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Derivation of Lie Groups for Some Higher Order Stochastic Differential Equations

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Abstract : In this paper, we give examples of the construction of Lie groups for some examples of second and third order stochastic differential equations of physical interest. Determining equations are derived for admitted Lie groups of transformations that transform Brownian motion terms into Brownian motion terms. The examples include both fiber-preserving and non-fiber-preserving transformations. It is proved by direct transformation that the derived Lie groups transform the original stochastic equations into stochastic equations with the same solutions.

Keywords : symmetry analysis; Lie groups; stochastic differential equations; admitted Lie group generators; determining equations.

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1 Introduction

General surveys of group analysis of differential equations have been given by many authors (see, e.g., [1–3]). Group analysis involves the study of symmetries of the equations, which means finding local groups of transformations that map a solution of the system of equations into a solution of the transformed system. Symmetry can make it possible to reduce the number of dependent and independent variables in the system, and also allows finding new solutions of the system and studying various parts of its solutions. Also, as originally shown by Noether (see, e.g., [2], chap.4), an important application of Lie group analysis is finding invariants for the system of equations. This application of group theory has been found to be essential in many areas of science and engineering in finding conservation laws that correspond to the group symmetry and the mathematical invariants that correspond to them. For example, both finite and Lie groups are used in classifying fundamental particles in physics, classsifying spectra of atoms and molecules in chemistry, analyzing electric radiation in electrical engineering etc.

In contrast to deterministic differential equations, there have been comparatively few attempts to apply symmetry techniques to stochastic differential equations or to find invariants or conservation laws associated with the systems. However, see [4, 5] for a study of approximate conservation laws for stochastic differential equations.

In the existing literature two main approaches have been used in applying group analysis to stochastic differential equations. The first approach (see, e.g., [6–9]) is based on fiber-preserving transformations of the form

$$\overline{x}_i = \varphi_i(t, x, a),\ \bar{t} = H(t, a)\ (i = 1, ..., n), \tag{1.1}$$

where t is the independent variable, x_i is a dependent variable and a is a canonical parameter for a Lie group of transformations. This approach has been applied to stochastic dynamical systems ([6, 9]), and to associated Fokker-Planck equations ([7, 8]). A weakness of this approach is that it is restricted to transformations in which the transformed independent variable is a function of the independent variable only.

The second approach ([4,5,10–12]) deals with symmetry transformations for a system of Itô differential equations in which the transformation of the independent and dependent variables are functions of both independent and dependent variables. This approach has been applied to scalar second-order stochastic differential systems ([11, 12]), to partial differential equations such as the heat equation [12], to Fokker-Planck equations ([5, 10]) and to Hamiltonian-Stratonovich dynamical control systems [5]. In these papers, there have also been attempts to involve Brownian motion in the transformation, but usually without proof that Brownian motion is transformed to Brownian motion.

In [13, 14], a new definition of an admitted Lie group of transformations for stochastic differential equations was given. These transformations included dependent as well as independent variables. In particular, the transformation of the

Brownian motion included both dependent and independent variables, and a strict proof was given that the transformed Brownian motion satisfied the properties of Brownian motion. This theory was then applied in [13–15] to derive Lie groups of transformations for first-order and some higher order stochastic differential equations.

In the present paper, the results given in [13, 14] are used to derive Lie groups for some higher-order stochastic differential equations of physical interest. We show how to construct determining equations for admitted Lie groups of transformations for second and third-order stochastic differential equations and give examples of the applications to selected second and third-order equations. Examples are given of both fiber-preserving and non-fiber-preserving transformations.

2 Transformations of Itô Integrals and Brownian motion

In this section, we summarize the mathematical tools required for defining the transformation of Brownian motion [13–16].

Let Ω be a set of elementary events ω, F be a σ-algebra of subsets of Ω, and P be a probability (or probability measure) on F. The triple $(\Omega,\ F,\ P)$ is called a probability space. It is assumed that a σ-algebra F is generated by a family of σ-algebras F_t, $(t \geq 0)$ such that

$$F_s \subset F_t \subset F \quad \forall s \leq t, \quad s, t \in I,$$

where $I = [0, T]$ and $T \in [0, \infty)$.

The flow of non-decreasing σ-algebras F_t is also called a filtration and the σ-algebra F is denoted by $F = (F_t)_{t\geq 0}$. The triple $(\Omega,\ F = (F_t)_{t\geq 0}, P)$ is called a filtrated probability space. Let $\{X(t) = X_1(t), ..., X_n(t)\}_{t\geq 0}$ be a stochastic process satisfying the system of n Itô equations with r Brownian motion terms given by (see, e.g., [16])

$$dX_i(t,\omega) = f_i(t, X(t,\omega))dt + \sum_{k=1}^{r} g_{ik}(t, X(t,\omega))dB_k(t,\omega), \quad (i = 1, ..., n), \quad (2.1)$$

with the initial condition $X(0) = X^0$. In (2.1), the $f_i(t, x)$ represent drift vectors, $g_{ik}(t, x)$ represent diffusion matrices and B_k $(k = 1, ..., r)$ are one-dimensional Brownian motions. To simplify notation, we will use the standard summation convention in the remainder of this paper that a repeated index denotes summation over that index.

As usual [16], the Itô equations (2.1) are to be interpreted in the sense that

$$X_i(t,\omega) = X_i^0(\omega) \ + \int_0^t f_i(s, X(s,\omega))ds \ + \int_0^t g_{ik}(s, X(s,\omega))dB_k(s,\omega), \quad (2.2)$$

for almost all $\omega \in \Omega$ and for each $t > 0$. In (2.2), the integral $\int_0^t f_i(s, X(s,\omega))ds$ represents a Riemann or Lebesgue integral and $\int_0^t g_{ik}(s, X(s,\omega))dB_k(s,\omega)$ is an Itô integral.

