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Metric Space of Subcopulas

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Abstract : Sklar's theorem states that any joint distribution function can be written as a composition of its marginal distributions and a subcopula. Structural study of the latter is therefore natural. In this work, we define a new metric on the space of subcopulas making the space of copula its subspace. This is done via suitably extended subcopulas to joint distribution functions. Relationship between this new metric and the previously defined metric on the space of subcopulas is also discussed.

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1 Introduction and Preliminaries

A (bivariate) *copula* is a joint distribution of two random variables uniformly distributed on the unit interval $[0,1]$. A subcopula is a restriction of a copula on some closed subset $\mathcal{A} \times \mathcal{B}$ of $[0,1]^2$ such that both zero and one belongs to both $\mathcal{A}$ and $\mathcal{B}$. Equivalently, a *subcopula* is a function $S : \mathcal{A} \times \mathcal{B} \to [0,1]$, where both $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $[0,1]$ containing zero and one, satisfies the following properties:

(G) $S(a,0) = 0 = S(0,b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

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(U) $S(a,1)=a$ and $S(1,b)=b$ for all $a\in\mathcal{A}$ and $b\in\mathcal{B}$, and

(2I) $V_S([a,b]\times[c,d])\geq 0$ where

$$V_S([a,b]\times[c,d]) = S(b,d)-S(a,d)-S(b,c)+S(a,c) \tag{1.1}$$

for all $a,c\in\mathcal{A}$ and $b,d\in\mathcal{B}$ such that $a\leq c$ and $b\leq d$.

For example, functions M and W defined by

$$M(u,v)=\min(u,v)$$

and

$$W(u,v)=\max(u+v-1,0)$$

for all $u,v\in[0,1]$ are (sub)copulas. Moreover, a function $S:[0,1]^2\to[0,1]$ is a subcopula if and only if $W\leq S\leq M$ on the domain of S and the condition **(2I)** holds.

Sklar Theorem [1] states that any (bivariate) joint distribution function of random variables can be written as a composition of a subcopula and its marginals. Moreover, this composition also determines the joint distribution. To be precise, let $\mathfrak{F}$ denote the space of distribution functions, $\mathfrak{H}$ denote the space of joint distribution functions and $\mathfrak{S}$ denote the set of subcopulas. Then there is a map $\alpha:\mathfrak{H}\to\mathfrak{S}\times\mathfrak{F}\times\mathfrak{F}$ such that the following statement holds.

$$\forall H\in\mathfrak{H}, \alpha(H)=(S,F,G)\Longleftrightarrow\begin{cases}\mathrm{dom}(S)=\overline{\mathrm{Range}(F)\times\mathrm{Range}(G)} & \text{and}\\ \forall x,y\in\mathbb{R}, H(x,y)=S(F(x),G(y))\end{cases}$$

where $\overline{\mathcal{A}}$ denotes the closure of the set $\mathcal{A}$. Clearly, F and G in the above identification are the marginals of H and S can be computed via $S(u,v)=H(F^-(u),G^-(v))$ for all $u\in\overline{\mathrm{Range}(F)}$ and $v\in\overline{\mathrm{Range}(G)}$. Here F^- denotes the *quantile function* associated with F, that is,

$$F^-(u)=\inf\{x\in\mathbb{R}:F(x)\geq u\}$$

for all $u\in[0,1]$. It follows that α defines a one-one correspondence between the set $\mathfrak{H}$ of joint distribution functions and the set $\hat{\mathfrak{H}}$ of triple $(S,F,G)\in\mathfrak{S}\times\mathfrak{F}\times\mathfrak{F}$ such that $\mathrm{dom}(S)=\overline{\mathrm{Range}(F)\times\mathrm{Range}(G)}$.

In [2], a metric on $\mathfrak{S}$ based on Hausdorff distance was introduced. Recall that the Hausdorff distance $h_d(A,B)$ between two closed subsets A and B of a compact metric space (X,d) is given by

$$h_d(A,B)=\max\left(\max_{y\in B}d(y,A),\max_{x\in A}d(x,B)\right).$$

With this, the space $K(X)$ of closed subsets of (X,d) is a compact metric space [3, Appendix B]. The convergence in $K(X)$ is characterized by the following theorem. Its proof is an immediate application of [3, Corollary C.6 and Theorem C.2 (iii)] (see also [3, Definition B.4 and Definition B.5] for definitions of related concepts).

