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Join-Semilattices of Integrable Set-Valued[1] Martingales

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Abstract : We derive order properties of set-valued (reversed) martingales. We show that the space of set-valued integrable (reversed) martingales with respect to a given filtration is a join-semilattice. Similar results are obtained for spaces of set-valued integrably bounded convergent martingales.

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1. Introduction

Let (Ω, Σ, μ) be a complete probability space and f a measurable function defined on Ω. The *positive part*, the *negative part* and the *absolute value* of f, given by $f^+ := f \vee 0$, $f^- := (-f) \vee 0$ and $|f| := f \vee (-f)$ respectively, are fundamental notions in the development of the theory of measurable functions, constructing the resulting integral $\int_\Omega f\, d\mu$ and deriving properties of the resulting $L^p(\mu)$-spaces.

We consider the situation where F is a function defined on Ω and with range contained in a hyperspace of a separable Banach space X. The natural ordering on hyperspaces of X is set inclusion. This yields a canonical ordering on sets of set-valued functions, namely the pointwise ordering. As many of the hyperspaces of X are also join-semilattices with respect to set inclusion, the corresponding set of set-valued functions is also a join-semilattice.

In the classical theory of scalar-valued martingales, Bochner considered partial ordering in the theory of martingales in [1]. Krickeberg used the positive part of a martingale (i.e. the minimal positive martingale above a given martingale) to obtain his famous decomposition of a submartingale in [10].

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Hiai and Umegaki studied functions which are set-valued and integrably bounded in [7]. Their work has found wide applications in many different areas of mathematics (cf. [2, 6, 11, 14, 17]). In particular, their approach to conditional expectations via selections (cf. [3, 7]), lead to many new results on set-valued martingales. Many of these results were documented by Li and Ogura in [15].

Our aim is to consider lattice properties of set-valued martingales. In particular, we address the issue of the *positive part* of a set-valued martingale.

Let (Σ_i) be an increasing sequence of sub-σ-fields of Σ. We consider sets of set-valued integrable (integrably bounded convergent) martingales with respect to (Σ_i) which are pointwise ordered. These sets are then shown to be join-semilattices and formulas are derived to explicitly calculate the join of two such martingales.

In §2 we take care of some preliminaries. The remaining part of the paper is roughly divided into two parts. The first part consists of §3 to §5. In this part the metric on the hyperspaces plays no role. The second part consists of §6 to §8. Here the metric on the hyperspaces plays a role. Some of the results in the second part can be obtained using techniques which do not involve measure theory, as in [4, 9, 12, 18, 19]. Here we follow the more traditional route of measure theory.

In §3, §4 and §5 we consider reversed integrable martingales which take their values in the hyperspace of nonempty closed subsets of X and integrable martingales which take their values in the hyperspace of nonempty, convex and closed subsets of X. We show that the sets of these martingales with respect to (Σ_i) are join-semilattices. In particular, we derive a formula to calculate the join of any two such martingales. This also enables us to calculate positive parts of such martingales.

In §6 to §8 we consider integrably bounded Δ-convergent martingales which take their values in the hyperspace of nonempty convex closed subsets of X. We show that the set of these martingales with respect to (Σ_i) is a join-semilattice. Moreover, this set can be endowed with a canonical metric under which it is complete. Our order theoretic approach yields, as a bonus, a necessary and sufficient condition for regular integrably bounded Δ-convergent martingales, which take their values in the hyperspace of nonempty, convex and closed subsets of X, to be Δ-convergent. To achieve our goal, we revisit the embedding theorem of Hörmander in §6. Here we show that the embedding of the hyperspace of nonempty convex compact subsets of a Banach space into a space $C(\Omega)$ of continuous functions on a compact Hausdorff space Ω, is join preserving.

For order properties of the space of integrably bounded $\mathbf{D}_\infty$-convergent fuzzy set-valued martingales with respect to (Σ_i), the reader is referred to [13].

Our work is inspired by the fact that Nguyen and Tran used order techniques in [16] to show that the space of upper semicontinuous functions on a locally compact topological space is metrizable (cf. also [20]).

2. Preliminaries

Let X be a Banach space and (Ω, Σ, μ) a complete probability space. We denote by $L^1(\mu, X)$ denote the space of (classes of a.e. equal) *Bochner integrable* functions

$f : \Omega \to X$. The *Bochner norm* on $L^1(\mu, X)$ is given by $\|f\|_1 = \int_\Omega \|f(\omega)\|_X d\mu$ (cf. [5]).

If Σ_1 is a sub σ-field of Σ, the *conditional expectation* of $f \in L^1(\mu, X)$ relative to Σ_1, denoted by $\mathbb{E}(f|\Sigma_1)$, is a Σ_1-measurable element of $L^1(\mu, X)$ which is given by

$$\int_A \mathbb{E}(f|\Sigma_1) d\mu = \int_A f d\mu \quad \text{for all } A \in \Sigma_1$$

(cf. [5]).

There are two natural operations on

$$\mathcal{P}_0(X) := \{A \subseteq X : A \text{ is nonempty}\},$$

namely addition and scalar multiplication, defined by

$$A + C := \{a + b : a \in A, b \in C\} \text{ and } \lambda A := \{\lambda a : a \in A\},$$

for all $A, C \in \mathcal{P}_0(X)$ and $\lambda \in R$. It is not always possible to find an additive inverse for a subset A of X. Thus, the set $\mathcal{P}_0(X)$ does not, in general, form a vector space with respect to the above defined addition and scalar multiplication. Let

$$\mathrm{f}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is norm closed}\}$$

and define $\oplus$ by

$$A \oplus C = \overline{A + C} \text{ for all } A, C \in \mathrm{f}(X),$$

where the closure is taken with respect to the norm on X. Then $\mathrm{f}(X)$ is closed under $\oplus$.

Recall that a partially ordered set $(P, \leq)$ is called a *join-semilattice* if the least upper bound of x and y, denoted $x \vee y$, exists for all $x, y \in P$.

The partially ordered set $(\mathrm{f}(X), \subseteq)$ is a join-semilattice with join $\vee$ given by

$$A \vee C = A \cup C \text{ for all } A, C \in \mathrm{f}(X).$$

3. The space $\mathcal{L}[\Sigma, \mathrm{f}(X)]$

Throughout the remainder of the paper, X is a separable Banach space and (Ω, Σ, μ) is a complete probability space.