Following the derivations in Øksendal ([16], chap.8) and Srihirun et al. [13, 14], we introduce transformations of the stochastic integrals in (2.2) as follows. Let $\eta(t,x)$ be a sufficiently many times continuously differentiable function and $\{X(t,\omega)\}_{t\geq 0}$ be a continuous and adapted stochastic process (see, also [1–3]). Since $\eta^2(t,x)$ is continuous, $\eta^2(t, X(t,\omega))$ is also an adapted process. We define

$$\beta(t,\omega) = \int_0^t \eta^2(s, X(s,\omega))ds, \ t \geq 0, \tag{2.3}$$

and for brevity write $\beta(t)$ instead of $\beta(t,\omega)$. The function $\beta(t)$ is called a random time change with time change rate $\eta^2(t, X(t,\omega))$. Note that $\beta(t)$ is also an F_t adapted process. Suppose now that $\eta(t,x) \neq 0$ for all (t,x). Then for each ω, the map $t \mapsto \beta(t)$ is strictly increasing. Next, we define (see, e.g., [14, 16])

$$\alpha(t,\omega) = \inf_{s\geq 0}\{s : \beta(s,\omega) > t\}, \tag{2.4}$$

and, for convenience, write $\alpha(t)$ instead of $\alpha(t,\omega)$. For almost all ω, the map $t \mapsto \alpha(t)$ is continuous, and

$$\beta(\alpha(t)) \ = \ t \ = \ \alpha(\beta(t)). \tag{2.5}$$

Then, since $\beta(t)$ is an F_t-adapted process, we have

$$\{\omega : \alpha(t) \leq s\} = \{\omega : t \leq \beta(s)\} \in F_s, \ \text{for all } t \geq 0 \text{ and } s \geq 0. \tag{2.6}$$

Hence $t \mapsto \alpha(t)$ is an F_s-stopping time for each t.

The following theorem [14] (see also [16], chap.8) will be crucial for defining the transformation of a Brownian motion.

Theorem 2.1. *Let $\eta(t,x)$ be a sufficiently many times continuously differentiable function and $\{X(t,\omega)\}_{t\geq 0}$ be a continuous and adapted stochastic process that is a solution of* (2.2)*. If $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion then*

$$\overline{B}(t) = \int_0^{\alpha(t)} \eta(s, X(s,\omega))dB(s), \quad t \geq 0 \tag{2.7}$$

is a standard Brownian motion $(\overline{B}(t), F_{\alpha(t)})$, where

$$F_{\alpha(t)} = \{A \in F : A \cup \{\omega : \alpha(t) \leq s\} \in F_s, \ \textit{for all } s \geq 0\}.$$

3 Lie Groups of Transformations for Stochastic Processes and Stochastic Differential Equations

This section is devoted to reviewing the group analysis method and its application to stochastic processes and in summarizing the theory developed in [14, 15] for obtaining Lie groups of transformations for stochastic differential equations. The theory will be used in section 5 to develop Lie groups for some second and third order stochastic differential equations of physical interest.

Assume that the set of transformations

$$\bar{t} = H(t,x,a), \quad \overline{x}_i = \varphi_i(t,x,a), \quad i = 1,\dots,n, \tag{3.1}$$

compose a one-parameter Lie group, where $H(t,x,a)$ and $\varphi_i(t,x,a)$ are sufficiently many times continuously differentiable functions, and a is a canonical parameter for the group. Let

$$h(t,x) = \frac{\partial}{\partial a}H(t,x,a)\bigg|_{a=0} \quad \text{and} \quad \xi_i(t,x) = \frac{\partial}{\partial a}\varphi_i(t,x,a)\bigg|_{a=0} \tag{3.2}$$

be the coefficients of the infinitesimal generator $h(t,x)\partial_t + \xi_i(t,x)\partial_{x_i}$ of a Lie group. Then $H(t,x,a)$ and the $\varphi_i(t,x,a)$ satisfy the Lie equations (see, e.g., [2, 17]),

$$\frac{\partial H}{\partial a} = h(H,\varphi_1,\dots,\varphi_n), \quad \frac{\partial \varphi_i}{\partial a} = \xi_i(H,\varphi_1,\dots,\varphi_n), \tag{3.3}$$

where the initial conditions at $a = 0$ are that

$$H(t,x,0) = t, \quad \varphi_i(t,x,0) = x_i, \quad i = 1,2,\dots,n. \tag{3.4}$$

In the following, we will use the standard notation $H_t(t,x,a) = \frac{\partial}{\partial t}H(t,x,a)$ for partial derivatives. Since $H_t(t,x,0) = 1$ and $H_t(t,x,a)$ are continuous functions, we must have $H_t(t,x,a) > 0$ in some neighborhood of $a = 0$. Therefore there exists a function $\eta(t,x,a)$ such that

$$\eta(t,x,a) = \sqrt{H_t(t,x,a)}, \qquad \eta(t,x,0) = \sqrt{H_t(t,x,0)} = 1, \tag{3.5}$$

which satisfies the conditions in section 2. Then, following the development in section 2, we let $\beta(t) = \int_0^t \eta^2(s,X(s,\omega),a)ds$ and $\alpha(t)$ be the inverse function of $\beta(t)$. Since $\beta(\alpha(\bar{t})) = \bar{t}$ for almost all ω, then

$$\eta^2(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a)\alpha_{\bar{t}}(\bar{t}) = 1 \quad \text{and} \quad \alpha_{\bar{t}}(s) = \eta^{-2}(\alpha(s),X(\alpha(s),\omega),a). \tag{3.6}$$

We now consider the application of Lie group theory to stochastic processes. Let $X(t,\omega)$ be a continuous and adapted stochastic process. A Lie group for $X(t,\omega)$ will be defined by a transformation of the form of (3.1). That is,

$$\overline{X}(\bar{t},\omega) = \varphi(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a), \tag{3.7}$$

and $\alpha(t)$ is the inverse function of $\beta(t)$. This gives an action of a Lie group (3.1) on stochastic processes. Then, replacing $\bar{t}$ by $\beta(t)$ and $\alpha(\bar{t})$ by t in (3.7), we obtain

$$\overline{X}(\beta(t),\omega) = \varphi(t,X(t,\omega),a).$$

It is useful to introduce a function $\tau(t,x) = \left.\dfrac{\partial}{\partial a}\eta(t,x,a)\right|_{a=0}$. Then, using (3.2) and (3.5), we have

$$\tau(t,x) = \frac{1}{2\eta(t,x,0)}\left.\frac{\partial}{\partial a}\eta^2(t,x,a)\right|_{a=0} = \frac{1}{2}\left.\frac{\partial}{\partial a}H_t(t,x,a)\right|_{a=0} = \frac{1}{2}h_t(t,x). \quad (3.8)$$

The functions $\tau(t,x)$ in (3.8) and $\xi_i(t,x)$ in (3.2) can be used to define a Lie group of transformations for stochastic processes with the infinitesimal generator

$$h(t,x)\partial_t + \xi_i(t,x)\partial_{x_i}. \quad (3.9)$$

The functions $H(t,x,a)$ and $\varphi_i(t,x,a)$ for a stochastic differential equation can be obtained from the infinitesimal generator formula by using the Lie equations (3.3) with the initial conditions (3.4).

4 Admitted Groups and Determinining Equations for Stochastic Differential Equations

We now summarize the method of Srihirun [13–15] for deriving Lie groups for systems of stochastic differential equations.

