2 \in \mathrm{inn}(t_2)$, then $M^2(x_2) = e^2_2$.

(ii) $\Rightarrow$ (i): It is clear that $\hat{\sigma}_{t_1,t_2}[F] = F$ where $F \in \{f, g\}$. Thus, if $t_2 \in \{f, g\}$, then σ_{t_1,t_2} is idempotent. Assume that in $P^1(t_2) = F_1 \cdots F_k$ where $F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$, there is the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with a subcoterm t'_2 of t_2. In this case, since $t_1 \in E_2$, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. If $t'_2 = g$, then $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[t'_2] = \hat{\sigma}_{t_1,t_2}[g] = t_2$. Assume that $t'_2 = g[x_1, x_2]$ where $x_1, x_2 \in \mathrm{cT}^{(2)}_{(2,2)}$. Let $e^2_1 \in \mathrm{inn}(t_2)$. Then $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[t'_2] = \hat{\sigma}_{t_1,t_2}[g[x_1, x_2]] = S(t_2, e^2_1, \hat{\sigma}_{t_1,t_2}[x_2])$. Thus, we have to replace e^2_1 in $\mathrm{inn}(t_2)$ of the coterm t_2 by e^2_1. Similarly, if $e^2_2 \in \mathrm{inn}(t_2)$, then we have to replace e^2_2 in $\mathrm{inn}(t_2)$ of the coterm t_2 by e^2_2. Therefore, σ_{t_1,t_2} is idempotent. □

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_1 = e^2_2$ and $t_2 = e^3_2[e^2_2, e^2_1[g[e^2_1, e^2_2], e^2_2], e^2_1]$.
- σ_{t_1,t_2} is not idempotent where $t_1 = e^2_2$ and $t_2 = g[e^2_2, e^2_1]$.

Similarly, in case of $t_2 \in E_2$, $t_1 \notin E_2$ and $\mathrm{co}(t_1) = 1$, we have as follows.

Proposition 3.3. *Let* $\sigma_{t_1,t_2} \in WP(2, 2) \setminus P(2, 2)$, $t_2 \in E_2$, $co(t_1) = 1$. *Then the following statements are equivalent:*

(i) σ_{t_1,t_2} *is idempotent.*

(ii) t_1 *is one of the following forms;*

- $t_1 \in \{f, g\}$,
- *in* $P^1(t_1) = F_1 \cdots F_k$ *where* $F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$, *there is the smallest positive integer* $1 \leq l \leq k$ *such that* $F_l = f$ *with a subcoterm* t_1' *of* t_1 *such that*
  - $t_1' = f$ *or*
  - $t_1' = f[x_1, x_2]$ *where* $x_1, x_2 \in cT^{(2)}_{(2,2)}$ *and the following conditions hold;*
    * *if* $e^2_1 \in inn(t_1)$, *then* $M^1(x_1) = e^2_1$,
    * *if* $e^2_2 \in inn(t_1)$, *then* $M^2(x_2) = e^2_2$.

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_2 = e^2_2$ and $t_1 = e^3_1[f, e^2_1, e^2_2]$.
- σ_{t_1,t_2} is not idempotent where $t_2 = e^2_2$ and $t_1 = e^2_1[e^2_2, f[e^2_2, e^2_2]]$.

Remark 3.4. *We observe that for* $\sigma_{t_1,t_2} \in WP(2, 2) \smallsetminus P(2, 2)$, $t_1 \in E_2$ *and* $co(t_2) > 1$, *if* σ_{t_1,t_2} *is idempotent, then* $inn(t_2) \cap E_2 \neq \emptyset$. *To see this, we suppose that* $inn(t_2) \cap E_2 = \emptyset$. *This implies that* $inn(t_2) \cap \{f, g\} \neq \emptyset$. *If* $f \in inn(t_2)$, *then* $t_1 \in inn(t_2)$. *If* $g \in inn(t_2)$, *then the coterm* $\hat{\sigma}_{t_1,t_2}$ *is longer that the coterm* t_2. *These contradict that* σ_{t_1,t_2}. *Hence,* $inn(t_2) \cap E_2 \neq \emptyset$. *Dually, for* $\sigma_{t_1,t_2} \in WP(2, 2) \smallsetminus P(2, 2)$, $t_2 \in E_2$ *and* $co(t_1) > 1$, *if* σ_{t_1,t_2} *is idempotent, then* $inn(t_2) \cap E_2 \neq \emptyset$.