Theorem 1.1. *Let (X, d) be a compact metric space. A sequence K_n converges to K in $(K(X), h_d)$ if and only if K_n converges to K in the sense of Painleve-Kuratowski, that is, if and only if the following holds.*

1. *For any $x_{n(k)} \in K_{n(k)}$ such that $n(k) \to \infty$ and $x_{n(k)} \to x$ as $k \to \infty$, x must belong to K.*
2. *For any $x \in K$, there is $x_n \in K_n$ such that $x_n \to x$.*

Denote d_2 the Euclidean distance and d_∞ the supremum distance. Define $\bar{d} : \mathfrak{S} \times \mathfrak{S} \to [0, \infty)$ by letting

$$\bar{d}(A, B) = h_{d_\infty}([A], [B]) + h_{d_2}(\operatorname{dom}(A), \operatorname{dom}(B)) \tag{1.2}$$

where $[A]$ denote the set of all copulas extending a subcopula A. For copulas A and B, $\bar{d}(A, B) = h_{d_\infty}([A], [B]) + h_{d_2}\left([0,1]^2, [0,1]^2\right) = d_\infty(A, B)$. Thus, the space of copula $(\mathfrak{C}, d_\infty)$ is a subspace of $\left(\mathfrak{S}, \bar{d}\right)$ and that $\left(\mathfrak{S}, \bar{d}\right)$ is a compact metric space [2, Corollary 4.5]. The convergences in $\left(\mathfrak{S}, \bar{d}\right)$ is characterized by the following result.

Theorem 1.2. [2, Theorem 4.4] *The sequence (S_n) converges to S in $\left(\mathfrak{S}, \bar{d}\right)$ if and only if the following two conditions hold.*

(C1) $\operatorname{dom}(S_n) \to \operatorname{dom}(S)$ *under the Hausdorff metric.*

(C2) *For all $x_n \in \operatorname{dom}(S_n)$ and $x \in \operatorname{dom}(S)$ such that $x_n \to x$, $S_n(x_n) \to S(x)$.*

In this work, will define another metric on $\mathfrak{S}$. We believe this new metric is more natural and is easier to compute since it does not rely on the distance between sets, yet, it has similar advantages as the metric $\bar{d}$.

2 Main Results

Denote

$$\hat{S}(u, v) = \sup\{S(s, t) : (s, t) \in \operatorname{dom}(S), s \le u, t \le v\}$$

where $\min(u, v) > 0$ and $\hat{S}(u, v) = 0$ when $\min(u, v) \le 0$ for all $u, v \in \mathbb{R}$ and $S \in \mathfrak{S}$. Since this extension all have the same domain, it is possible to define supremum distance between them, that is, define

$$\hat{d}(S, T) = d_\infty(\hat{S}, \hat{T})$$

for all $S, T \in \mathfrak{S}$. It turns out that the topology induced by $\left(\mathfrak{S}, \hat{d}\right)$ is stronger than that induced by $\left(\mathfrak{S}, \bar{d}\right)$. It follows that $\alpha^{-1} : \hat{\mathfrak{H}} \to \mathfrak{H}$ is continuous. The proofs of these results are carefully explained below.

2.1 Distribution Form of Subcopulas

For any closed set $\mathcal{A} \subseteq [0,1]$ containing zero and one, denote $l(\mathcal{A})$ the set of all $x \in (0,1]$ such that $(x-\epsilon, x] \cap \mathcal{A} = \{x\}$ for some small $\epsilon > 0$ and $r(\mathcal{A})$ the set of all $x \in [0,1)$ such that $[x, x+\epsilon) \cap \mathcal{A} = \{x\}$ for some small $\epsilon > 0$. Denote also

$$D_{\mathcal{A}}(x) = \sup \{a \in \mathcal{A} : a \leq x\}$$

when $x \geq 0$ and $D_{\mathcal{A}}(x) = 0$ otherwise. Then $D_{\mathcal{A}}$ is non-decreasing, right continuous, and $D_{\mathcal{A}}(x) = x$ for all $x \in \mathcal{A}$. Thus, $D_{\mathcal{A}}$ is a distribution function. We will called $D_{\mathcal{A}}$ the *subuniform distribution* with support $\mathcal{A}$. Clearly, $D_{\mathcal{A}} \neq D_{\mathcal{A}}^-$ unless $\mathcal{A} = [0,1]$. Also, discontinuity points of $D_{\mathcal{A}}$ are exactly those that belongs to $l(\mathcal{A})$ while the discontinuity points of $D_{\mathcal{A}}^-$ are exactly those that belongs to $r(\mathcal{A})$. For example, the following figure show graphs of both $D_{\mathcal{A}}$ and $D_{\mathcal{A}}^-$ when $\mathcal{A} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