Definition 4.1. ([13, 14]) A Lie group of transformations (3.1) is called admitted by a system of n stochastic differential equations (2.1) if for any solution $X(t,\omega)$ of (2.1) the functions $\xi_i(t,x)$ $(i = 1,\dots,n)$ and $\tau(t,x)$ satisfy the following determining equations:

$$\begin{aligned}
&\xi_{i,t}(t,X(t,\omega)) + f_j\xi_{i,j}(t,X(t,\omega)) + \frac{1}{2}g_{jk}g_{lk}\xi_{i,jl}(t,X(t,\omega)) \\
&\quad - 2f_{i,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds - f_{i,j}\xi_j(t,X(t,\omega)) - 2f_i\tau(t,X(t,\omega)) = 0, \\
&g_{jk}\xi_{i,j}(t,X(t,\omega)) - 2g_{ik,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds \\
&\quad - g_{ik}\tau(t,X(t,\omega)) - g_{ik,j}\xi_j(t,X(t,\omega)) = 0, \qquad (i = 1,...,n; k = 1,...,r), \quad (4.1)
\end{aligned}$$

where, for example, the notation used for partial derivatives is

$$\xi_{i,t} = \frac{\partial \xi_i}{\partial t}, \quad \xi_{i,j} = \frac{\partial \xi_i}{\partial X_j}, \quad \xi_{i,jl} = \frac{\partial^2 \xi_i}{\partial X_j \partial X_l}, \quad (4.2)$$

and a repeated index again means summation over the index.

The determining equations (4.1) are constructed so that the Lie group of transformations (3.1) transforms any solution of equations (2.1) or (2.2) into a solution of the same equations.

The symmetry of the stochastic differential equation can be checked by applying the time change formula for Riemann integrals (see, e.g., [18]) to the Itô integral. According to the time change formula, if φ is a continuously differentiable function on a closed and bounded interval $[a, b]$ with $\varphi'(x) \neq 0$ for all $x \in [a, b]$, and if $[c, d] = \varphi([a, b])$ and f is integrable on $[c, d]$, then

$$\int_c^d f(t)dt = \int_a^b f(\varphi(x))|\varphi'(x)|dx. \tag{4.3}$$

The time change formula for Itô integrals given in [19] is a non-anticipating function e with

$$P(\int_0^t e^2 ds + \int_0^t \eta^2 ds < \infty, t \geq 0) = 1, \tag{4.4}$$

which satisfies the formula

$$\int_0^{\alpha(t)} e(s,\omega)dB(s) = \int_0^t e(\alpha(s),\omega)\frac{1}{\eta(\alpha(s), X(\alpha(s),\omega), a)}d\overline{B}(s).$$

We note that the determining equations in (4.1) can easily be adapted for a second or third order equation in explicit form by the standard process of converting the higher-order equation to a system of first-order equations. For example, consider the third-order Itô equation

$$\dddot{X}(t) = f(t, X(t), \dot{X}(t), \ddot{X}(t)) + g(t, X(t), \dot{X}(t), \ddot{X}(t))\frac{dB(t)}{dt}, \tag{4.5}$$

where f and g are given functions and B is a Brownian motion. Equation (4.5) can be rewritten as the system of first-order Itô integral equations

$$\begin{aligned} X(t,\omega) &= X(0,\omega) + \int_0^t Y(s,\omega)ds, \quad Y(t,\omega) = Y(0,\omega) + \int_0^t Z(s,\omega)ds, \\ Z(t,\omega) &= Z(0,\omega) + \int_0^t f(s, X(s,\omega), Y(s,\omega))ds \\ &\quad + \int_0^t g(s, X(s,\omega), Y(s,\omega))dB(s). \end{aligned} \tag{4.6}$$

Comparing this system of first-order equations with (2.1) and (2.2), we have $n = 3$, $r = 1$ and

$$\begin{aligned} &f_1(t, X, Y, Z) = Y, \quad f_2(t, X, Y, Z) = Z, \quad f_3(t, X, Y, Z) = f(t, X, Y), \\ &g_{11}(t, X, Y, Z) = g_{21}(t, X, Y, Z) = 0, \qquad g_{31}(t, X, Y, Z) = g(t, X, Y). \end{aligned} \tag{4.7}$$

The determining equations for the functions ξ_i(t,x,y,z), $i = 1, 2, 3$, and τ(t,x,y,z) can then be obtained from the determining equations (4.1).

5 Results and Examples

In this section, we present some examples of group analysis of second and third-order stochastic differential equations of physical interest.

5.1 Second-Order Stochastic Differential Equations

Example 5.1. (Stochastic Mathieu Equation). Consider the stochastic generalization of the Mathieu equation given in (5.1). This generalization has been studied by [20] and plays an important role in the study of stability of excited oscillators.

$$\ddot{X}(t) + \varepsilon\beta\dot{X}(t) + X(t) = -\varepsilon X(t)\frac{dB(t)}{dt}, \tag{5.1}$$

where ε and β are constants and $0 < \varepsilon < 1$. The corresponding Itô integral equations are:

$$X(t,\omega) = X(0,\omega) + \int_0^t Y(s,\omega)ds, \tag{5.2}$$

$$Y(t,\omega) = Y(0,\omega) - \int_0^t (X(s,\omega) + \varepsilon\beta Y(s,\omega))ds - \varepsilon\int_0^t X(s,\omega)dB(s). \tag{5.3}$$

For (5.2) and (5.3), the functions corresponding to f_i and g_{ik} $(i = 1, 2,\ k = 1)$ in (2.1) and (2.2) are:

$$\begin{aligned} f_1(t,X,Y) = Y, \quad f_2(t,X,Y) &= -X - \varepsilon\beta Y, \\ g_{11}(t,X,Y) = 0, \quad g_{21}(t,X,Y) &= -\varepsilon X. \end{aligned} \tag{5.4}$$

The determining equations for this system can then be obtained from (4.1). We obtain

$$\begin{aligned} &\xi_{1,t} + y\xi_{1,x} - (x+\varepsilon\beta y)\xi_{1,y} + \frac{1}{2}\varepsilon^2x^2\xi_{1,yy} - \xi_2 - 2y\tau = 0, \\ &\xi_{2,t} + y\xi_{2,x} - (x+\varepsilon\beta y)\xi_{2,y} + \frac{1}{2}\varepsilon^2x^2\xi_{2,yy} + \xi_1 + \varepsilon\beta\xi_2 + 2(x+\varepsilon\beta y)\tau = 0, \\ &\xi_{1,y} = 0, \quad \xi_{2,y} - \tau - \frac{1}{x}\xi_1 = 0. \end{aligned} \tag{5.5}$$

