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Minimal S_β-open sets and Maximal S_β-closed sets in topological spaces

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Abstract : In this work, new classes of sets called minimal S_β-open set and maximal S_β-closed set in topological spaces which were subclasses of S_β-open and S_β-closed sets respectively are introduced. We proved that the complement of minimal S_β-open set was maximal S_β-closed set. Some properties of the new concept of both sets were studied.

Keywords : S_β-open sets, S_β-closed sets, minimal S_β-open sets, maximal S_β-closed sets.

1 Introduction and preliminaries

In the year 2001 and 2003 [1,2], Nakaoka and Oda initiated minimal open (rep. closed) sets which are subclass of open (resp. closed) sets in topological spaces. In the year 1963 [3], Levine introduced semi open sets. In the year 1983 [4], E-Moonsef, El-Deeb and Mahmoud formilated β- open sets. In the year 2013 [5], Khalaf Moonsef and Ahmed created and S_β-open sets.

In the present paper, the author presents minimal S_β-open sets and maximal S_β-closed sets in topological spaces and some of their basic properties obtain form the study. Throughout this paper, a space X represent the topological space on which no separation axioms are assumed unles explicitly stated. For a subset A of topological space X, $cl(A)$ and $int(A)$ denote the closure of A and interior of A repectively. The complement of A in X denoted by $X \backslash A$. Now, we recall the following definitions and characterizations.

Definition 1.1. A proper nonempty open subset U of a topological space X is said to be: (i) a minimal open set [1] if any open set contained in U is $\emptyset$ or U, and (ii) a maximal open set [1] if any open set containing U is X or U.

Definition 1.2. A proper nonempty closed subset F of a topological space X is said to be: (i) a minimal closed set [2] if any closed set contained in F is $\emptyset$ or F, and (ii) a maximal closed set [2] if any closed set containing F is X or F.

Definition 1.3. A subset A of a topological space X is called: (i) a semi-open [3] if $A \subseteq cl(int(A))$, and (ii) a β-open [4] if $A \subseteq cl(int(cl(A)))$. The complement of semi-open (resp. β-open) set is semi-closed (resp.β-closed).

Definition 1.4. A semi-open subset A of a topological space X is called a S_β-open set [5] if each $x \in A$ there exists β-closed set F such that $x \in F \subseteq A$. A subset B of a topological space X is called β-closed if $X \backslash B$ is S_β-open.

Definition 1.5. A subset A of a topological space X is said to be a regular open set [7] if $A = int(cl(A))$. It is called a regular closed if $X \backslash A$ is regular open.

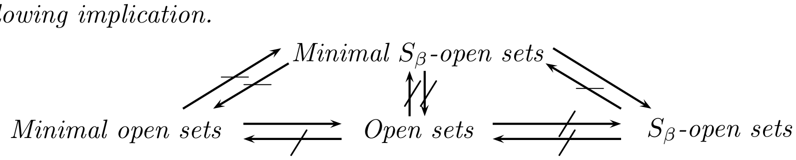
2 Minimal S_β-open sets

Definition 2.1. A proper nonempty S_β-open subset G of a topological space X is said to be a minimal S_β-open set if any S_β-open set contained in G is $\emptyset$ or G.

Example 2.2. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{b, c\}, X\}$, we have:
Open sets are $\emptyset, X$ and $\{b, c\}$; Minimal open set is $\{b, c\}$; and
S_β-open sets are $\emptyset$ and X, resulting in no existing of minimal S_β-open set.

Example 2.3. Let $X = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, we have: Open sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}$, and $\{a, c\}$; Minimal open sets are $\{a\}$, and $\{b\}$;
S_β-open sets are $\emptyset, X, \{b\}$ and $\{a, c\}$; and Minimal S_β-open sets are $\{b\}$ and $\{a, c\}$.

Remark 2.4. *From the findings and by the above two examples, we have the following implication.*



Lemma 2.5. (i) *Let G be a minimal S_β-open set and H be a S_β-open set which $G \cap H$ is an S_β-open set. Then $G \cap H = \emptyset$ or $G \subseteq H$.*

(ii) *Let G and H be minimal S_β-open sets such that $G \cap H$ be an S_β-open set. Then $G \cap H = \emptyset$ or $G = H$.*

Proof. (i) Let G be a minimal S_β-open set, H be a S_β-open set, and $G \cap H$ be an S_β-open set. If $G \cap H \neq \emptyset$ then we have to show that $G \subseteq H$. Since $G \cap H \neq \emptyset$ then $G \cap H \subseteq G$ and $G \cap H \subseteq H$. But G be a minimal S_β-open set, so we have $G \cap H = \emptyset$ or $G \cap H = G$. Thus $G \cap H = G$. So $G \subseteq H$.

(ii) Let G and H be minimal S_β-open sets which $G \cap H$ is an S_β-open set. If $G \cap H = \emptyset$, then the proof is complete. If $G \cap H \neq \emptyset$ then we have to prove that $G = H$. Suppose $G \cap H \neq \emptyset$ then by (i), we see that $G \subseteq H$ and $H \subseteq G$. Therefore $G = H$.