A function $F : \Omega \to \mathrm{f}(X)$ is Σ-*measurable* provided that there exists a sequence (f_i) such that each function $f_i : \Omega \to X$ is

(M1) μ-measurable; i.e., each f_i is of the form $\sum_{j=1}^{n_i} x_j \chi_{A_j}$ where $A_j \in \Sigma$ and $x_j \in X$ (cf. [5, p.41]),
(M2) a *selection* of F; i.e. $f_i(\omega) \in F(\omega)$ a.e. for all $\omega \in \Omega$ and $i \in \mathbb{N}$, and
(M3) $F(\omega) = \overline{\{f_i(\omega) : i \in \mathbb{N}\}}$ for all $\omega \in \Omega$, where the closure is the norm closure in X

(cf. [7, 15]). Let

$$\mathbf{M}[\Sigma, \mathrm{f}(X)] := \{F : \Omega \to \mathrm{f}(X) : F \text{ is } \Sigma\text{-measurable}\}.$$

If $F_1, F_2 \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$ and $\lambda \in \mathbb{R}_+$, define $F_1 \oplus F_2$, λF_1 and $\overline{\mathrm{co}} F_1$, respectively, by

$$(F_1 \oplus F_2)(\omega) = F_1(\omega) \oplus F_2(\omega),$$

$$(\lambda F_1)(\omega) = \lambda(F_1(\omega)) \text{ and } (\overline{\text{co}}F_1)(\omega) = \overline{\text{co}}(F_1(\omega))$$

for all $\omega \in \Omega$. (Here $\overline{\text{co}}(F_1(\omega))$ denotes the norm closure in X of the convex hull $\text{co}(F_1(\omega))$ of $F(\omega)$).

For all $F_1, F_2 \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$, define

$$F_1 \leq F_2 \iff F_1(\omega) \subseteq F_2(\omega) \text{ a.e. for all } \omega \in \Omega.$$

Theorem 3.1. (cf. [7, 15]) *The space* $\mathbf{M}[\Sigma, \mathrm{f}(X)]$ *has the following properties:*

(a) *If* $F_1 \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$*, then* $\overline{\text{co}}F_1 \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$.
(b) $(\mathbf{M}[\Sigma, \mathrm{f}(X)], \leq)$ *is a partially ordered set.*
(c) *If* $(F_i) \subseteq \mathbf{M}[\Sigma, \mathrm{f}(X)]$ *and* F *is defined by* $F(\omega) = \overline{\bigcup_{i=1}^{\infty} F_i(\omega)}$ *for all* $\omega \in \Omega$*, then* $F \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$.

As a special case of Theorem 3.1(c), it follows that if F and G are Σ-measurable, then the join $F \vee G$ of F and G, given by $(F \vee G)(\omega) = F(\omega) \cup G(\omega)$ for all $\omega \in \Omega$, is Σ-measurable. Thus, $(\mathbf{M}[\Sigma, \mathrm{f}(X)], \vee)$ is a join semi-lattice.

If $F \in \mathbf{M}[\Sigma, \mathrm{f}(X)]$, then F is called *integrable* provided that $S_F^1 \neq \emptyset$, where

$$S_F^1 := \{f \in L^1(\mu, X) : f(\omega) \in F(\omega) \text{ a.e.}\}.$$

Let

$$\mathcal{L}[\Sigma, \mathrm{f}(X)] := \{F \in \mathbf{M}[\Sigma, \mathrm{f}(X)] : S_F^1 \neq \emptyset\}.$$

Let Σ_0 be a sub-σ-field of Σ. If $F : \Omega \to \mathrm{f}(X)$ is Σ_0-measurable, let

$$S_F^1(\Sigma_0) = \{f \in L^1(\Sigma_0, \mu, X) : f(\omega) \in F(\omega) \text{ a.e.}\}.$$

Hiai and Umegaki proved in [7] that, if $F \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$, then there exists a unique $G \in \mathcal{L}[\Sigma_0, \mathrm{f}(X)]$ such that

$$S_G^1(\Sigma_0) = \overline{\{\mathbb{E}(f|\Sigma_0) : f \in S_F^1\}},$$

where the closure is taken in $L^1(\Sigma, \mu, X)$, and $\mathbb{E}(f|\Sigma_0)$ denotes the conditional expectation of $f\colon \Omega \to X$ with respect to Σ_0.

As is customary, we denote G by $\mathcal{E}[F|\Sigma_0]$ and call $\mathcal{E}[F|\Sigma_0]$ the *conditional expectation* of $F\colon \Omega \to \mathrm{f}(X)$ relative to Σ_0 (cf. [7, 15]).

Theorem 3.2. (cf. [7, 15]) *Let* Σ_0 *be a sub-*σ*-field of* Σ. *If* $F \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$*, then the conditional expectation* $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}[\Sigma_0, \mathrm{f}(X)]$ *of* F *with respect to* Σ_0 *has the following properties:*

(E1) *If* $F \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$*, then* $\mathcal{E}[\overline{\text{co}}F|\Sigma_0] = \overline{\text{co}}\mathcal{E}[F|\Sigma_0]$.
(E2) *If* $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$*, then* $F_1 \leq F_2$ *implies* $\mathcal{E}[F_1|\Sigma_0] \leq \mathcal{E}[F_2|\Sigma_0]$.
(E3) *If* $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{f}(X)]$ *is an increasing sequence, i.e.,* $F_i(\omega) \subseteq F_{i+1}(\omega)$ *a.e. for all* $i \in \mathbb{N}$*, and* $F(\omega) := \overline{\bigcup_{i=1}^{\infty} F_i(\omega)}$ *for all* $\omega \in \Omega$*, then*

$$\mathcal{E}[F|\Sigma_0](\omega) = \overline{\bigcup_{i=1}^{\infty} \mathcal{E}[F_i|\Sigma_0](\omega)} \text{ a.e..}$$

4. Reversed martingales in $\mathcal{L}[\Sigma, \mathrm{f}(X)]$

Definition 4.1. *Let $(\Sigma_i)_{i\in-\mathbb{N}}$ be a sequence of sub-σ-fields of Σ such that $\Sigma_{i-1} \subseteq \Sigma_i$ for all $i \in -\mathbb{N}$. Then $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ is called a* reversed martingale (*respectively,* reversed submartingale, reversed supermartingale) *provided that $F_i \in \mathcal{L}[\Sigma_i, \mathrm{f}(X)]$ and $F_{i-1}(\omega) = \mathcal{E}[F_i|\Sigma_{i-1}](\omega)$ (respectively, $F_{i-1}(\omega) \subseteq \mathcal{E}[F_i|\Sigma_{i-1}](\omega)$, $F_{i-1}(\omega) \supseteq \mathcal{E}[F_i|\Sigma_{i-1}](\omega)$) a.e. $\omega \in \Omega$ and for all $i \in -\mathbb{N}$.*

If $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ is a reversed martingale, then $\mathcal{E}[F_{-1}|\Sigma_i] = F_i$ for all $i \in -\mathbb{N}$. Hence $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ is of the form $(\mathcal{E}[F_{-1}|\Sigma_i], \Sigma_i)_{i\in-\mathbb{N}}$.