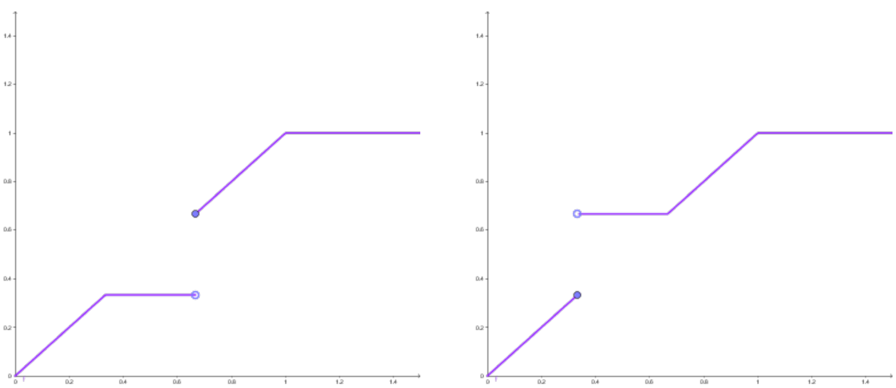

Figure 1: Graphs of $D_{\mathcal{A}}$ (left) and $D_{\mathcal{A}}^-$ (right) when $\mathcal{A} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

Note that a subuniform distributions is not a uniform distribution unless its support is the unit interval since the former is not continuous while the latter is.

For any $S \in \mathfrak{S}$ and $u, v \in \mathbb{R}$, denote $\hat{S}(u,v) = 0$ when $\min(u,v) \leq 0$ and

$$\hat{S}(u,v) = \sup \{S(s,t) : (s,t) \in \operatorname{dom}(S), s \leq u, t \leq v\}$$

otherwise. Clearly, $\hat{S}$ is a right-continuous extension of S. Since S satisfies **(2I)**, $\hat{S}$ is actually a joint distribution function. We will called $\hat{S}$ the *distribution form* of S. The marginal distributions of $\hat{S}$ are $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ where $\mathcal{A} \times \mathcal{B}$ is the domain of S. Also, $\alpha(\hat{S}) = (S, D_{\mathcal{A}}, D_{\mathcal{B}})$. One advantage of using distribution forms is that their domains are always $\mathbb{R}^2$ while the domain of subcopulas might vary. For example, it is possible to consider supremum distance between two distribution forms of two subcopulas even when the supremum distance between those two subcopulas themselves are not well-defined due to domain inequality.

Another extension that has the same domain advantage is the function S^- defined by $S^-(u,v) = H(F^-(u), G^-(v))$ for all $u, v \in [0,1]$ where $\alpha(H) = (S, F, G)$. Equivalently,

$$S^-(u,v) = \inf\{S(s,t) : (s,t) \in \mathrm{dom}(S), s \geq u, t \geq v\}$$

whenever $u, v \leq 1$ and $S^-(u,v) = \min(u,v,1)$ otherwise. Due to the fact that F^- and G^- are not necessary right continuous, S^- is also not necessary right continuous which means S^- is not necessary a joint distribution function. Thus, we dismiss S^- from consideration.

Another advantage of distribution forms is the following change of variable formula.

Proposition 2.1. *Let H be a joint distribution and $\alpha(H) = (S, F, G)$. If H is the joint distribution of random variables X and Y, then $\hat{S}$ is the joint distribution of random variables $F(X)$ and $G(Y)$. If $\hat{S}$ is a joint distribution function of random variables U and V, then H is the joint distribution of random varibles $F^-(U)$ and $G^-(V)$. Particularly,*

$$\int h(u,v)\mathrm{d}\hat{S}(u,v) = \int h(F(x), G(y))\mathrm{d}H(x,y)$$

for any integrable function h.