We have used the Maple program to find the solution of the determining equations (5.5) and obtained the following solution:

$$\xi_1 = C_1x, \quad \xi_2 = C_1y, \quad \tau = 0. \tag{5.6}$$

From (3.8), $h_t(t,x) = 2\tau(t,x) = 0$, and therefore we can choose $h(t,x) = 0$. A basis of admitted generators ξ_1, ξ_2 corresponding to (5.6) can be obtained by setting $C_1 = 1$. Then, using (3.9), we obtain an infinitesimal generator for the Lie group as:

$$h(t,x,y)\partial_t + \xi_1(t,x,y)\partial_x + \xi_2(t,x,y)\partial_y = x\partial_x + y\partial_y. \tag{5.7}$$

On integrating the Lie equations (3.3) and (3.4) for the admitted generator in (5.7), we obtain the following Lie group of transformations.

$$\begin{aligned}
&\bar{t} = H(t, x(t), y(t), a) = t,\\
&\overline{x}(\beta(t)) = \varphi_1(t, x(t), y(t), a) = x(t)e^a,\\
&\overline{y}(\beta(t)) = \varphi_2(t, x(t), y(t), a) = y(t)e^a, \qquad \eta(t, x(t), a) = 1,\\
&\beta(t) = \int_0^t \eta^2(s, x(s), y(s), a)ds = t = \bar{t}, \quad \alpha(\bar{t}) = \alpha(\beta(t)) = t.
\end{aligned} \tag{5.8}$$

We now show that solutions of the transformed Itô equations of (5.2) and (5.3) have the same solutions as the original equations.

Assume that $\overline{X}(\bar{t}) = \overline{X}(\beta(t))$ and $\overline{Y}(\bar{t}) = \overline{Y}(\beta(t))$ are solutions of the transformed Itô equations of (5.2) and (5.3), i.e.,

$$\overline{X}(\bar{t}, \omega) = \overline{X}(0, \omega) + \int_0^{\bar{t}} \overline{Y}(\overline{s}, \omega)d\overline{s}, \tag{5.9}$$

$$\begin{aligned}
\overline{Y}(\bar{t}, \omega) &= \overline{Y}(0, \omega) - \int_0^{\bar{t}} (\varepsilon\beta\overline{Y}(\overline{s}, \omega) + \overline{X}(\overline{s}, \omega))d\overline{s}\\
&\quad -\varepsilon \int_0^{\bar{t}} \overline{X}(\overline{s}, \omega)d\overline{B}(\overline{s}).
\end{aligned} \tag{5.10}$$

Using the transformations in (5.8), we can transform equation (5.9) into

$$e^a X(t, \omega) = e^a X(0, \omega) + e^a \int_0^t Y(s, \omega)ds, \tag{5.11}$$

and therefore (5.2) is satisfied.

From Theorem 2.1 and the transformation (5.8) the Brownian motion $B(t)$ is transformed to the Brownian motion

$$\overline{B}(\bar{t}) = \int_0^{\alpha(\bar{t})} \eta(s, X(s, \omega), a)dB(s) = \int_0^t 1dB(s), \tag{5.12}$$

and hence $d\overline{B}(\bar{t}) = dB(t)$. Therefore the transformations of the terms in (5.10) are:

$$\begin{aligned}
&\overline{Y}(\bar{t}, \omega) = Y(t, \omega)e^a, \qquad \overline{Y}(0, \omega) = Y(0, \omega)e^a,\\
&\int_0^{\bar{t}} (\varepsilon\beta\overline{Y}(\overline{s}, \omega) + \overline{X}(\overline{s}, \omega))d\overline{s} = \int_0^t (\varepsilon\beta Y(s, \omega)e^a + X(s, \omega)e^a)ds,\\
&-\varepsilon \int_0^{\bar{t}} \overline{X}(\overline{s}, \omega)d\overline{B}(\overline{s}) = -\varepsilon \int_0^t X(s, \omega)e^a dB(s).
\end{aligned} \tag{5.13}$$

The transformation of (5.10) is then

$$e^a Y(t, \omega) = e^a Y(0, \omega) - e^a \int_0^t (\varepsilon\beta Y(s, \omega) + X(s, \omega))ds - e^a\varepsilon \int_0^t X(s, \omega)dB(s), \tag{5.14}$$

and therefore (5.3) is satisfied.

To complete the proof, we note that the transformed Itô equations are equivalent to the transformed stochastic differential equation

$$\ddot{\overline{X}}(\bar{t}) + \varepsilon\beta\dot{\overline{X}}(\bar{t}) + \overline{X}(\bar{t}) = -\varepsilon\overline{X}(\bar{t})\frac{d\overline{B}(\bar{t})}{d\bar{t}},$$

and therefore the Lie group of transformations (5.8) transforms any solution of (5.1) into a solution of the same equation.

Example 5.2. Consider the stochastic Liénard equation in (5.15) ([21], p.158). This equation is a model for random vibrations in a spring-mass system. We let $X(t)$ be the displacement of the mass from its equilibrium position, $Y(t) = \frac{dX}{dt} = \dot{X}(t)$ be the velocity, m be the mass, b be a damping factor, k a spring constant and γ and λ be constants associated with the stochastic disturbance.

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = \sqrt{2\gamma^2\lambda}\frac{dB(t)}{dt}. \tag{5.15}$$

The corresponding Itô integral equations are:

$$X(t,\omega) = X(0,\omega) + \int_0^t Y(s,\omega)ds, \tag{5.16}$$

$$Y(t,\omega) = Y(0,\omega) - \int_0^t (\frac{k}{m}X(s,\omega) + \frac{b}{m}Y(s,\omega))ds + \frac{\sqrt{2\gamma^2\lambda}}{m}\int_0^t dB(s). \tag{5.17}$$

For (5.16) and (5.17), the functions corresponding to f_i and g_{ik} $(i = 1, 2,\ k = 1)$ in (2.1) and (2.2) are:

$$\begin{aligned} &f_1(t,X,Y) = Y, \qquad f_2(t,X,Y) = -\frac{k}{m}X - \frac{b}{m}Y, \\ &g_{11}(t,X,Y) = 0, \qquad g_{21}(t,X,Y) = \frac{\sqrt{2\gamma^2\lambda}}{m}. \end{aligned} \tag{5.18}$$