Denote by $M_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$ the set of all reversed martingales $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$. For all $(F_i, \Sigma_i)_{i\in-\mathbb{N}}, (G_i, \Sigma_i)_{i\in-\mathbb{N}} \in M_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$, define

$$(F_i, \Sigma_i)_{i\in-\mathbb{N}} \le (G_i, \Sigma_i)_{i\in-\mathbb{N}} \Longleftrightarrow F_i \le G_i \text{ for each } i \in -\mathbb{N}.$$

Our aim is to show that $\mathcal{M}_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$ is a join-semilattice.

Lemma 4.2. *If $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ and $(G_i, \Sigma_i)_{i\in-\mathbb{N}}$ are reversed submartingales, then $\big(\mathcal{E}[F_{-1} \vee G_{-1}\,|\Sigma_i]), \Sigma_i\big)_{i\in-\mathbb{N}}$ is the minimal reversed martingale with*

$$F_i \le \mathcal{E}[F_{-1} \vee G_{-1}\,|\Sigma_i] \text{ and } G_i \le \mathcal{E}[F_{-1} \vee G_{-1}\,|\Sigma_i] \text{ for all } i \in -\mathbb{N}.$$

Proof. As $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ and $(G_{-i}, \Sigma_{-i})_{i\in-\mathbb{N}}$ are reversed submartingales, it follows from (E2) that

$$F_i \le \mathcal{E}[F_{-1}|\Sigma_i] \le \mathcal{E}[F_{-1} \vee G_{-1}\,|\Sigma_i] \text{ and } G_i \le \mathcal{E}[G_{-1}|\Sigma_i] \le \mathcal{E}[F_{-1} \vee G_{-1}\,|\Sigma_i]$$

for all $i \in -\mathbb{N}$.

To prove the minimality, let $(H_i, \Sigma_i)_{i\in-\mathbb{N}}$ be a reversed martingale with $F_i \le H_i$ and $G_i \le H_i$ for all $i \in -\mathbb{N}$. Then, $F_i \vee G_i \le H_i$ for all $i \in -\mathbb{N}$. In particular, $F_{-1} \vee G_{-1} \le H_{-1}$; thus, by (E2), we get that $\mathcal{E}[F_{-1} \vee G_{-1}|\Sigma_i] \le \mathcal{E}[H_{-1}|\Sigma_i] = H_i$ for all $i \in -\mathbb{N}$. This completes the proof. □

The following is the main result of this section.

Theorem 4.3. *The space $\mathcal{M}_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$ has the following properties:*

(a) *$\mathcal{M}_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$ is a join-semilattice; i.e., if $(F_i, \Sigma_i)_{i\in-\mathbb{N}}$ and $(G_i, \Sigma_i)_{i\in-\mathbb{N}}$ are reversed martingales, then*

$$(F_i, \Sigma_i)_{i\in-\mathbb{N}} \vee (G_i, \Sigma_i)_{i\in-\mathbb{N}} = \big(\mathcal{E}[(F_{-1} \vee G_{-1})|\Sigma_i], \Sigma_i\big)_{i\in-\mathbb{N}}.$$

In particular,

$$(F_i, \Sigma_i)^+_{i\in-\mathbb{N}} = (\mathcal{E}[F_{-1}{}^+|\Sigma_i], \Sigma_i)_{i\in-\mathbb{N}}.$$

(b) *$\mathcal{M}_{\mathrm{rev}}(\mathcal{L}[\Sigma, \mathrm{f}(X)], \Sigma_i)$ is order closed; i.e., if $(F_i^{(\alpha)}, \Sigma_i)_{i\in-\mathbb{N}}$ is an increasing sequence (in α, with $\alpha \in \mathbb{N}$,) of reversed martingales, then $(G_i, \Sigma_i)_{i\in-\mathbb{N}}$, where G_i, for all $i \in -\mathbb{N}$, is defined by*

$$G_i(\omega) = \overline{\bigcup_{\alpha=1}^{\infty} F_i^{(\alpha)}(\omega)} \text{ for all } \omega \in \Omega,$$

is the minimal reversed martingale such that $F_i^\alpha \le G_i$ *for all* $\alpha \in \mathbb{N}$ *and* $i \in -\mathbb{N}$*. If, in addition,* $\{0\} \subseteq F_i^{(\alpha)}$ *for all* $\alpha, \in \mathbb{N}$ *and* $i \in -\mathbb{N}$*, then* $\{0\} \subseteq G_i$ *for all* $i \in -\mathbb{N}$*.*

Proof. (a) This follows directly from Lemma 4.2, as reversed martingales are also reversed submartingales.

(b) Let $(F_i^{(\alpha)}, \Sigma_i)_{i\in-\mathbb{N}}$ be an increasing sequence (in $\alpha \in \mathbb{N}$) of reversed martingales. It is readily verified, by using Theorem 3.1 (c), that $G_i \in \mathcal{L}[\Sigma_i, \mathrm{f}(X)]$ for all $i \in -\mathbb{N}$. For $m, n \in -\mathbb{N}$ and $m \ge i$, we get

$$\mathcal{E}[G_m|\Sigma_i](\omega) = \overline{\bigcup_{\alpha=1}^{\infty} \mathcal{E}[F_m^{(\alpha)}|\Sigma_i](\omega)} = \overline{\bigcup_{\alpha=1}^{\infty} F_i^{(\alpha)}(\omega)} = G_i(\omega) \text{ a.e.}$$

for all $\omega \in \Omega$. This shows that $(G_i, \Sigma_i)_{i\in-\mathbb{N}}$ is a reversed martingale. The minimality of $(G_i, \Sigma_i)_{i\in-\mathbb{N}}$ follows easily from the definition of G_i. If, in addition, $\{0\} \subseteq F_i^{(\alpha)}(\omega)$ for all $\alpha \in \mathbb{N}$ and $i \in -\mathbb{N}$, then $\{0\} \subseteq G_i(\omega)$ for all $\omega \in \Omega$. □

5. Martingales in $\mathcal{L}[\Sigma, \mathrm{f}(X)]$

In the preceding paragraph, we dealt with $-\mathbb{N}$ as our index set. Here we deal with $\mathbb{N}$ as index set.