Proof. Assume H is the joint distribution of random variables X and Y. Using the fact that $F(x) = F(F^-(F(x)))$, $\mathbb{P}(X \leq x) = \mathbb{P}(X \leq F^-F(x))$ for all $x \in \mathbb{R}$. It follows that

$$\begin{aligned}\mathbb{P}\left(X \neq F^-(F(X))\right) &\leq \mathbb{P}\left(X \in \left\{x \in \mathbb{R} : F^-F(x) < x\right\}\right) \\ &= 0,\end{aligned}$$

that is, $X = F^-(F(X))$ with probability one. Similarly, $Y = G^-(G(Y))$ with probability one. Using monotonicity of F and G, we have

$$\begin{aligned}\mathbb{P}(X \leq x, Y \leq y) &\leq \mathbb{P}(F(X) \leq F(x), G(Y) \leq G(y)) \\ &= \mathbb{P}\left(FF^-F(X) \leq F(x), GG^-G(Y) \leq G(y)\right) \\ &\leq \mathbb{P}\left(F^-F(X) \leq x, G^-G(Y) \leq y\right) \\ &= \mathbb{P}(X \leq x, Y \leq y),\end{aligned}$$

that is,

$$\mathbb{P}(F(X) \leq F(x), G(Y) \leq G(y)) = H(x,y) = S(F(x), G(y))$$

for all $x, y \in \mathbb{R}$. Combining this with the fact that

$$\mathbb{P}(F(X) \in \mathrm{Range}(F), G(Y) \in \mathrm{Range}(G)) = 1,$$

$\hat{S}$ is the joint distribution function of $F(X)$ and $G(Y)$. Particularly,

$$\int h(u,v)\mathrm{d}\hat{S}(u,v) = \mathbb{E}h(F(X),G(Y)) = \int h(F(x),G(y))\mathrm{d}H(x,y)$$

for any integrable function h.

Last, assume that $\hat{S}$ is the joint distribution function of U and V. Then U and V have the same joint distribution as $F(X)$ and $G(Y)$. This means $F^{-}(U)$ and $G^{-}(V)$ have the same joint distribution as $F^{-}F(X) = X$ and $G^{-}G(Y) = Y$. $\square$

2.2 Supremum Distance between Subuniform Distributions

In this part, we will study the metric space $(\mathfrak{U}, d_\infty)$ where $\mathfrak{U}$ is the set of subuniform distributions defined in the previous section. Since subuniform distributions can be identified by their supports, it is natural to compare the convergences in $(\mathfrak{U}, d_\infty)$ with the convergences of their supports under Hausdorff metric.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be closed subsets of $[0,1]$ containing both zero and one. Then*

$$h_{d_2}(\mathcal{A},\mathcal{B}) \le d_\infty(D_\mathcal{A}, D_\mathcal{B}).$$

Proof. Assume that $\mathcal{A} \neq \mathcal{B}$; otherwise, the result is obvious. Without loss of generality, we may assume that $h_{d_2}(\mathcal{A},\mathcal{B}) = \max_{a\in\mathcal{A}} d_2(a,\mathcal{B})$. Let $x \in \mathcal{A}$ and $y \in \mathcal{B}$ be such that $|x-y| = \min_{b\in\mathcal{B}}|x-b| = h_{d_2}(\mathcal{A},\mathcal{B})$. Since $h_{d_2}(\mathcal{A},\mathcal{B}) > 0$, $x \notin \mathcal{B}$. Particularly, $x \neq y$.

Case 1. $y < x$.

Suppose there is $z \in \mathcal{B} \cap (y,x]$. Then $x - z < x - y = \min_{b\in\mathcal{B}}|x-b| \le x - z$ which is a contradiction. Thus, $\mathcal{B}\cap(y,x] = \emptyset$. It follows that

$$d_\infty(D_\mathcal{A}, D_\mathcal{B}) \ge D_\mathcal{A}(x) - D_\mathcal{B}(x) = x - y.$$

Case 2. $y > x$

Suppose there is $z \in \mathcal{B}\cap(2x-y,x]$. Then $x - z < x-(2x-y) = y-x = \min_{b\in\mathcal{B}}|x-b| \le x-z$ which is a contradiction. Thus, $\mathcal{B}\cap(2x-y,x] = \emptyset$. It follows that $D_\mathcal{B}(x) \le 2x-y$. Thus,

$$d_\infty(D_\mathcal{A}, D_\mathcal{B}) \ge |D_\mathcal{A}(x) - D_\mathcal{B}(x)| \ge x-(2x-y) = y-x.$$

In either cases, $h_{d_2}(\mathcal{A},\mathcal{B}) \le d_\infty(D_\mathcal{A}, D_\mathcal{B})$. $\square$