The system of determining equations (4.1) thus becomes

$$\begin{aligned} &\xi_{1,t} + y\xi_{1,x} - (\frac{k}{m}x + \frac{b}{m}y)\xi_{1,y} + \frac{\gamma^2\lambda}{m^2}\xi_{1,yy} - \xi_2 - 2y\tau \ = \ 0, \\ &\xi_{2,t} + y\xi_{2,x} - (\frac{k}{m}x + \frac{b}{m}y)\xi_{2,y} + \frac{\gamma^2\lambda}{m^2}\xi_{2,yy} + \frac{k}{m}\xi_1 + \frac{b}{m}\xi_2 \\ &\qquad +2(\frac{k}{m}x + \frac{b}{m}y)\tau \ = 0, \\ &\xi_{1,y} = 0,\ \xi_{2,y} - \tau \ = \ 0. \end{aligned} \tag{5.19}$$

We have used the Maple program to find the solutions of the determining equations (5.19) and obtained the following solutions:

$$\xi_1 = C_1e^{r_1t} + C_2e^{r_2t},\ \xi_2 = C_1r_1e^{r_1t} + C_2r_2e^{r_2t},\ \tau = 0, \tag{5.20}$$

where $r_1 = \frac{1}{2}(\frac{-b+(b^2-4km)^{\frac{1}{2}}}{m})$ and $r_2 = \frac{1}{2}(\frac{-b-(b^2-4km)^{\frac{1}{2}}}{m})$. Since $\tau = 0$ for these solutions, we can choose $h(t,x) = 0$. A basis of the admitted generators corresponding to (5.20) can be obtained by setting $(C_1, C_2) = (1, 0)$ and $(C_1, C_2) = (0, 1)$. Then, on substituting for h, ξ_1, ξ_2 into (3.9), we obtain the following two admitted generators.

$$e^{r_1 t}\partial_x + r_1 e^{r_1 t}\partial_y, \quad e^{r_2 t}\partial_x + r_2 e^{r_2 t}\partial_y. \tag{5.21}$$

On integrating the Lie equations (3.3) and (3.4) for the first admitted generator in (5.21), we obtain the following Lie group of transformations.

$$\begin{aligned}
&\bar{t} = H(t, x(t), y(t), a) = t,\\
&\overline{x}(\beta(t)) = \varphi_1(t, x(t), y(t), a) = x(t) + ae^{r_1 t},\\
&\overline{y}(\beta(t)) = \varphi_2(t, x(t), y(t), a) = y(t) + ar_1 e^{r_1 t},\\
&\eta(t, x(t), y(t), a) = 1,\\
&\beta(t) = \int_0^t \eta^2(s, x(s), y(s), a)ds = t = \bar{t}, \quad \alpha(\bar{t}) = \alpha(\beta(t)) = t,
\end{aligned} \tag{5.22}$$

and, from Theorem 2.1, $d\overline{B}(\bar{t}) = dB(t)$. The equations for the second admitted generator in (5.21) are similar, but with r_1 replaced by r_2.

The proof that the Lie group transformations in (5.22) transforms solutions of (5.15) into solutions of the same equation is similar to the proof given in Example 5.1, and therefore we will not give the details here (for details, see [22]).

Examples 5.1 and 5.2 are both fiber-preserving transformations since the transformation of t is not a function of the independent variables. The following example shows the application of the theory to a non-fiber-preserving transformation.

Example 5.3. (Non-Fiber-Preserving Transformation). Consider the equation (5.23) [23],

$$\ddot{X}(t) = F(x) - \mu G^2(x)\dot{X}(t) + \varepsilon G(x)\frac{dB}{dt}. \tag{5.23}$$

This second-order stochastic differential equation (5.23) describes the position of a particle subject to a deterministic forcing term $F(x)$, a damping term $-\mu G^2(x)\dot{X}(t)$ and a random forcing term $\varepsilon G(x)\frac{dB}{dt}$. In (5.23), $-\mu G^2(x)$ is the coefficient of the damping term and $\varepsilon G(x)$ is the amplitude of the random forcing term.

We consider the special case $F(x) = X^2(t)$ and $G(x) = X(t)$. Then equation (5.23) becomes

$$\ddot{X}(t) = X^2(t) - \mu X^2(t)\dot{X}(t) + \varepsilon X(t)\frac{dB(t)}{dt}. \tag{5.24}$$

The corresponding Itô integral equations for the system of first-order equations are then

$$X(t,\omega) = X(0,\omega) + \int_0^t Y(s,\omega)ds, \tag{5.25}$$

$$Y(t,\omega) = Y(0,\omega) + \int_0^t (X^2(s,\omega) - \mu X^2(s,\omega)Y(s,\omega))ds + \varepsilon \int_0^t X(s,\omega)dB(s). \tag{5.26}$$

For (5.25) and (5.26), the functions corresponding to f_i and g_{ik} $(i = 1, 2,\ k = 1)$ in (2.1) and (2.2) are:

$$\begin{aligned} &f_1(t, X, Y) = Y, \qquad f_2(t, X, Y) = X^2 - \mu X^2 Y, \\ &g_{11}(t, X, Y) = 0, \qquad g_{21}(t, X, Y) = \varepsilon X. \end{aligned} \tag{5.27}$$

The system of determining equations (4.1) therefore becomes

$$\begin{aligned} &\xi_{1,t} + y\xi_{1,x} + (x^2 - \mu x^2 y)\xi_{1,y} + \frac{1}{2}\varepsilon^2 x^2 \xi_{1,yy} - \xi_2 - 2y\tau = 0, \\ &\xi_{2,t} + y\xi_{2,x} + (x^2 - \mu x^2 y)\xi_{2,y} + \frac{1}{2}\varepsilon^2 x^2 \xi_{2,yy} \\ &\qquad - (2x - 2\mu xy)\xi_1 + \mu x^2 \xi_2 - 2(x^2 - \mu x^2 y)\tau = 0, \\ &\xi_{1,y} = 0, \quad x\xi_{2,y} - x\tau - \xi_1 = 0. \end{aligned} \tag{5.28}$$