Definition 5.1. *Let* $(F_i)_{i\in\mathbb{N}} \subseteq \mathcal{L}[\Sigma, \mathrm{f}(X)]$ *and* $(\Sigma_i)_{i\in\mathbb{N}}$ *an increasing sequence of sub-*σ*-fields of* Σ*. Then* $(F_i, \Sigma_i)_{i\in\mathbb{N}}$ *is called a* martingale *(respectively,* submartingale*) in* $\mathcal{L}[\Sigma, \mathrm{f}(X)]$ *provided that* $F_i \in \mathcal{L}[\Sigma_i, \mathrm{f}(X)]$ *and* $F_i(\omega) = (\subseteq)\ \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ *a.e. for all* $i \in \mathbb{N}$*.*

Lemma 5.2. *Let* (Σ_i) *be an increasing sequence of sub-*σ*-fields of* Σ*. If* (F_i, Σ_i) *and* (G_i, Σ_i) *are martingales, then* $(F_i \vee G_i, \Sigma_i)$ *is a submartingale, where*

$$(F_i \vee G_i)(\omega) = F_i(\omega) \cup G_i(\omega)$$

a.e. for all $\omega \in \Omega$*.*

Proof. If F_i and G_i are Σ_i-measurable, then $F_i \vee G_i$ is also Σ_i-measurable. Also, if F_i and G_i are integrable, then $F_i \vee G_i$ is also integrable. Furthermore,

$$\begin{aligned} \mathcal{E}[F_{i+1} \vee G_{i+1}\,|\Sigma_i](\omega) &\supseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \cup \mathcal{E}[G_{i+1}\,|\Sigma_i](\omega) \\ &= F_i(\omega) \cup G_i(\omega) \\ &= (F_i \vee G_i)(\omega) \text{ a.e.} \end{aligned}$$

for all $i \in \mathbb{N}$. Thus, $\mathcal{E}[F_{i+1} \vee G_{i+1}\,|\Sigma_i] \ge F_i \vee G_{i+1}$ for all $i \in \mathbb{N}$. □

Let

$$\mathrm{cf}(X) = \{A \in \mathrm{f}(X) : A \text{ is convex}\}.$$

Then $(\mathrm{cf}(X), \subseteq)$ is a join-semilattice; the join $\vee$ is given by

$$A \vee C = \overline{\mathrm{co}}(A \cup C) \text{ for all } A, C \in \mathrm{cf}(X),$$

where $\overline{\mathrm{co}}(A \cup C)$ denotes the norm closure in X of the closed convex hull of $A \cup C$.

Define

$$\mathcal{L}[\Sigma, \mathrm{cf}(X)] = \{F \in \mathcal{L}[\Sigma, \mathrm{f}(X)] : F(\omega) \in \mathrm{cf}(X) \text{ a.e.}\}.$$

Theorem 5.3. *If $F \in \mathcal{L}[\Sigma, \mathrm{cf}(X)]$, the the conditional expectation $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}[\Sigma_0, \mathrm{cf}(X)]$ of F has the following additional properties:*

(E4) *If $F \in \mathcal{L}[\Sigma_0, \mathrm{cf}(X)]$, then $\mathcal{E}[F|\Sigma_0] = F$.*

(E5) *If Σ_1 and Σ_2 are sub-σ-fields of Σ such that $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma$ and $F \in \mathcal{L}[\Sigma, \mathrm{cf}(X)]$, then $\mathcal{E}[\,\mathcal{E}[F|\Sigma_2]\,|\,\Sigma_1] = \mathcal{E}[F|\Sigma_1]$.*

Lemma 5.4. *Let (Σ_i) be an increasing sequence of sub-σ-fields of Σ. If (F_i, Σ_i) and (G_i, Σ_i) are martingales with values in $\mathrm{cf}(X)$, then $(F_i \vee G_i, \Sigma_i)$ is a sub-martingale with values in $\mathrm{cf}(X)$, where $(F_i \vee G_i)(\omega) = \overline{\mathrm{co}}(F_i(\omega) \cup G_i(\omega))$ a.e. for all $\omega \in \Omega$.*

Proof. Let U_i be defined by $U_i(\omega) = F_i(\omega) \cup G_i(\omega)$. If F_i and G_i are Σ_i-measurable, then U_i is also Σ_i-measurable, and so $\overline{\mathrm{co}}\, U_i = F_i \vee G_i$ is also Σ_i-measurable. Also, if F_i and G_i are integrable, then U_i is also integrable, and so $\overline{\mathrm{co}} U_i = F_i \vee G_i$ is also integrable.

It follows from Lemma 5.2 that $\mathcal{E}[U_{i+1}|\Sigma_i] \geq U_i$ for all $i \in \mathbb{N}$. Thus, for all $i \in \mathbb{N}$ and for all $\omega \in \Omega$,

$$\begin{aligned}
\mathcal{E}[(F_{i+1} \vee G_{i+1})\,|\Sigma_i](\omega) &= \mathcal{E}[\overline{\mathrm{co}}\, U_{i+1}|\Sigma_i](\omega) \\
&= \overline{\mathrm{co}}\, \mathcal{E}[U_{i+1}|\Sigma_i](\omega) \\
&\supseteq \overline{\mathrm{co}}\, U_i(\omega) \\
&= (F_i \vee G_i)(\omega) \text{ a.e.},
\end{aligned}$$

from which we get that $\mathcal{E}[(F_{i+1} \vee G_{i+1})\,|\Sigma_i] \geq (F_i \vee G_i)$ for all $i \in \mathbb{N}$. □

Our first aim is to derive an analogue of (E3) for $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{cf}(X)]$.

Lemma 5.5. *Let Y be a Banach space and $A \subseteq Y$. Then $\overline{\mathrm{co}}(\overline{A}) = \overline{\mathrm{co}} A$.*

Proof. The easy proof is left to the reader. □

Theorem 5.6. *If $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{f}(X)]$ is an increasing sequence, i.e., $F_i(\omega) \subseteq F_{i+1}(\omega)$ a.e. for all $i \in \mathbb{N}$, and $F(\omega) := \overline{\mathrm{co}}\big(\bigcup_{i=1}^{\infty} F_i(\omega)\big)$ for all $\omega \in \Omega$, then*

$$\mathcal{E}[F|\Sigma_0](\omega) = \overline{\mathrm{co}}\Big(\bigcup_{i=1}^{\infty} \mathcal{E}[F_i|\Sigma_0](\omega)\Big) \text{ a.e.}$$

Proof. It follows from Theorem 3.1 (e), Lemma 5.5 and Theorem 3.1 (c) that $\overline{\mathrm{co}}\big(\bigcup_{i=1}^{\infty} \mathcal{E}[F_i|\Sigma_0](\omega)\big)$ is Σ-measurable.