Using the same technique, we can also prove that $h_{d_2}(\mathcal{A},\mathcal{B}) \le d_\infty\left(D^-_\mathcal{A}, D^-_\mathcal{B}\right)$ for all closed subsets $\mathcal{A}$ and $\mathcal{B}$ of $[0,1]$ containing both zero and one. The converse does not hold, however. Consider for example the sequence $\mathcal{A}_n = \left[0,\frac{1}{n}\right]\cup\left[1-\frac{1}{n},1\right]$. Then $h_{d_2}(\mathcal{A}_n,\mathcal{A}_\infty) = \frac{1}{n}$ where $\mathcal{A}_\infty = \{0,1\}$. However,

$$d_\infty(D_{\mathcal{A}_n}, D_{\mathcal{A}_\infty}) \ge D_{\mathcal{A}_n}(1-\frac{1}{n}) - D_{\mathcal{A}_\infty}(1-\frac{1}{n}) = 1-\frac{1}{n}$$

and

$$d_\infty\left(D^-_{\mathcal{A}_n}, D^-_{\mathcal{A}_\infty}\right) \geq D^-_{\mathcal{A}_\infty}(\frac{1}{n}) - D_{\mathcal{A}_n}(\frac{1}{n}) = 1 - \frac{1}{n}.$$

Thus, we can conclude that the correspondence $D_\mathcal{A} \mapsto \mathcal{A}$ is continuous while its inverse is not. This example can also be adapted to show that uniform convergences of subuniform distributions does not implies uniform convergences of their corresponding quantiles and vice versa. For example, let $\mathcal{B}_n = \left[0, \frac{1}{n}\right] \cup \{1\}$. Then $d_\infty\left(D_{\mathcal{B}_n}, D_{\mathcal{A}_\infty}\right) \to 0$ while $d_\infty\left(D^-_{\mathcal{B}_n}, D^-_{\mathcal{A}_\infty}\right) \to 1$. Similarly, $\mathcal{C}_n = \{0\} \cup \left[1 - \frac{1}{n}, 1\right]$ implies $d_\infty\left(D^-_{\mathcal{C}_n}, D^-_{\mathcal{A}_\infty}\right) \to 0$ while $d_\infty\left(D_{\mathcal{C}_n}, D_{\mathcal{A}_\infty}\right) \to 1$.

Next, we will show that $(\mathfrak{U}, d_\infty)$ is a closed subspace of $(\mathfrak{F}, d_\infty)$. This will also implies $(\mathfrak{U}, d_\infty)$ is compact since $(\mathfrak{F}, d_\infty)$ is.

Corollary 2.3. *The space* $(\mathfrak{U}, d_\infty)$ *is complete. Equivalently,* $(\mathfrak{U}, d_\infty)$ *is a closed subspace of* $(\mathfrak{F}, d_\infty)$.

Proof. Let $\mathcal{A}_n$ be closed subsets of $[0,1]$ containing both zero and one such that $(D_{\mathcal{A}_n})$ is a Cauchy sequence in $(\mathfrak{U}, d_\infty)$. Then, $(\mathcal{A}_n)$ is Cauchy under Hausdorff metric by Proposition (2.2). Thus, $\mathcal{A}_n \to \mathcal{A}$ for some closed subset $\mathcal{A}$ of $[0,1]$. Since all $\mathcal{A}_n$ contains both zero and one, $\mathcal{A}$ also contains both zero and one. Since $(\mathfrak{F}, d_\infty)$ is compact, $D_{\mathcal{A}_n} \to F$ for some $F \in \mathfrak{F}$. We will show that $F = D_\mathcal{A}$. Since all $D_{\mathcal{A}_n}$ agree outside the unit interval, $F = D_\mathcal{A}$ outside the unit interval too.

Next, consider a continuity point $x \in [0,1]$ of F. If $x \in \mathcal{A}$, there is a sequence $x_n \in \mathcal{A}_n$ such that $x_n \to x$. Thus, $F(x) = \lim_{n\to\infty} D_{\mathcal{A}_n}(x_n) = \lim_{n\to\infty} x_n = x = D_\mathcal{A}(x)$. If $x \notin \mathcal{A}$, set $x_n = \max \mathcal{A}_n \cap [0, x)$. Since $(x_n) \subseteq [0,1]$, it has a convergence subsequence, say $x_{n(k)} \to y$ as $k \to \infty$. It follows that $y \in \mathcal{A}$ and $y < x$. Suppose there is $z \in \mathcal{A} \cap (y, x)$. Then there is $z_n \in \mathcal{A}_n$ such that $z_n \to z$. Since $z < x$, $z_n < x$ for large n. Thus, $z_{n(k)} \leq x_{n(k)}$ for large k which directly implies $z \leq y$, a contradiction. Thus, $\mathcal{A} \cap (y, x) = \emptyset$ and $D_\mathcal{A}(x) = D_\mathcal{A}(y)$. Now, $D_\mathcal{A}(y) = F(y) \leq F(x) = \lim_{n\to\infty} D_{\mathcal{A}_n}(x_n) = \lim_{n\to\infty} x_n = y = D_\mathcal{A}(y)$. Hence $F(x) = D_\mathcal{A}(x)$ also.