□

Proposition 2.6. *Let G be a minimal S_β-open set. If x is an element of G, and N is an S_β-open neighborhood of x which $G \cap H$ is an S_β-open, then $G \subseteq N$.*

Proof. Let G be a minimal S_β-open set containing an element x, and let N be an S_β-open neighborhood of x. Then we have an $G \cap N \neq \emptyset$. So by Lemma 2.5(i) therefore $G \subseteq N$. □

Proposition 2.7. *Let G be a minimal S_β-open set,*

$$\bigcap\{N|N \text{ is an } S_\beta\text{-open neighborhood of } x\} \text{ is an } S_\beta\text{-open}.$$

Then $G = \bigcap\{N|N$ is an S_β-open neighborhood of $x\}$ for any x of N.

Proof. Let $x \in G$ and G be a minimal S_β-open set. So G is an S_β-open neighborhood of x. Then we have $\bigcap\{N|N$ is an S_β-open neighborhood of x$\} \subseteq G$. So by Proposition 2.6 we see that $G \subseteq \bigcap\{N|N$ is an S_β-open neighborhood of x$\}$. Therefore $G = \bigcap\{N|N$ is an S_β-open neighborhood of x$\}$. □

Theorem 2.8. *Let X be a topological space and the intersection of two S_β-open sets in X be an S_β-open set. If G be a nonempty S_β-open subset in X, then the following three conditions are equivalent.*

(i) *G is a minimal S_β-open set.*

(ii) *$G \subseteq S_\beta cl(H)$ for any nonempty subset H of G.*

(iii) *$S_\beta cl(G) = S_\beta cl(H)$ for any nonempty subset H of G.*

Proof. (i) ⇒ (ii) Let G be a minimal S_β-open set and H be a nonempty S_β-open subset of G. Let $x \in G$ by Proposition 2.6 for any S_β-open N containing x, $H \subseteq G \subseteq N$, which implies $H \subseteq G$. Now $H = H \cap G \subseteq H \cap N$. Since $N \neq \emptyset$, therefore $H \cap N \neq \emptyset$ implies $x \in S_\beta cl(H)$. It follows that $G \subseteq S_\beta cl(H)$.

(ii) ⇒ (iii) Let H be a nonempty subset of G that is $H \subseteq G$ which implies $S_\beta cl(H) \subseteq S_\beta cl(G)$. By assumtion we have $S_\beta cl(G) \subseteq S_\beta cl(S_\beta cl(H)) = S_\beta cl(H)$. Therefore $S_\beta cl(G) = S_\beta cl(H)$.

(iii) ⇒ (i) Form (iii) we have $S_\beta cl(G) = S_\beta cl(H)$ for any nonempty subset H of G. Suppose G is not a minimal S_β-open set then there exist a nonempty

S_β-open set U such that $U \subseteq G$ and $U \neq G$. Now there exist an element y in G such that $y \notin U$ which implies $y \in X \backslash U$. Hence we obtain that $S_\beta cl(y) \subseteq S_\beta cl(X\backslash U) = X\backslash U$, because $X\backslash U$ is a S_β-closed set in X. Since $U \subseteq G \subseteq S_\beta cl(G)$ and $S_\beta cl(y) \nsubseteq U$, there exists an element z in $S_\beta cl(y)$ such that $z \notin U$. It follows that $S_\beta cl(y) \neq S_\beta cl(G)$. □

Definition 2.9 ([7])**.** A topological space (X, τ) is said to be locally indiscrete if every open subset of X is closed.

Definition 2.10. A topological space (X, τ) is said to be locally finite S_β space if each of its elements is contained in a finite S_β-open set.

Lemma 2.11 ([5])**.** *Every regular closed sets is S_β-open set.*

Lemma 2.12. *A topological space (X, τ) is a locally indiscrete if A is an open set in X then A is an S_β-open set in X.*

Proof. Let A be an open subset in X. If $A = \emptyset$, then there is nothing to prove. But if $A \neq \emptyset$ then we have to prove that A is an S_β-open. Since X is locally indiscrete, this implies that $A = clA = cl(intA)$. So that A is regular closed, by Lemma 2.11 therefore $A \in S_\beta(O(X))$. □

Theorem 2.13. *Let X be a locally indiscrete space and G be an open subset in X. If G is a minimal S_β-open set, then it is a minimal open set.*

Proof. Let G is an open set set in a locally indiscrete X, we need to prove that G is a minimal open set. Suppose that G is not a minimal open set, then $G \neq \emptyset$ and there exist an open set H such that $H \subset G$ and $H \neq G$. Since X is locally indiscrete, this implies that $H = clH = cl(intH)$. So that H is regular closed, therefore, by lemma 2.11 $H \in S_\beta(O(X))$. Thus we get that G is an S_β-open set containing H and $H \neq G$ and $G \neq \emptyset$, which is contradiction. Hence G is a minimal open set. □

Theorem 2.14. *Let G be a minimal S_β-open set and x an element of $X\backslash G$. If define $G_x = \cap\{N|N$ is an S_β-open neighborhood of $x\}$ and $G \cap N$ be a minimal S_β-open then $G_x \cap N = \emptyset$ or $N \subset G_x$.*