Define $V : \Omega \to \mathrm{f}(X)$ by $V(\omega) = \overline{\bigcup_{i=1}^{\infty} F_i(\omega)}$ for all $\omega \in \Omega$. By Lemma 5.5, we have $F(\omega) = \overline{\mathrm{co}}\, V(\omega)$ for all $\omega \in \Omega$. By properties (E1) and (E3), and another application of Lemma 5.5, we get

$$\begin{aligned}\mathcal{E}[F|\Sigma_0](\omega) &= \overline{\text{co}}\,\mathcal{E}[V|\Sigma_0](\omega)\\ &= \overline{\text{co}}\,\Big(\overline{\bigcup_{i=1}^{\infty}\mathcal{E}[F_i\,|\,\Sigma_0\,](\omega)}\Big)\\ &= \overline{\text{co}}\,\Big(\bigcup_{i=1}^{\infty}\mathcal{E}[F_i\,|\,\Sigma_0\,](\omega)\Big)\text{ a.e..}\end{aligned}$$

This completes the proof. □

Let

$$\mathcal{M}(\mathcal{L}[\Sigma,\text{f}(X)],\Sigma_i)=\{(F_i,\Sigma_i):(F_i,\Sigma_i)\text{ is a martingale in }\mathcal{L}[\Sigma,\text{f}(X)]\}.$$

For all $(F_i,\Sigma_i,(G_i,\Sigma_i)\in\mathcal{M}(\mathcal{L}[\Sigma,\text{f}(X)],\Sigma_i)$, define

$$(F_i,\Sigma_i)\leq(G_i,\Sigma_i)\Longleftrightarrow F_i(\omega)\subseteq G_i(\omega)\text{ a.e. for all }\omega\in\Omega,\ i\in\mathbb{N}.$$

Then $\mathcal{M}(\mathcal{L}[\Sigma,\text{f}(X)],\Sigma_i)$ is a partially ordered set, as is easily verified.

The following is our first main result. The proof uses property (E3), which restricts us to nonempty convex closed subsets of X:

Theorem 5.7. *Let (Σ_i) be an increasing sequence of sub-σ-fields of Σ. Then*

(a) *$\mathcal{M}(\mathcal{L}[\Omega,\text{cf}(X)],\Sigma_i)$ is a join-semilattice; i.e., if (F_i,Σ_i) and (G_i,Σ_i) are martingales with values in $\text{cf}(X)$, then $(F_i,\Sigma_i)\vee(G_i,\Sigma_i)$ is a martingale with values in $\text{cf}(X)$. In particular, $(F_i,\Sigma_i)\vee(G_i,\Sigma_i)=(S_i,\Sigma_i)$, where*

$$S_i(\omega)=\overline{\text{co}}\left(\bigcup_{m\geq i}\mathcal{E}\big[F_m\vee G_m\big|\Sigma_i\big](\omega)\right)\ \textit{for all }\omega\in\Omega.$$

In particular, if (F_i,Σ_i) is a martingale with values in $\text{cf}(X)$, then $(F_i,\Sigma_i)^+:=(P_i,\Sigma_i)$, where

$$P_i(\omega)=\overline{\text{co}}\left(\bigcup_{m\geq i}\mathcal{E}\big[F_m^+|\Sigma_i\big](\omega)\right)\ \textit{for all }\omega\in\Omega,$$

is the minimal martingale with values in $\text{cf}(X)$ such that $F_i\leq P_i$ and $\{0\}\leq P_i$ for all $i\in\mathbb{N}$.

(b) *$\mathcal{M}(\mathcal{L}[\Sigma,\text{f}(X)],\Sigma_i)$ is order closed; i.e., if $(F_i^{(\alpha)},S_i)_{\alpha=1}^{\infty}$ is an increasing sequence (in α) of martingales with values in $\text{cf}(X)$, then (G_i,Σ_i), where each $G_i\colon\Omega\to\text{cf}(X)$ is defined by*

$$G_i(\omega)=\overline{\text{co}}\left(\bigcup_{\alpha=1}^{\infty}F_i^{(\alpha)}(\omega)\right)\ \textit{for all }\omega\in\Omega,$$

is the minimal $\text{cf}(X)$-valued martingale such that $F_i^{(\alpha)}\leq G_i$ for all α. If, in addition, $\{0\}\subseteq F_i^{(\alpha)}$ for all α,i, then $\{0\}\subseteq G_i$.

Proof. (a) Fix $i \in \mathbb{N}$ and consider the set $\{\mathcal{E}[F_m \vee G_m|\Sigma_i] : m \geq i\}$. This set is increasing, since

$$\begin{aligned}\mathcal{E}[F_m \vee G_m|\Sigma_i] &= \mathcal{E}\left[\mathcal{E}[F_{m+1}|\Sigma_m] \vee \mathcal{E}[G_{m+1}|\Sigma_m]\middle|\Sigma_i\right] \\ &\leq \mathcal{E}\left[\mathcal{E}[F_{m+1} \vee G_{m+1}|\Sigma_m]\middle|\Sigma_i\right] \\ &= \mathcal{E}[F_{m+1} \vee G_{m+1}|\Sigma_i]\end{aligned}$$

for each $m \geq i$ (where the inequality in step two is obtained by using property (E2). Define S_i by

$$S_i(\omega) = \overline{\text{co}}\left(\bigcup_{m\geq i} \mathcal{E}\big[F_m \vee G_m|\Sigma_i](\omega)\right) \text{ for all } \omega \in \Omega.$$

We claim that (S_i, Σ_i) is a martingale. It is readily verified that $S_i \in \mathcal{L}[\Sigma_i, \mathrm{f}(X)]$ for all $i \in \mathbb{N}$. Furthermore, by property (E5),

$$\begin{aligned}\mathcal{E}[S_i|\Sigma_k](\omega) &= \overline{\text{co}}\left(\bigcup_{m\geq i} \mathcal{E}\left[\mathcal{E}[F_m \vee G_m|\Sigma_i]\,\middle|\Sigma_k\right](\omega)\right) \\ &= \overline{\text{co}}\left(\bigcup_{m\geq k} \mathcal{E}\left[F_m \vee G_m|\Sigma_k](\omega)\right) \\ &= S_k(\omega),\end{aligned}$$

a.e. for all $\omega \in \Omega$ and for any $k \leq i$. This proves that (S_i, Σ_i) is a martingale.

To conclude that $(S_i, \Sigma_i) = (F_i, \Sigma_i) \vee (G_i, \Sigma_i)$, let (Z_i, Σ_i) be a martingale, with $(Z_i) \subseteq \mathcal{L}[\Omega, \mathrm{cf}(X)]$, such that $(Z_i, \Sigma_i) \geq (F_i, \Sigma_i)$ and $(Z_i, \Sigma_i) \geq (G_i, \Sigma_i)$. Then

$$Z_i(\omega) \supseteq \overline{\text{co}}(F_i(\omega) \cup G_i(\omega)) \text{ a.e.}$$

for all $\omega \in \Omega$ and for all $i \in \mathbb{N}$. Thus, it follows from

$$Z_i(\omega) = \mathcal{E}[Z_m|\Sigma_i](\omega) \supseteq \mathcal{E}[F_m \vee G_m|\Sigma_i](\omega) \text{ a.e.}$$

for each $m \geq i$, that $Z_i \geq S_i$ for each $i \in \mathbb{N}$; i.e., $(Z_i, \Sigma_i) \geq (S_i, \Sigma_i)$.