Last, consider when $x \in [0,1]$ is not a continuity point of F. Since there are only countably many such points, there is a sequence (x_n) of continuity points of F such that $x_n \searrow x$. Since both F and $D_\mathcal{A}$ are right continuious, $F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} D_\mathcal{A}(x_n) = D_\mathcal{A}(x)$. □

We end this section with the comparison between the Hausdorff distance of domains of subcopulas and the supremum distance of their corresponding marginal distributions. Since the domain of a subcopula is a product of closed sets and its corresponding marginal distributions are subuniform, the statement takes the following form.

Proposition 2.4. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *and* $\mathcal{D}$ *be closed subsets of* $[0,1]$ *containing both zero and one. Then*

$$\max\left(h_{d_2}\left(\mathcal{A}, \mathcal{B}\right), h_{d_2}\left(\mathcal{C}, \mathcal{D}\right)\right) \leq h_{d_2}\left(\mathcal{A} \times \mathcal{C}, \mathcal{B} \times \mathcal{D}\right) \leq \sqrt{h_{d_2}\left(\mathcal{A}, \mathcal{B}\right)^2 + h_{d_2}\left(\mathcal{C}, \mathcal{D}\right)^2}.$$

Particularly, $h_{d_2}\left(\mathcal{A} \times \mathcal{C}, \mathcal{B} \times \mathcal{D}\right) \leq d_\infty\left(D_\mathcal{A}, D_\mathcal{B}\right) + d_\infty\left(D_\mathcal{C}, D_\mathcal{D}\right)$.

Proof. Notice that $\min_{b\in\mathcal{B}}|a-b| \le \min_{(b,d)\in\mathcal{B}\times\mathcal{D}}\sqrt{|a-b|^2+|c-d|^2}$ for all $(a,c)\in\mathcal{A}\times\mathcal{C}$. It follows that

$$\begin{aligned}\max_{a\in\mathcal{A}}\min_{b\in\mathcal{B}}|a-b| &\le \max_{(a,c)\in\mathcal{A}\times\mathcal{C}}\ \min_{(b,d)\in\mathcal{B}\times\mathcal{D}}\sqrt{|a-b|^2+|c-d|^2}\\ &= \max_{x\in\mathcal{A}\times\mathcal{C}}\ \min_{y\in\mathcal{B}\times\mathcal{D}} d_2(x,y).\end{aligned}$$

By symmetry, $\max_{b\in\mathcal{B}}\min_{a\in\mathcal{A}}|a-b| \le \max_{y\in\mathcal{B}\times\mathcal{D}}\min_{x\in\mathcal{A}\times\mathcal{C}} d_2(x,y)$. Hence $h_{d_2}(\mathcal{A},\mathcal{B}) \le h_{d_2}(\mathcal{A}\times\mathcal{C},\mathcal{B}\times\mathcal{D})$. The proof that $h_{d_2}(\mathcal{C},\mathcal{D}) \le h_{d_2}(\mathcal{A}\times\mathcal{C},\mathcal{B}\times\mathcal{D})$ can be done similarly.

To prove the second inequality, we may assume $h_{d_2}(\mathcal{A}\times\mathcal{C},\mathcal{B}\times\mathcal{D}) = \max_{x\in\mathcal{A}\times\mathcal{C}} d_2(x,\mathcal{B}\times\mathcal{D})$. Let $x=(x_1,x_2)\in\mathcal{A}\times\mathcal{C}$ and $y=(y_1,y_2)\in\mathcal{B}\times\mathcal{D}$ be such that $d_2(x,y)=d_2(x,\mathcal{B}\times\mathcal{D})=h_{d_2}(\mathcal{A}\times\mathcal{C},\mathcal{B}\times\mathcal{D})$. Denote $a_i=\min(x_i,y_i)$ and $b_i=\max(x_i,y_i)$. If there is $y_3\in\mathcal{B}\cap(2a_1-b_1,b_1)$, then