We have used the Maple program to find the solution of the determining equations (5.28) and obtained the following solution:

$$\xi_1 = \frac{C_1}{x^2},\ \xi_2 = 0,\ \tau = -\frac{C_1}{x^3}. \tag{5.29}$$

For this solution $h_t(t, x) = 2\tau(t, x) = -2\dfrac{C_1}{x^3}$ and therefore $h(t, x) = -\dfrac{2C_1 t}{x^3}$. Then, on substituting for h, ξ_1, ξ_2 into (3.9) and choosing $C_1 = 1$, we obtain the following admitted generator

$$-\frac{2t}{x^3}\partial_t + \frac{1}{x^2}\partial_x.$$

To find the Lie group of transformations corresponding to this generator, we substitute the generator into the Lie equations (3.3) and (3.4) to obtain

$$\frac{\partial H}{\partial a} = \frac{-2H}{\varphi_1^3}, \quad \frac{\partial \varphi_1}{\partial a} = \frac{1}{\varphi_1^2}, \quad \frac{\partial \varphi_2}{\partial a} = 0, \tag{5.30}$$

with the initial conditions at $a = 0$ of $H = t,\ \ \varphi_1 = x,\ \ \varphi_2 = y$.
Then, solving (5.30), we obtain the Lie group of transformations:

$$\begin{aligned} &\bar{t} = H(t, x(t), y(t), a) = t(1 + 3ax(t)^{-3})^{-\frac{2}{3}}, \\ &\overline{x}(\beta(t)) = \varphi_1(t, x(t), y(t), a) = (x(t)^3 + 3a)^{\frac{1}{3}}, \\ &\overline{y}(\beta(t)) = \varphi_2(t, x(t), y(t), a) = y(t), \\ &\eta(t, x(t), y(t), a) = (1 + 3ax^{-3})^{-\frac{1}{3}}, \\ &\beta(t) = \int_0^t \eta^2(s, x(s), y(s), a)ds, \quad \alpha(\beta(t)) = t, \quad t \geq 0. \end{aligned} \tag{5.31}$$

We now show that the Lie group of transformations (5.31) transforms a solution of the Itô equations (5.25) and (5.26) into a solution of the same equations. For ease of writing, we define a time variable $T=\beta(t)$. Then, we have $\alpha(T)=t$ and

$$dT = \frac{d(\beta(t))}{dt}dt = \eta^2(t,x(t),y(t),a)dt = (1+3ax(t)^{-3})^{-\frac{2}{3}}dt. \tag{5.32}$$

From Theorem 2.1, the transformation for the Brownian motion is

$$\begin{aligned} \overline{B}(T) &= \overline{B}(\beta(t)) = \int_0^{\alpha(\beta(t))} \eta(s,x(s),y(s),\omega)dB(s,x(s),y(s),\omega) \\ &= \int_0^t (1+3aX^{-3}(s,\omega))^{-\frac{1}{3}}dB(s), \end{aligned} \tag{5.33}$$

and therefore $d\overline{B}(T) = (1+3aX^{-3}(t,\omega))^{-\frac{1}{3}}dB(t)$.

Applying Itô's formula to the function $\varphi_1(t,x,y,a) = (x^3+3a)^{\frac{1}{3}}$ and using the transformations (5.31) and (5.32), we have

$$\begin{aligned} \overline{X}(T,\omega) &= \varphi_1(t,X(t,\omega),Y(t,\omega),a) \\ &= \varphi_1(0,X(0,\omega),Y(0,\omega),a) + \int_0^t Y(s,\omega)X^2(s,\omega)(X^3(s,\omega)+3a)^{-\frac{2}{3}}ds \\ &= \overline{X}(0,\omega) + \int_0^t Y(s,\omega)(1+3aX^{-3}(s,\omega))^{-\frac{2}{3}}ds \\ &= \overline{X}(0) + \int_0^T \overline{Y}(\overline{s},\omega)d\overline{s}, \end{aligned} \tag{5.34}$$

and therefore equation (5.25) transforms correctly.

Applying Itô's formula to the function $\varphi_2(t,X(t,\omega),Y(t,\omega),a)$, we have

$$\begin{aligned} \overline{Y}(T,\omega) &= \varphi_2(t,X(t,\omega),Y(t,\omega),a) \\ &= \varphi_2(0,X(0,\omega),Y(0,\omega),a) + \int_0^t (X^2(s,\omega)-\mu X^2(s,\omega)Y(s,\omega))ds \\ &\quad + \int_0^t \varepsilon X(s,\omega)dB(s). \end{aligned} \tag{5.35}$$

Using the transformations in equations (5.31) and (5.32), we have $\overline{Y}(T,\omega) = Y(t,\omega)$ and

$$\overline{X}^2(T,\omega)dT = (X(t,\omega)^3+3a)^{\frac{2}{3}}(1+3aX(t,\omega)^{-3})^{-\frac{2}{3}}dt = X(t,\omega)^2dt. \tag{5.36}$$

Also, from (5.33), we have

$$\overline{X}(T)d\overline{B}(T) = (X(t)^3+3a)^{\frac{1}{3}}(1+3aX^{-3}(t))^{-\frac{1}{3}}dB(t) = X(t)dB(t). \tag{5.37}$$

Then substituting the above equations into (5.35), we have

$$\overline{Y}(T,\omega) = \overline{Y}(0,\omega) + \int_0^T (\overline{X}^2(\overline{s},\omega) - \mu\overline{X}^2(\overline{s},\omega)\overline{Y}(\overline{s},\omega))d\overline{s} + \int_0^T \varepsilon\overline{X}(\overline{s},\omega)d\overline{B}(\overline{s}), \tag{5.38}$$

and therefore (5.26) transforms correctly.

The transformed Itô equations (5.34) and (5.38) are equivalent to the transformed differential equation

$$\ddot{\overline{X}}(\beta(t),\omega) = \overline{X}^2(\beta(t),\omega) - \mu\overline{X}^2(\beta(t),\omega)\dot{\overline{X}}(\beta(t),\omega) + \varepsilon\overline{X}(\beta(t),\omega)\frac{d\overline{B}}{d\beta(t)}. \tag{5.39}$$

Therefore, the transformations (5.31) transform any solution of (5.24) into a solution of the same equation.

5.2 Third-Order Stochastic Differential Equations

We consider a non-autonomous oscillating problem which can be modeled by the following third-order stochastic differential equation [24].

$$\dddot{X}(t) + \gamma\ddot{X}(t) + b^2\dot{X}(t) + \gamma b^2 X(t) = \varepsilon^{k_0} f_0(\mu_0 t, X(t), \dot{X}(t), \ddot{X}(t)) + f_\varepsilon(\mu_0 t, X(t), \dot{X}(t), \ddot{X}(t), \varepsilon)\frac{dB(t)}{dt}, \tag{5.40}$$

where γ, b, k_0, ε, $(\varepsilon \neq 0)$ and μ_0 are constants. We will consider two examples of this model.