(b) Let $(F_i^{(\alpha)}, \Sigma_i)$ be an increasing sequence (in (α)) of martingales with values in $\mathrm{cf}(X)$. For each $i \in \mathbb{N}$, define $G_i : \Omega \to \mathrm{cf}(X)$ by

$$G_i(\omega) = \overline{\text{co}}\left(\bigcup_{\alpha=1}^{\infty} F_i^{(\alpha)}(\omega)\right) \text{ for all } \omega \in \Omega.$$

Then $G_i \in \mathcal{L}[\Sigma_i, \mathrm{f}(X)]$ for all $i \in \mathbb{N}$, and for $m \geq i$ we get

$$\begin{aligned}\mathcal{E}[G_m|\Sigma_i](\omega) &= \overline{\text{co}}\left(\bigcup_{\alpha=1}^{\infty} \mathcal{E}[F_m^{(\alpha)}|\Sigma_i](\omega)\right) \\ &= \overline{\text{co}}\left(\bigcup_{\alpha=1}^{\infty} F_i^{(\alpha)}(\omega)\right) \\ &= G_i(\omega)\end{aligned}$$

a.e. for all $\omega \in \Omega$. Thus, $\mathcal{E}[G_m|\Sigma_i] = G_i$ for all $m \geq i$. This shows that (G_i, Σ_i) is a martingale with values in $\mathrm{cf}(X)$. Moreover, if $\{0\} \subseteq F_i^{(\alpha)}$ for all α, i, then $\{0\} \subseteq G_i(\omega)$ for all $\omega \in \Omega$. In this case, (G_i, Σ_i) is a positive martingale with values in $\mathrm{cf}(X)$. □

6. The space $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$

If $A \in \mathcal{P}_0(X)$ and $x \in X$, the *distance* between x and A is defined by

$$d(x, A) = \inf\{\|x - y\|_X : y \in A\}.$$

Let

$$\mathrm{bf}(X) = \{A \in \mathrm{f}(X) : A \text{ is bounded}\}.$$

Define d_H for all $A, B \in \mathrm{bf}(X)$ by

$$d_H(A, B) = \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A).$$

Then d_H is a metric on $\mathrm{bf}(X)$, which is called the *Hausdorff metric*, and $(\mathrm{bf}(X), d_H)$ is a complete metric space. In the special case where $B = \{0\}$, let

$$\|A\|_H = d_H(A, \{0\});$$

in general $\|\cdot\|_H$ is not a norm. Let

$$\mathrm{cbf}(X) = \{A \in \mathrm{bf}(X) : A \text{ is convex}\}.$$

Then $\mathrm{cbf}(X)$ is a closed subspace of $\mathrm{cbf}(X)$ (cf. [15]).

If $F\colon \Omega \to \mathrm{f}(X)$ is measurable, then F is called *integrably bounded* provided that there exists $\rho \in L^1(\mu)$ such that $\|x\|_X \leq \rho(\omega)$ for all $x \in F(\omega)$ and for all $\omega \in \Omega$. In this case, $F(\omega) \in \mathrm{bf}(X)$ a.e. for all $\omega \in \Omega$ and

$$\|F(\omega)\|_H = \sup\{\|x\|_X : x \in F(\omega)\} \leq \rho(\omega) \text{ for all } \omega \in \Omega.$$

We denote by $\mathcal{L}^1[\Sigma, \mathrm{f}(X)]$ the set of all (equivalence classes of a.e. equal) measurable $F\colon \Omega \to \mathrm{f}(X)$ which are integrably bounded. For all $F_1, F_2 \in \mathcal{L}^1[\Sigma, \mathrm{f}(X)]$ define

$$\Delta(F_1, F_2) = \int_\Omega d_H\big(F_1(\omega), F_2(\omega)\big) d\mu.$$

Then $(\mathcal{L}^1[\Sigma, \mathrm{f}(X)], \Delta)$ is a complete metric space (cf. [15]).

We recall from [15] that, if we define $\leq$ for each $F, G \in \mathcal{L}^1[\Sigma, \mathrm{f}(X)]$ by

$$F \leq G \iff F(\omega) \subseteq G(\omega) \text{ a.e. for all } \omega \in \Omega,$$

then $(\mathcal{L}^1[\Sigma, \mathrm{f}(X)], \leq)$ is a partially ordered set. Moreover, it is readily verified that $(\mathcal{L}^1[\Sigma, \mathrm{f}(X)], \leq)$ is a join-semilattice; the join $A \vee C$ of $A, C \in \mathcal{L}^1[\Sigma, \mathrm{f}(X)]$ is given by

$$(A \vee C)(\omega) = A(\omega) \cup C(\omega) \text{ a.e. for all } \omega \in \Omega.$$

Let

$$\mathcal{L}^1[\Sigma, \mathrm{cf}(X)] = \{F \in \mathcal{L}^1[\Sigma, \mathrm{f}(X)] : F(\omega) \in \mathrm{cf}(X) \text{ a.e.}\}.$$

It is readily seen that $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ is a partially ordered set with respect to the ordering induced by the ordering on $\mathcal{L}^1[\Sigma, \mathrm{f}(X)]$. The partially ordered set

$\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ is also a join-semilattice; the join $A \vee C$ of $A, C \in \mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ is given by

$$(A \vee C)(\omega) = \overline{\mathrm{co}}\,(A(\omega) \cup C(\omega)) \text{ a.e. for all } \omega \in \Omega.$$

Furthermore, $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ is a closed subspace of $\mathcal{L}^1[\Sigma, \mathrm{f}(X)]$ (cf. [15]).

We need compapability properties between the ordering $\leq$ and the metric Δ on $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$. For this purpose, we first revisit Hörmaner's embedding theorem for $\mathrm{cbf}(X)$ (cf. [8, 15]).

Suppose M be a metric space and let

$$C_b(M) := \{f : \Omega \to R : f \text{ is continuous and bounded}\}.$$

Endow the latter with pointwise addition and scalar multiplication and norm given by

$$\|f\|_\infty := \sup\{|f(s)| : t \in M\}.$$

Then $C_b(M)$ is a Banach space.

Let X^* denote the continuous dual of X. For every bounded subset C of X and each $x^* \in X^*$, let

$$s(x^*, C) := \sup\{x^*(x) : x \in C\}.$$

Theorem 6.1. (Hörmander) *There exist a metric space S and a map $J : \mathrm{cbf}(X) \to C_b(S)$ such that*

(a) $J(\alpha A + \beta C) = \alpha J(A) + \beta J(C)$ *for all* $A, C \in \mathrm{cbf}(X)$ *and* $\alpha, \beta \in R_+$,
(b) $d_H(A, C) = \|J(A) - J(C)\|_\infty$ *for all* $A, C \in \mathrm{cbf}(X)$.