$$d_2(x,(y_3,y_2)) < d_2(x,y) = d_2(x,\mathcal{B}\times\mathcal{D})$$

which is a contradiction. Thus, $\mathcal{B}\cap(2a_1-b_1,b_1)=\emptyset$.
Similarly, $\mathcal{D}\cap(2a_2-b_2,b_2)=\emptyset$. This implies

$$\begin{aligned}|x_1-y_1| &= d_2(x_1,\mathcal{B}) \le d_2(\mathcal{A},\mathcal{B}),\\ |x_2-y_2| &= d_2(x_2,\mathcal{D}) \le d_2(\mathcal{C},\mathcal{D}),\end{aligned}$$

and hence $d_2(x,y) \le \sqrt{h_{d_2}(\mathcal{A},\mathcal{B})^2 + h_{d_2}(\mathcal{C},\mathcal{D})^2}$.

Since $\sqrt{h_{d_2}(\mathcal{A},\mathcal{B})^2 + h_{d_2}(\mathcal{C},\mathcal{D})^2} \le h_{d_2}(\mathcal{A},\mathcal{B}) + h_{d_2}(\mathcal{C},\mathcal{D}) \le d_\infty(D_\mathcal{A},D_\mathcal{B}) + d_\infty(D_\mathcal{C},D_\mathcal{D})$, we are done. □

2.3 Supremum Distance between Distribution Forms of Subcopulas

For any $S,T\in\mathfrak{S}$, denote $\hat{d}(S,T)=d_\infty(\hat{S},\hat{T})$. Then $\left(\mathfrak{S},\hat{d}\right)$ is a metric space. For any closed sets $\mathcal{A},\mathcal{B}\subseteq[0,1]$ containing both zero and one, denote $\mathfrak{S}(\mathcal{A}\times\mathcal{B})$ the set of subcopulas with domain $\mathcal{A}\times\mathcal{B}$. It is easy to see that $\hat{d}(S,T) = d_\infty(S,T) = \bar{d}(S,T)$ whenever S and T have the same domain. Thus, $\left(\mathfrak{S},\hat{d}\right)$ contains $\left(\mathfrak{S}(\mathcal{A}\times\mathcal{B}),\bar{d}\right) = \left(\mathfrak{S}(\mathcal{A}\times\mathcal{B}),d_\infty\right)$ as a subspace for all such $\mathcal{A}\times\mathcal{B}$. The space $\left(\mathfrak{S},\hat{d}\right)$ and $\left(\mathfrak{S},\bar{d}\right)$, however, are not topologically equivalent. Consider the following example.

Example 2.5. Let $\mathcal{A}_n=\left[0,\frac{1}{n}\right]\cup\left[1-\frac{1}{n},1\right]$. Then $h_{d_2}(\mathcal{A}_n,\mathcal{A}_\infty)\to 0$ where $\mathcal{A}_\infty=\{0,1\}$. However, $d_\infty(D_{\mathcal{A}_n},D_{\mathcal{A}_\infty})\to 1$. Define $\hat{S}_n(u,v)=D_{\mathcal{A}_n}(u)D_{\mathcal{A}_n}(v)$ for all $u,v\in[0,1]$ and define S_n to be the restriction of $\hat{S}_n$ on $\mathcal{A}_n\times\mathcal{A}_n$. Then S_n is a subcopula for all n. It follows that $\hat{d}(S_n,S_\infty)\to 1$. However, the condition **(C2)** holds since $h_{d_2}(\mathcal{A}_n,\mathcal{A}_\infty)\to 0$ which immediately implies $\bar{d}(S_n,S_\infty)\to 0$.

Theorem 2.6. *The convergence in* $\left(\mathfrak{S},\hat{d}\right)$ *implies the convergence in* $\left(\mathfrak{S},\bar{d}\right)$.

Proof. Let $S_n, S \in \mathfrak{S}$ be such that $\hat{d}(S_n,S) = d_\infty(\hat{S}_n,\hat{S}) \to 0$. Denote $\mathrm{dom}(S) = \mathcal{A}\times\mathcal{B}$ and $\mathrm{dom}(S_n) = \mathcal{A}_n\times\mathcal{B}_n$. Since $d_\infty\left(D_{\mathcal{A}_n},D_{\mathcal{A}}\right) \le d_\infty(\hat{S}_n,\hat{S})$,

$$d_\infty\left(D_{\mathcal{A}_n},D_{\mathcal{A}}\right)\to 0.$$

Similarly, we can show that $d_\infty\left(D_{\mathcal{B}_n},D_{\mathcal{B}}\right)\to 0$. Hence $h_{d_2}\left(\mathcal{A}_n\times\mathcal{B}_n,\mathcal{A}\times\mathcal{B}\right)\to 0$ by Proposition (2.4).