Example 5.4. As the first example of (5.40), we assume that $\gamma = b = 0$ and $f_0 = 0, f_\varepsilon = \varepsilon x^2$. The third order equation is then:

$$\dddot{X}(t) \;=\; \varepsilon X^2\frac{dB(t)}{dt}. \tag{5.41}$$

The corresponding Itô integral equations for (5.41) are:

$$X(t,\omega) \;=\; X(0,\omega) + \int_0^t Y(s,\omega)ds, \tag{5.42}$$

$$Y(t,\omega) \;=\; Y(0,\omega) + \int_0^t Z(s,\omega)ds, \tag{5.43}$$

$$Z(t,\omega) \;=\; Z(0,\omega) + \varepsilon\int_0^t X^2(s,\omega)dB(s). \tag{5.44}$$

Comparing (5.44) with (2.1) and (2.2), we have $n = 3$, $r = 1$ and

$$\begin{aligned} &f_1(t,X,Y,Z) = Y, \quad f_2(t,X,Y,Z) = Z, \quad f_3(t,X,Y,Z) = 0,\\ &g_{11}(t,X,Y,Z) \;=\; g_{21}(t,X,Y,Z) \;=\; 0, \quad g_{31}(t,X,Y,Z) = \varepsilon X^2. \end{aligned} \tag{5.45}$$

The system of determining equations (4.1) thus becomes

$$\begin{aligned}
\xi_{1,t} + y\xi_{1,x} + z\xi_{1,y} + \frac{1}{2}\varepsilon^2 x^4 \xi_{1,zz} - \xi_2 - 2y\tau &= 0,\\
\xi_{2,t} + y\xi_{2,x} + z\xi_{2,y} + \frac{1}{2}\varepsilon^2 x^4 \xi_{2,zz} - \xi_3 - 2z\tau &= 0,\\
\xi_{3,t} + y\xi_{3,x} + z\xi_{3,y} + \frac{1}{2}\varepsilon^2 x^4 \xi_{3,zz} &= 0,\\
\xi_{1,z} = 0,\ \xi_{2,z} = 0,\ x\xi_{3,y} + x\xi_{3,z} - x\tau - 2\xi_1 &= 0. \qquad (5.46)
\end{aligned}$$

We have used the Maple program to find the solution of the determining equations (5.46) and obtained the following solution:

$$\xi_1 = 5C_1 x,\ \xi_2 = 7C_1 y,\ \xi_3 = 9C_1 z,\ \tau = -C_1. \qquad (5.47)$$

For this solution, $h(t,x,y,z) = -2\int_0^t C_1 ds = -2C_1 t$. Then, substituting (5.47) into (3.9) and choosing $C_1 = 1$, we obtain the following admitted generator.

$$-2t\partial_t + 5x\partial_x + 7y\partial_y + 9z\partial_z.$$

The Lie group of transformations corresponding to this generator are the solutions of the Lie equations

$$\frac{\partial H}{\partial a} = -2H, \quad \frac{\partial \varphi_1}{\partial a} = 5\varphi_1, \quad \frac{\partial \varphi_2}{\partial a} = 7\varphi_2, \quad \frac{\partial \varphi_3}{\partial a} = 9\varphi_3,$$

with the initial conditions at $a = 0$ of $H = t,\ \varphi_1 = x,\ \varphi_2 = y,\ \varphi_3 = z$.

The transformations which correspond to this generator are then

$$\begin{aligned}
&\bar{t} = te^{-2a},\ \ \bar{x}(\bar{t}) = x(t)e^{5a},\ \bar{y}(\bar{t}) = y(t)e^{7a},\ \ \bar{z}(\bar{t}) = z(t)e^{9a},\\
&\eta(t,x,a) = e^{-a},\ \ \beta(t) = \int_0^t \eta^2(s,x,a)ds = te^{-2a} = \bar{t},\\
&\alpha(\bar{t}) = \alpha(\beta(t)) = t,\ \ t \geq 0. \qquad (5.48)
\end{aligned}$$

We now show that if the equations for the transformed variables of the form of the Itô equations (5.42), (5.43) and (5.44) are satisfied, then the equations (5.42), (5.43) and (5.44) are also satisfied. Then, since the transformation is invertible, solutions of the transformed equations will be transformations of the solutions of the original Itô equations.

Assume that $\overline{X}(\bar{t})$, $\overline{Y}(\bar{t})$ and $\overline{Z}(\bar{t})$ are solutions of the Itô equations (5.42), (5.43) and (5.44) for the transformed variables, i.e.,

$$\overline{X}(\bar{t},\omega) = \overline{X}(0,\omega) + \int_0^{\bar{t}} \overline{Y}(\bar{s},\omega)d\bar{s}, \qquad (5.49)$$

$$\overline{Y}(\bar{t},\omega) = \overline{Y}(0,\omega) + \int_0^{\bar{t}} \overline{Z}(\bar{s},\omega)d\bar{s}, \qquad (5.50)$$

$$\overline{Z}(\bar{t},\omega) = \overline{Z}(0,\omega) + \varepsilon\int_0^{\bar{t}} \overline{X}^2(\bar{s},\omega)d\overline{B}(\bar{s}), \qquad (5.51)$$

and from Theorem 2.1

$$\overline{B}(\bar{t}) = \int_0^{\alpha(\bar{t})} e^{-a} dB(s) = \int_0^t e^{-a} dB(s), \quad \text{i.e.} \quad d\overline{B}(\bar{t}) = e^{-a} dB(t). \tag{5.52}$$

Then, using the transformations in (5.48), we obtain

$$X(t,\omega)e^{5a} = X(0,\omega)e^{5a} + \int_0^t Y(s,\omega)e^{7a}e^{-2a}ds, \tag{5.53}$$

$$Y(t,\omega)e^{7a} = Y(0,\omega)e^{7a} + \int_0^t Z(s,\omega)e^{9a}e^{-2a}ds, \tag{5.54}$$

$$Z(t,\omega)e^{9a} = Z(0,\omega)e^{9a} + \varepsilon \int_0^t X^2(s,\omega)e^{10a}e^{-a}dB(s). \tag{5.55}$$

Therefore, the Itô equations (5.49), (5.50) and (5.51) for the transformed variables $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ transform into the original Itô equations (5.42), (5.43) and (5.44) for the original variables X, Y, Z.

To complete the proof, we note that the transformed Itô equations (5.49), (5.50) and (5.51) are equivalent to the transformed stochastic differential equation

$$\dddot{\overline{X}}(\bar{t}) = \varepsilon \overline{X}^2 \frac{d\overline{B}(\bar{t})}{d\bar{t}}, \tag{5.56}$$

and therefore the Lie group of transformations (5.48) transforms any solution of (5.41) into a solution of the same equation.