It is shown in [15] that $S = \{x^* \in X^* : \|x^*\| = 1\}$, endowed with the topology of bounded convergence, and J, defined by $J(A) = s(\,\cdot\,, A)$ for all $A \in \mathrm{cbf}(X)$, satisfy the required properties.

If M is a metric space, there is a canonical ordering on $C_b(M)$, namely the pointwise ordering. If we endow $C_b(M)$ with the pointwise ordering, then $C_b(M)$ is a partially ordered set and also a join-semilattice. The join $f \vee g$ of $f, g \in C_b(M)$ is given by

$$(f \vee g)(s) = \max\{f(s), g(s)\} \text{ for all } s \in M.$$

It is well known that $C_b(M)$ is a vector lattice and

$$\|f_1 \vee g_1 - f_2 \vee g_2\|_\infty \leq \|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty$$

for all $f_1, f_2, g_1, g_2 \in C_b(M)$ (cf. [21]).

We claim that the map J in Theorem 6.1 is join preserving. The following is used to prove our claim.

Lemma 6.2. *For all nonempty bounded subsets A and C of X and for all $x^* \in X^*$,*

$$s\left(x^*, \overline{\mathrm{co}}\big(A \cup C\big)\right) = \max\{s(x^*, A), s(x^*, C)\}.$$

Proof. Direct verification yields that

$$s(x^*, A \cup C) = \max\{s(x^*, A), s(x^*, C)\}$$

for all nonempty bounded subsets A and C of X and for all $x^* \in X^*$. It is also readily verified that for any nonempty bounded subset A of X and for all $x^* \in X^*$,

$$s(x^*, \overline{\text{co}}A) = s(x^*, A).$$

To complete the proof, let A and B be nonempty bounded subsets of X and $x^* \in X^*$. Then, by the preceding parts of the proof,

$$s\big(x^*, \overline{\text{co}}(A \cup C)\big) = s(x^*, A \cup C) = \max\{s(x^*, A), s(x^*, C)\}.$$

□

As a consequence, we can make the following addition to Theorem 6.1:

(c) $J\big(\overline{\text{co}}(A \cup C)\big) = \max\{J(A), J(C)\}$ for all $A, C \in \text{ck}(X)$.

We are now in a position to derive compabability properties between the ordering $\leq$ and the metric Δ on $\mathcal{L}^1[\Sigma, \text{cf}(X)]$.

Theorem 6.3. *Let* $\{F_i, G_i, F, G : i \in N\} \subseteq \mathcal{L}^1[\Sigma, \text{cbf}(X)]$. *If* $\lim_{i\to\infty} \Delta(F_i, F) = 0$ *and* $\lim_{i\to\infty} \Delta(G_i, G) = 0$*, then*

(a) $\lim_{i\to\infty} \Delta\left(F_i \vee G_i), (F \vee G)\right) = 0$*, and*

(b) $F \leq G$*, provided that* $F_i \leq G_i$ *for all* $i \in \mathbb{N}$.

Proof. (a) Let S be a metric space and $J\colon \text{cbf}(X) \to C(S)$ be as in Theorem 6.1. Let $\omega \in \Omega$. As mentioned above,

$$\begin{aligned} &\|j\left(F_i(\omega)\right) \vee j\left(G_i(\omega)\right) - j\left(F(\omega)\right) \vee j\left(G(\omega)\right)\|_\infty \\ &\leq \quad \|j\left(F_i(\omega)\right) - j\left(F(\omega)\right)\|_\infty + \|j\left(G_i(\omega)\right) - j\left(G(\omega)\right)\|_\infty, \end{aligned}$$

from which we get that

$$\begin{aligned} &d_H\left(\overline{\text{co}}\left(F_i(\omega) \cup G_i(\omega)\right), \overline{\text{co}}(F(\omega) \cup G(\omega))\right) \\ &\leq \quad d_H(F_i(\omega), F(\omega)) + d_H(G_i(\omega), G(\omega)). \end{aligned}$$

It follows from the definition of Δ that

$$\Delta\left((F_i \vee G_i), (F \vee G)\right) \leq \Delta(F_i, F) + \Delta(G_i, G).$$

Hence, $\lim_{i\to\infty} \Delta\left((F_i \vee G_i), (F \vee G)\right) = 0$.

(b) Since $0 = \lim_{i\to\infty} \Delta(G, G_i)$ and $F_i \leq G_i$ for all $i \in \mathbb{N}$, it follows from

$$0 = \lim_{i\to\infty} \Delta\left((F \vee G), (F_i \vee G_i)\right) = \lim_{i\to\infty} \Delta\left((F \vee G), G_i\right),$$

that $G = F \vee G$. Hence, $F \leq G$. □

7. Martingales in $\mathcal{L}^1[\Sigma, \text{cf}(X)]$

Theorem 7.1. (cf. [7, 15]) *Let* Σ_0 *be a sub-*σ*-field of* Σ. *If* $F \in \mathcal{L}^1[\Sigma, \text{cf}(X)]$, *then the conditional expectation* $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Sigma_0, \text{cf}(X)]$ *of* F *with respect to* Σ_0 *has the following property:*

(E6) *If* $F_1, F_2 \in \mathcal{L}^1[\Sigma, \text{cf}(X)]$*, then* $\Delta(\mathcal{E}[F_1|\Sigma_0], \mathcal{E}[F_2|\Sigma_0]) \leq \Delta(F_1, F_2)$.

We recall the following terminology from [15]:

Definition 7.2. (cf. [15]) *Let* (Σ_i) *be a increasing sequence of sub* σ*-fields of* Σ. *Then* (F_i, Σ_i) *is called a* set-valued martingale (*a* set-valued submartingale) *in* $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$, *provided that* $F_i \in \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$, *and* $F_i = (\subseteq)\, \mathcal{E}[F_{i+1}|\Sigma_i]$ *for all* $i \in N$.

Definition 7.3. *If* (F_i, Σ_i) *in a set-valued martingale in* $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ *and if there exists* $F \in \mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ *such that* $F_i = \mathcal{E}[F|\Sigma_i]$ *for all* $i \in \mathbb{N}$, *then* (F_i, Σ_i) *is called a* regular *martingale.*

In general, regular set-valued martingales are not Δ-convergent (cf. [7] or [15, p.133]). The following two results, which clarify this issue, can also be derived for abstract metric space martingales without the use of measure theory, as in [12, Theorems 5.3 and 5.4].

Lemma 7.4. *Let* $F \in \mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$. *Then* $\lim_{i\to\infty} \Delta(\mathcal{E}[F|\Sigma_i], F) = 0$ *if and only if* $F \in \overline{\bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$, *where the closure is taken with respect to* Δ.