Now, let $(u_n,v_n)\in\mathcal{A}_n\times\mathcal{B}_n$ be such that $(u_n,v_n)\to(u,v)\in\mathcal{A}\times\mathcal{B}$. Then $u = D_{\mathcal{A}}(u)$ and $|u_n - D_{\mathcal{A}_n}(u)| \le |u_n-u| + |D_{\mathcal{A}}(u)-D_{\mathcal{A}_n}(u)| \to 0$. Similarly, $|v_n - D_{\mathcal{B}_n}(v)|\to 0$. Since

$$\begin{aligned}
|S_n(u_n,v_n)-S(u,v)| &\le |S_n(u_n,v_n)-S_n(D_{\mathcal{A}_n}(u),D_{\mathcal{B}_n}(v))|\\
&\quad + |S_n(D_{\mathcal{A}_n}(u),D_{\mathcal{B}_n}(v))-S(u,v)|\\
&\le |u_n-D_{\mathcal{A}_n}(u)| + |v_n-D_{\mathcal{B}_n}(v)| + \left|\hat{S}_n(u,v)-\hat{S}(u,v)\right|\\
&\le |u_n-D_{\mathcal{A}_n}(u)| + |v_n-D_{\mathcal{B}_n}(v)| + d_\infty\left(\hat{S}_n,\hat{S}\right),
\end{aligned}$$

$S_n(u_n,v_n)\to S(u,v)$. By Theorem (1.2), $\bar{d}(S_n,S)\to 0$. □

We end this work with the following result stating that convergences in $\left(\mathfrak{S},\hat{d}\right)$ is equivalent to convergences in $\left(\mathfrak{S},\bar{d}\right)$ and convergences of their marginals.

Theorem 2.7. *For any* $S\in\mathfrak{S}\left(\mathcal{A}\times\mathcal{C}\right)$ *and* $T\in\mathfrak{S}\left(\mathcal{B}\times\mathcal{D}\right)$, *then*

$$\hat{d}(S,T)\le\bar{d}(S,T)+d_\infty\left(D_{\mathcal{A}},D_{\mathcal{B}}\right)+d_\infty\left(D_{\mathcal{C}},D_{\mathcal{D}}\right).$$

As a consequence, convergences in $\left(\mathfrak{S},\bar{d}\right)$ *and convergences of their marginals together implies convergences in* $\left(\mathfrak{S},\hat{d}\right)$.

Proof. Pick a copula C extending S and a copula D extending T such that $d_\infty\left(C,D\right) = h_{d_\infty}\left([S],[T]\right)$. For any $x,y\in\mathbb{R}$,

$$\begin{aligned}
\left|\hat{S}\left(x,y\right)-\hat{T}(x,y)\right| &= \left|S\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)-T\left(D_{\mathcal{B}}\left(x\right),D_{\mathcal{D}}\left(y\right)\right)\right|\\
&= \left|C\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)-D\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)\right|\\
&\le \left|C\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)-D\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)\right|\\
&\quad + \left|D\left(D_{\mathcal{A}}\left(x\right),D_{\mathcal{C}}\left(y\right)\right)-D\left(D_{\mathcal{B}}\left(x\right),D_{\mathcal{D}}\left(y\right)\right)\right|\\
&\le d_\infty\left(C,D\right)+\left|D_{\mathcal{A}}\left(x\right)-D_{\mathcal{B}}\left(x\right)\right|+\left|D_{\mathcal{C}}\left(y\right)-D_{\mathcal{D}}\left(y\right)\right|\\
&\le h_{d_\infty}\left([S],[T]\right)+d_\infty\left(D_{\mathcal{A}},D_{\mathcal{B}}\right)+d_\infty\left(D_{\mathcal{C}},D_{\mathcal{D}}\right)\\
&\le \bar{d}(S,T)+d_\infty\left(D_{\mathcal{A}},D_{\mathcal{B}}\right)+d_\infty\left(D_{\mathcal{C}},D_{\mathcal{D}}\right).
\end{aligned}$$

By taking supremum over all $x,y\in\mathbb{R}$, $\hat{d}(S,T)\le\bar{d}(S,T)+d_\infty\left(D_{\mathcal{A}},D_{\mathcal{B}}\right)+d_\infty\left(D_{\mathcal{C}},D_{\mathcal{D}}\right)$. □

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