By the above observation, we obtain the following result.

Proposition 3.5. *Let* $\sigma_{t_1,t_2} \in WP(2, 2) \smallsetminus P(2, 2)$, $t_{j_1} \in E_2$, $co(t_{j_2}) > 1$ *where* j_1 *and* j_2 *are distinct elements in* $\{1, 2\}$. *We have that if* σ_{t_1,t_2} *is idempotent, then* $inn(t_{j_2}) \cap E_2 \neq \emptyset$.

In case that $t_1 \in E_2$, $t_2 \notin E_2$ and co$(t_2) > 1$, we obtain the following propositions.

Proposition 3.6. *Let* $\sigma_{t_1,t_2} \in WP(2, 2) \smallsetminus P(2, 2)$, $t_1 = e^2_m$, $m = 1, 2$, $co(t_2) > 1$, $t_2 = F[s_1, s_2]$ *with* $P^m(t_2) = F_1 \cdots F_k$ *for some natural number* k *where* $F, F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$ *and* $inn(t_2) \cap E_2 = \{e^2_j\}$, $j = 1, 2$. *Then the following statements are equivalent:*

(i) σ_{t_1,t_2} *is idempotent.*

(ii) In $P^m(t_2)$, *there exists the smallest positive integer* $l \in \{1, \ldots, k\}$ *such that* $F_l = g$ *with a subcoterm* t_2', *and one of the following conditions hold;*

- *if* $inn(t_2) \cap \{f, g\} \neq \emptyset$, *then* $t_2' = g$,
- *if* $inn(t_2) \cap \{f, g\} = \emptyset$, *then* $t_2' = g[s_1', s_2']$, $s_1', s_2' \in cT^{(2)}_{(2,2)}$ *such that the set of cooperation symbols occurring in* $P^m(s_j')$ *is* $\{f\}$ *or* $\emptyset$.

Proof. (i) $\Rightarrow$ (ii): Since $P^m(t_2) = F_1 \cdots F_k$ where $F_l \in \{f, g\} \cup E$, $1 \leq l \leq k$ for some natural number k, then there exists $q \in \{1, \ldots, k\}$ such that $F_q = g$ since otherwise $\hat{\sigma}_{t_1,t_2}[t_2] \in E_2$, which is a contradiction. Let $l \in \{1, \ldots, k\}$ be the smallest positive integer such that $F_l = g$ with the subcoterm t_2' of t_2 where $t_2' = g$ or $t_2' = g[s_1', s_2']$, $s_1', s_2' \in cT^{(2)}_{(2,2)}$. Let $\text{inn}(t_2) \cap \{f, g\} \neq \emptyset$. Suppose that $t_2' = g[s_1', s_2']$ where $s_1', s_2' \in cT^{(2)}_{(2,2)}$. Since σ_{t_1,t_2} is idempotent, we consider

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= \hat{\sigma}_{t_1,t_2}[g[s_1', s_2']] \\ &= S^2_2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']) \\ &= S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']). \end{aligned}$$

If f occurs in $\text{inn}(t_2)$, then we have to replace f in $\text{inn}(t_2)$ of the coterm t_2 by $f[e^2_j, e^2_j]$. Thus, $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']) \neq t_2$, which is a contradiction. If g occurs in $\text{inn}(t_2)$, then the coterm $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2'])$ must be longer that the coterm t_2. These implies that $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']) \neq t_2$, which is a contradiction. Thus, we obtain $t_2' = g$.

Let $\text{inn}(t_2) \cap \{f, g\} = \emptyset$. Then we have that $t_2' = g[s_1', s_2']$ and $s_1', s_2' \in cT^{(2)}_{(2,2)}$. Suppose that the nonempty set of cooperation symbols occurring in $P^m(s_j')$ is not $\{f\}$. Since σ_{t_1,t_2} is idempotent, we consider

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= \hat{\sigma}_{t_1,t_2}[g[s_1', s_2']] \\ &= S^2_2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']) \\ &= S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2']). \end{aligned}$$

This implies that the coterm $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s_1'], \hat{\sigma}_{t_1,t_2}[s_2'])$ must be longer than the coterm t_2. This follows that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, which is a contradiction. Therefore $t_2' = g[s_1', s_2']$, $s_1', s_2' \in cT^{(2)}_{(2,2)}$ such that the set of cooperation symbols occurring in $P^m(s_j')$ is $\{f\}$ or $\emptyset$.