Proof. Since N is an S_β-open, by Lemma 2.5(i) we have $G \cap N = \emptyset$ or $G \subset N$. If $G \subset N$ for any S_β-open neighborhood N of x, then

$$G \subset \cap\{N|N \text{ is an } S_\beta\text{-open neighborhood of } x\}.$$

Therefore $G \subset G_x$. Otherwise there exists an S_β-open neighborhood N of x such that $G \cap N = \emptyset$. Then we have $G \cap G_x = \emptyset$. □

Theorem 2.15. *Let G be a nonempty finite S_β-open set. Then there exists at least one minimal S_β-open set H such that $H \subset G$.*

Proof. Let G be a nonempty finite S_β-open set. If G is a minimal S_β-open set, we may set $H = G$. If G is not a minimal S_β-open set, then there exists an S_β-open set G_1 such that $\emptyset \neq G_1 \subset G$. If G_1 is a minimal S_β-open set, we may set $H = G_1$. If G_1 is not a minimal S_β-open set, then there exists an S_β-open set G_2 such that $\emptyset \neq G_2 \subset G_1 \subset G$. Continuing this process, we have a sequence of S_β-open sets

$$G \supset G_1 \supset G_2 \supset \ldots \supset G_i \supset \ldots.$$

Since G is a finite S_β-open set, this process repeats only finitely. Then, finally we get a minimal S_β-open set $H = G_n$ for some positive integer n. Therefore $H \subset G$. □

Corollary 2.16. *Let X be a locally finite S_β space and G is a nonempty S_β-open set. Then there exsits at least one minimal S_β-open set H such that $H \subset G$.*

Proof. Let G is a nonempty S_β-open set. So there exsits an element x of G. Since X be a locally finite S_β space, we have a finite S_β-open set G_x such that $x \in G_x$. Since $G \cap G_x$ is a finite S_β-open set, we get a minimal S_β-open set H such that $H \subset G \cap G_x \subset G$ by Theorem 2.15. □

Theorem 2.17. *Let G_λ be an S_β-open set for any $\lambda \in \Lambda$ and the intersection of two S_β-open sets is an S_β-open and H be a nonempty finite S_β- open set. Then $H \cap (\cap_{\lambda\in\Lambda} G_\lambda)$ is a finite S_β- open set.*

Proof. Since H is a nonempty finite S_β- open set, there exists an integer n such that
$H \cap (\cap_{\lambda\in\Lambda} G_\lambda) = H \cap (\cap_{i=1}^{n} G_{\lambda_i})$ and hence we have the result. □

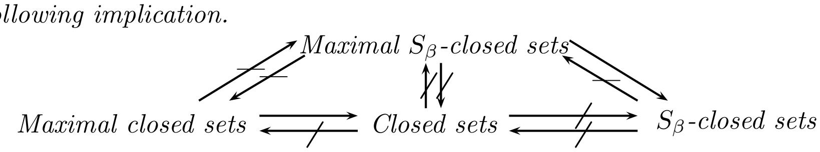
3 Maximal S_β-closed sets

Definition 3.1. A proper nonempty S_β- closed subset F of a topological space X is said to be a maximal S_β-open set if and only if any S_β-cloed set which contains F is X or F.

Example 3.2. Let $X = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, therefore,

Closed sets are $\emptyset, X, \{b, c\}, \{a, c\}, \{c\}$, and $\{b\}$.
Maximal closed set are $\{a, c\}$ and $\{b, c\}$.
S_β-closed sets are $\emptyset, X, \{a, c\}$ and $\{b\}$.
Maximal S_β-closed sets are $\{a, c\}$ and $\{b\}$.

Remark 3.3. *From the known results and by the above example 3.2 we have the following implication.*



Theorem 3.4. *Let X be a topological space and G be a subset in X, then G is a minimal S_β-open set if and only if $X \backslash G$ is a maximal S_β-closed set.*

Proof. ($\Rightarrow$) Let G be a minimal S_β-open set. Then by definition it is clear that G is a S_β-open. Therefore $X \backslash G$ is a S_β-closed. We have to show that $X \backslash G$ is a maximal S_β-closed. Suppose $X \backslash G$ is not a maximal S_β-closed set, there exist an S_β-closed subset F of X such that $X \backslash G \subseteq F$. Hence $X \backslash F$ is an S_β-open and $X \backslash F \subseteq G$ and this contradict being G is a minimal S_β-open.

($\Leftarrow$) Let $X \backslash G$ be a maximal S_β-closed subset of X, then we have G is an S_β-open set. Suppose that there is a nonemty S_β-open subset H of X such that $H \subseteq G$. So that $X \backslash G \subseteq X \backslash H$ but $X \backslash H$ is a proper S_β-closed subset of X. Contradiction to the assumtion of being $X \backslash G$ is a maximal S_β-closed. Therefore G is a minimal S_β-open. □

Remark 3.5. *The result of theorem 3.4, we have that basic properties of maximal S_β-closed sets paralleling to those of the minimal S_β-open sets.*

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