Example 5.5. As a second example of (5.40), we assume that $k_0 = 1$, $f_0 = 1, f_\varepsilon = \varepsilon$ and then equation (5.40) becomes

$$\dddot{X}(t) + \gamma\ddot{X}(t) + b^2\dot{X}(t) + \gamma b^2 X(t) = \varepsilon + \varepsilon\frac{dB(t)}{dt}. \tag{5.57}$$

For (5.57), the functions corresponding to (4.6) are $f = \varepsilon - \gamma z - b^2 y - \gamma b^2 x$ and $g = \varepsilon$. The equivalent system of first-order Itô integral equations is then:

$$X(t,\omega) = X(0,\omega) + \int_0^t Y(s,\omega)ds, \tag{5.58}$$

$$Y(t,\omega) = Y(0,\omega) + \int_0^t Z(s,\omega)ds, \tag{5.59}$$

$$\begin{aligned} Z(t,\omega) &= Z(0,\omega) - \int_0^t (\gamma b^2 X(s,\omega) + b^2 Y(s,\omega) + \gamma Z(s,\omega) - \varepsilon)ds \\ &\quad + \varepsilon \int_0^t dB(s). \end{aligned} \tag{5.60}$$

Comparing (5.60) with (2.1) and (2.2), we have $n = 3$, $r = 1$ and

$$\begin{aligned} f_1(t,X,Y,Z) &= Y, \qquad f_2(t,X,Y,Z) = Z, \\ f_3(t,X,Y,Z) &= -\gamma b^2 X - b^2 Y - \gamma Z + \varepsilon, \\ g_{11}(t,X,Y,Z) &= g_{21}(t,X,Y,Z) = 0, \quad g_{31}(t,X,Y,Z) = \varepsilon. \end{aligned} \tag{5.61}$$

The system of determining equations (4.1) thus becomes

$$\xi_{1,t} + y\xi_{1,x} + z\xi_{1,y} + (\varepsilon - \gamma z - b^2 y - \gamma b^2 x)\xi_{1,z} + \frac{1}{2}\varepsilon^2\xi_{1,zz} - \xi_2 - 2y\tau = 0,$$
$$\xi_{2,t} + y\xi_{2,x} + z\xi_{2,y} + (\varepsilon - \gamma z - b^2 y - \gamma b^2 x)\xi_{2,z} + \frac{1}{2}\varepsilon^2\xi_{2,zz} - \xi_3 - 2z\tau = 0,$$
$$\xi_{3,t} + y\xi_{3,x} + z\xi_{3,y} + (\varepsilon - \gamma z - b^2 y - \gamma b^2 x)\xi_{3,z} + \frac{1}{2}\varepsilon^2\xi_{3,zz} + \gamma b^2\xi_1 + b^2\xi_2 + \gamma\xi_3 - 2(\varepsilon - \gamma z - b^2 y - \gamma x)\tau = 0,$$

$$\xi_{1,z} = 0,\ \xi_{2,z} = 0,\ \xi_{3,y} - \xi_{3,z} - \tau = 0. \tag{5.62}$$

We have used the Maple program to find the solutions of the determining equations (5.62) and obtained the following solutions:

$$\begin{aligned}
\xi_1 &= C_1\sin(bt) + C_2\cos(bt) + C_3 e^{-\gamma t},\\
\xi_2 &= C_1 b\cos(bt) - C_2 b\sin(bt) - C_3\gamma e^{-\gamma t},\\
\xi_3 &= -C_1 b^2\sin(bt) - C_2 b^2\cos(bt) + C_3\gamma^2 e^{-\gamma t}, \quad \tau = 0.
\end{aligned} \tag{5.63}$$

Three sets of admitted generators corresponding to (5.63) can be obtained by choosing $(C_1, C_2, C_3) = (1, 0, 0)$, $(C_1, C_2, C_3) = (0, 1, 0)$, $(C_1, C_2, C_3) = (0, 0, 1)$. The sets are:

$$\begin{aligned}
&1) \qquad \sin(bt)\partial_x + b\cos(bt)\partial_y - b^2\sin(bt)\partial_z,\\
&2) \qquad \cos(bt)\partial_x - b\sin(bt)\partial_y - b^2\cos(bt)\partial_z,\\
&3) \qquad e^{-\gamma t}\partial_x - \gamma e^{-\gamma t}\partial_y + \gamma^2 e^{-\gamma t}\partial_z.
\end{aligned} \tag{5.64}$$

Then, integrating the Lie equations (3.3) and (3.4) for the three sets of generators, we obtain the following Lie groups of transformations for the three generators.

1) $\bar{t} = t,\ \bar{x}(\bar{t}) = x(t) + a\sin(bt),\ \bar{y}(\bar{t}) = y(t) + ab\cos(bt),$
$\bar{z}(\bar{t}) = z(t) - ab^2\sin(bt), \quad \eta(t, x, a) = 1,$
$$\beta(t) = \int_0^t \eta^2(s, x, a)ds = t = \bar{t}, \quad \alpha(\bar{t}) = \alpha(\beta(t)) = t, \quad t \geq 0. \tag{5.65}$$

2) $\bar{t} = t,\ \bar{x}(\bar{t}) = x(t) + a\cos(bt),\ \bar{y}(\bar{t}) = y(t) - ab\sin(bt),$
$\bar{z}(\bar{t}) = z(t) - ab^2\cos(bt), \quad \eta(t, x, a) = 1,$
$$\beta(t) = \int_0^t \eta^2(s, x, a)ds = t = \bar{t}, \quad \alpha(\bar{t}) = \alpha(\beta(t)) = t, \quad t \geq 0. \tag{5.66}$$

3) $\bar{t} = t,\ \bar{x}(\bar{t}) = x(t) + ae^{-\gamma t},\ \bar{y}(\bar{t}) = y(t) - a\gamma e^{-\gamma t},$
$\bar{z}(\bar{t}) = z(t) + a\gamma^2 e^{-\gamma t}, \quad \eta(t, x, a) = 1,$
$$\beta(t) = \int_0^t \eta^2(s, x, a)ds = t = \bar{t}, \quad \alpha(\bar{t}) = \alpha(\beta(t)) = t, \quad t \geq 0. \tag{5.67}$$

The proofs that the transformations for the three generators transform (5.57) into a solution of the same equation is similar to the method in Example 5.4 and will be omitted (for details, see [22]).

6 Conclusion

The definition of an admitted Lie group of transformations for stochastic differential equations has been applied to second and third-order stochastic differential equations. A correct approach for generalization of group analysis to higher order stochastic differential equations has been developed and applied to derive generators for Lie groups for symmetry transformation of some second and third-order stochastic differential equations of physical interest. The theory has been applied to derive generators for three examples of second-order equations and two examples of third order equations. Examples have been given for both fiber-preserving and non-fiber-preserving transformations. Proofs have been given to show that for two fiber-preserving transformations and one non-fiber-preserving transformation, the transformations transform a solution of a higher-order stochastic differential equation into a solution of the same equation.

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