(ii) $\Rightarrow$ (i): Without loss of generality, we assume that $m, j = 1$. Then, in $P^1(t_2)$, there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with a subcoterm t_2'. Assume that $\text{inn}(t_2) \cap \{f, g\} \neq \emptyset$. Then $t_2' = g$. This clearly implies that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Assume that $\text{inn}(t_2) \cap \{f, g\} = \emptyset$. Then $t_2' = g[s_1', s_2']$, $s_1', s_2' \in cT^{(2)}_{(2,2)}$ such that the set of cooperation symbols occurring in $P^1(s_1')$ is $\{f\}$ or $\emptyset$. Since $\text{inn}(t_2) \cap E_2 = \{e^2_1\}$, we obtain that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. $\square$

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_1 = e^2_1$ and $t_2 = e^2_2[e^2_1, f[g[e^2_1, e^2_1], g[e^2_1, f[e^2_1, e^2_1]]]]$.

- σ_{t_1,t_2} is not idempotent where $t_1 = e_2^2$ and $t_2 = g[e_2^2, g[e_2^2, e_2^2]]$.

Similarly, we can prove the following proposition.

Proposition 3.7. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_2 = e_m^2$, $m = 1,2$, $co(t_1) > 1$, $t_1 = F[s_1, s_2]$ with $P^m(t_1) = F_1 \cdots F_k$ for some natural number k where $F, F_i \in \{f,g\} \cup E$, $1 \leq i \leq k$ and $inn(t_1) \cap E_2 = \{e_j^2\}$, $j = 1,2$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} is idempotent.

(ii) In $P^m(t_1)$, there exists the smallest positive integer $l \in \{1,\ldots,k\}$ such that $F_l = f$ with a subcoterm t_1', and one of the following conditions hold;

- *if $inn(t_1) \cap \{f,g\} \neq \emptyset$, then $t_1' = f$,*
- *if $inn(t_1) \cap \{f,g\} = \emptyset$, then $t_1' = f[s_1', s_2']$, $s_1', s_2' \in cT_{(2,2)}^{(2)}$ such that the set of cooperation symbols occurring in $P^m(s_j')$ is $\{g\}$ or $\emptyset$.*

For example, let f and g be cooperation symbols of type $(2,2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_2 = e_1^2$ and $t_1 = g[f[e_2^2, e_2^2], f[e_2^2, e_2^2]]$.
- σ_{t_1,t_2} is not idempotent where $t_2 = e_2^2$ and $t_1 = f[g, g[e_1^2, e_1^2]]$.

Next, we give a characterization that a weak projection cohypersubstitution σ_{t_1,t_2} is idempotent where $\text{inn}(t_i) = E_2$, $i = 1,2$.

Proposition 3.8. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_1 = e_1^2$, $co(t_2) > 1$, $t_2 = F[s_1, s_2]$ with $P^1(t_2) = F_1 \cdots F_k$ for some natural number k where $F, F_i \in \{f,g\} \cup E$, $1 \leq i \leq k$ and $inn(t_2) = E_2$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} is idempotent.

(ii) In $P^1(t_2)$, there exists the smallest positive integer $l \in \{1,\ldots,k\}$ such that $F_l = g$ with a subcoterm t_2' of t_2, and one of the following conditions holds;

- *$t_2' = g$,*
- *if $t_2' = g[s_1', s_2']$ where $s_1', s_2' \in cT_{(2,2)}^{(2)}$, then the following conditions hold;*
  - *the set of cooperation symbols occurring in $P^1(s_1')$ is $\{f\}$ or $\emptyset$, and $M^1(s_1') = e_1^2$ or $M^1(s_1') = f$,*
  - *the set of cooperation symbols occurring in $P^1(s_2')$ is $\{f\}$ or $\emptyset$ and $M^1(s_2') = e_2^2$.*

Proof. (i) $\Rightarrow$ (ii): Since $P^1(t_2) = F_1 \cdots F_k$ where $F_l \in \{f,g\} \cup E$, $1 \leq l \leq k$ for some natural number k, then there exists $q \in \{1,\ldots,k\}$ such that $F_q = g$ since otherwise $\hat{\sigma}_{t_1,t_2}[t_2] \in E_2$, which is a contradiction. Let $l \in \{1,\ldots,k\}$ be the smallest positive integer such that $F_l = g$ with the subcoterm t_2' of t_2 where $t_2' = g$ or $t_2' = g[s_1', s_2']$, $s_1', s_2' \in \text{cT}_{(2,2)}^{(2)}$. Assume that $t_2' \neq g$. Then $t_2' = g[s_1', s_2']$,

$s'_1, s'_2 \in cT^{(2)}_{(2,2)}$. Suppose that the nonempty set of cooperation symbols occurring in $P^1(s'_1)$ is not $\{f\}$. Since σ_{t_1,t_2} is idempotent,

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t'_2] \\ &= \hat{\sigma}_{t_1,t_2}[g[s'_1, s'_2]] \\ &= S^2_2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2]) \\ &= S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2]). \end{aligned}$$

This implies that the coterm $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2])$ must be longer that the coterm t_2. This follows that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, which is a contradiction. Hence, the set of cooperation symbols occurring in $P^1(s'_1)$ is $\{f\}$ or $\emptyset$. Suppose that $M^1(s'_1) = e^2_2$. Then we have to replace e^2_1 in $\text{inn}(t_2)$ of the coterm t_2 by e^2_2. This implies that $t_2 \neq \hat{\sigma}_{t_1,t_2}[t_2]$, which is a contradiction. Suppose that $M^1(s'_1) = g$. Then the coterm $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2])$ must be longer that the coterm t_2. This implies that $t_2 \neq \hat{\sigma}_{t_1,t_2}[t_2]$, which is a contradiction. If $M^1(s'_1) = e^2_1$ or $M^1(s'_1) = f$, then we replace e^2_1 in $\text{inn}(t_2)$ of the coterm t_2 by e^2_1. Hence, $M^1(s'_1) = e^2_1$ or $M^1(s'_1) = f$. Similarly, we can show that the set of cooperation symbols occurring in $P^1(s'_2)$ is $\{f\}$ or $\emptyset$. Suppose that $M^1(s'_2) = e^2_1$ or $M^1(s'_2) = f$. Then we have to replace e^2_2 in $\text{inn}(t_2)$ of the coterm t_2 by e^2_1. This implies that $t_2 \neq \hat{\sigma}_{t_1,t_2}[t_2]$. Suppose that $M^1(s'_2) = g$. Then the coterm $S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2])$ must be longer that coterm t_2. Hence, for any cases we obtain $M^1(s'_2) = e^2_2$.

(ii) $\Rightarrow$ (i): In $P^1(t_2)$, there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with a subcoterm t'_2. It is clear that if $t'_2 = g$, then $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[t'_2] = \hat{\sigma}_{t_1,t_2}[g] = t_2$. Now, if $t'_2 \neq g$ we consider

$$\begin{aligned} \hat{\sigma}_{t_1,t_2}[t_2] &= \hat{\sigma}_{t_1,t_2}[t'_2] \\ &= \hat{\sigma}_{t_1,t_2}[g[s'_1, s'_2]] \\ &= S^2_2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2]) \\ &= S^2_2(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \hat{\sigma}_{t_1,t_2}[s'_2]). \end{aligned}$$

If $M^1(s'_1) = e^2_1$ and $M^1(s'_2) = e^2_2$, then we have to replace e^2_1, e^2_2, f and g in $\text{inn}(t_2)$ by e^2_1, e^2_2, f and g, respectively. So, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. If $M^1(s'_1) = f$ and $M^1(s'_2) = e^2_2$, then we have to replace e^2_1, e^2_2, f and g in $\text{inn}(t_2)$ by e^2_1, e^2_2, f and g, respectively. So, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. □

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_1 = e^2_1$ and $t_2 = e^2_1[f[g[f, e^2_2], g[f[e^2_2, e^2_1], f]], g[g, f]]$.
- σ_{t_1,t_2} is not idempotent where $t_1 = e^2_1$ and $t_2 = f[g[e^2_2, e^2_1], f[e^2_1, e^2_1]]$.

Similarly, we can prove the following propositions.

Proposition 3.9. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_1 = e_2^2$, $co(t_2) > 1$, $t_2 = F[s_1, s_2]$ with $P^2(t_2) = F_1 \cdots F_k$ for some natural number k where $F, F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$ and $inn(t_2) = E_2$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} *is idempotent.*

(ii) In $P^2(t_2)$, there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with a subcoterm t_2', and one of the following conditions hold;

- $t_2' = g$,
- *if $t_2' = g[s_1', s_2']$ where $s_1', s_2' \in cT^{(2)}_{(2,2)}$, then the following conditions hold;*
  - *the set of cooperation symbols occurring in $P^2(s_1')$ is $\{f\}$ or $\emptyset$, and $M^2(s_1') = e_1^2$,*
  - *the set of cooperation symbols occurring in $P^2(s_2')$ is $\{f\}$ or $\emptyset$ and $M^2(s_2') = e_2^2$ or $M^2(s_2') = g$.*

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_1 = e_2^2$ and $t_2 = g[f[g, e_1^2], f[e_2^2, f[e_1^2, e_2^2]]]$.
- σ_{t_1,t_2} is not idempotent where $t_1 = e_2^2$ and $t_2 = e_1^1[f[g, f[e_1^2, e_2^2]]]$.

Proposition 3.10. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_2 = e_1^2$, $co(t_2) > 1$, $t_1 = F[s_1, s_2]$ with $P^1(t_1) = F_1 \cdots F_k$ for some natural number k where $F, F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$ and $inn(t_1) = E_2$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} *is idempotent.*

(ii) In $P^1(t_1)$, there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = f$ with a subcoterm t_1', and one of the following conditions hold;

- $t_1' = f$,
- *if $t_1' = f[s_1', s_2']$ where $s_1', s_2' \in cT^{(2)}_{(2,2)}$, then the following conditions hold;*
  - *the set of cooperation symbols occurring in $P^1(s_1')$ is $\{g\}$ or $\emptyset$, and $M^1(s_1') = e_1^2$ or $M^1(s_1') = f$,*
  - *the set of cooperation symbols occurring in $P^1(s_2')$ is $\{g\}$ or $\emptyset$ and $M^1(s_2') = e_2^2$.*

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_2 = e_1^2$ and $t_1 = g[g[f, e_1^2], f[e_2^2, g]]$.
- σ_{t_1,t_2} is not idempotent where $t_2 = e_1^2$ and $t_1 = g[f[e_2^2, e_1^2], g[e_1^2, e_2^2]]$.

Proposition 3.11. *Let $\sigma_{t_1,t_2} \in WP(2,2) \setminus P(2,2)$, $t_2 = e_2^2$, $co(t_2) > 1$, $t_1 = F[s_1, s_2]$ with $P^2(t_1) = F_1 \cdots F_k$ for some natural number k where $F, F_i \in \{f, g\} \cup E$, $1 \leq i \leq k$ and $inn(t_1) = E_2$. Then the following statements are equivalent:*

(i) σ_{t_1,t_2} *is idempotent.*

(ii) *In $P^2(t_1)$, there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = f$ with a subcoterm t_1', and one of the following conditions hold;*

- $t_1' = f$,
- *if $t_1' = f[s_1', s_2']$ where $s_1', s_2' \in cT^{(2)}_{(2,2)}$, then the following conditions hold;*
  - *the set of cooperation symbols occurring in $P^2(s_1')$ is $\{g\}$ or $\emptyset$, and $M^2(s_1') = e_1^2$,*
  - *the set of cooperation symbols occurring in $P^2(s_2')$ is $\{g\}$ or $\emptyset$ and $M^2(s_2') = e_2^2$ or $M^2(s_2') = f$.*

For example, let f and g be cooperation symbols of type $(2, 2)$. Then we have

- σ_{t_1,t_2} is idempotent where $t_2 = e_2^2$ and $t_1 = e_1^2[g[f[e_1^2, e_1^2], f[e_1^2, e_2^2]], f[g[e_1^2, e_1^2], f[e_1^2, e_2^2]]]$.
- σ_{t_1,t_2} is not idempotent where $t_2 = e_2^2$ and $t_1 = e_1^2[g[f[e_1^2, e_1^2], f[e_2^2, e_2^2]], f[g[e_1^2, e_1^2], f[e_1^2, e_2^2]]]$.

Acknowledgements : The authors would like to thank the referees for his comments and suggestions on the manuscript. The research was supported by Development and Promotion of Science and Technology Talents Project (DPST), Department of Mathematics, Faculty of Science, Khon Kaen University.

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(Received 26 March 2017)
(Accepted 30 June 2017)