Proof. Suppose $\lim_{i\to\infty} \Delta(\mathcal{E}[F|\Sigma_i], F) = 0$. Since $\mathcal{E}[F|\Sigma_i] \in \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$, for each $i \in \mathbb{N}$, we have $F \in \overline{\bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$. To prove the converse, suppose that $F \in \overline{\bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$. Then there exists a sequence $(F_n) \subseteq \bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$ such that $\lim_{i\to\infty} \Delta(F_i, F) = 0$. Thus, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so that $\Delta(F_n, F) < \varepsilon/2$. Since $\mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)] \subseteq \mathcal{L}^1[\Sigma_{i+1}, \mathrm{cf}(X)]$ for all $i \in \mathbb{N}$, there exists an $I_n \in \mathbb{N}$ such that $i \geq I_n$ implies $F_n \in \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$ and

$$\begin{aligned}
\Delta(\mathcal{E}[F|\Sigma_i], F) &\leq \Delta(\mathcal{E}[F|\Sigma_i], F_n) + \Delta(F_n, F) \\
&\leq \Delta(\mathcal{E}[F|\Sigma_i], \mathcal{E}[F_n|\Sigma_i]) + \Delta(F_n, F) \\
&\leq \Delta(F, F_n) + \Delta(F_n, F) \\
&< \varepsilon,
\end{aligned}$$

which completes the proof. □

The following is a necessary and sufficient condition for a regular set-valued martingale to be convergent:

Theorem 7.5. *Let* (F_i, Σ_i) *be a martingale in* $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$. *Then* (F_i) *is* Δ-*convergent if and only there exists* $F \in \overline{\bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$ *such that* $F_i = \mathcal{E}[F|\Sigma_i]$ *for all* $i \in \mathbb{N}$.

Proof. Suppose $F \in \mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ and $\lim_{i\to\infty} \Delta(F_i, F) = 0$. Since (F_i, Σ) is a martingale in $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ and (E6) holds, it follows that

$$\Delta(F_k, \mathcal{E}[F|\Sigma_k]) = \Delta(\mathcal{E}[F_i|\Sigma_k], \mathcal{E}[F|\Sigma_k]) \leq \Delta(F_i, F)$$

for all $k \leq i$. This, together with $\lim_{i\to\infty} \Delta(F_i, F) = 0$, yields $n_0 \in \mathbb{N}$ such that $F_i = \mathcal{E}[F|\Sigma_i]$ for all $i \geq n_0$. But, for $i \leq n_0$, we have from (E6)

$$\Delta(\mathcal{E}[F|\Sigma_i], F_i) = \Delta(\mathcal{E}\left[\mathcal{E}[F|\Sigma_{n_0}]\,|\Sigma_i\right], \mathcal{E}[F_{n_0}|\Sigma_i) \leq \Delta(\mathcal{E}[F|\Sigma_{n_0}], F_{n_0}) < \varepsilon.$$

Thus, $F_i = \mathcal{E}[F|\Sigma_i]$ for all $i \in \mathbb{N}$. Since $\mathcal{E}[F|\Sigma_i] \in \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \Delta(F_i, F) = 0$, we get that $F \in \overline{\bigcup_{i=1}^{\infty} \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$.

Conversely, by the assumption and Lemma 7.4, we have

$$\Delta(\mathcal{E}[F|\Sigma_i], F) = \Delta(F_i, F) \to 0,$$

which completes the proof. □

Reversed martingales in the present setting are simple objects, as the following result shows:

Theorem 7.6. *Let $(F_i, \Sigma_i)_{i\in -\mathbb{N}}$ be a set-valued in reversed martingale in $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$. Then the following statements are equivalent:*

(a) *$(F_i)_{i\in-\mathbb{N}}$ is Δ-convergent.*
(b) *There exists $F \in \bigcap_{i\in-\mathbb{N}}\mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$ such that $F_i = \mathcal{E}[F|\Sigma_i]$ for all $i \in -\mathbb{N}$.*
(c) *There exists $F \in \mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ such that $F_i = F$ for all $i \in -\mathbb{N}$.*

Proof. (a)⇒(b) As in the proof of Theorem 7.5, it follows that $F_i = \mathcal{E}[F|\Sigma_i]$ for all $i \in -\mathbb{N}$. Since $\mathcal{E}[F|\Sigma_i] \in \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$ for all $i \in -\mathbb{N}$ and $\lim_{i\to-\infty} \Delta(F_i, F) = 0$, we get that $F \in \bigcap_{i\in-\mathbb{N}}\mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]$. The implications (b)⇒(c)⇒(a) are trivial. □

8. The space $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$

Suppose that (Σ_i) is an increasing sequence of sub-σ-fields of Σ. Denote by $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$ the set of all set-valued martingales (F_i, Σ_i) in $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ for which (F_i) is Δ-convergent.

Define $\Delta_\mathcal{M}$ by

$$\Delta_\mathcal{M}\left((F_i, \Sigma_i), (G_i, \Sigma_i)\right) = \sup_{i\in N} \Delta(F_i, G_i).$$

for all $(F_i, \Sigma_i), (G_i, \Sigma_i) \in \mathcal{M}\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$. Then $\big(\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i), \Delta_\mathcal{M}\big)$ is a metric space, as simple verification shows. The completeness of the latter space follows as a bonus from Theorem 8.1 below.

Endow $\mathcal{L}^1[\Sigma, \mathrm{cf}(X)]$ with an order relation, by defining

$$(F_i, \Sigma_i) \le (G_i, \Sigma_i) \iff F_i \le G_i \text{ for all } i \in \mathbb{N}$$

for all $(F_i, \Sigma_i), (G_i, \Sigma_i) \in \mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$. It is then readily verified that $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$ is a partially ordered set.

The aim of the remaining part of this section is to derive further properties of the space $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i)$.

Theorem 8.1. *The map $L : \mathcal{M}_\Delta(\mathcal{L}^1[\Sigma, \mathrm{cf}(X)], \Sigma_i) \to \overline{\bigcup_{i=1}^\infty \mathcal{L}^1[\Sigma_i, \mathrm{cf}(X)]}$, defined by*

$$L\big((F_i, \Sigma_i)\big) = \lim_{i\to\infty} F_i,$$

has the following properties:

(a) *L is a bijection and* $L^{-1}\colon \overline{\bigcup_{i=1}^{\infty}\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]} \to \mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$ *is given by*

$$L^{-1}(F) = (\mathcal{E}[F|\Sigma_i],\Sigma_i).$$

(b) *Both L and L^{-1} are order preserving.*

(c) *L is an isometry.*

Proof. (a) To see that L is a surjection, let $F \in \overline{\bigcup_{i=1}^{\infty}\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]}$. Then, by Lemma 7.4, $\lim_{i\to\infty}\Delta(\mathcal{E}[F|\Sigma_i],F) = 0$ and $(\mathcal{E}[F|\Sigma_i],\Sigma_i)$ is a set-valued martingale in $\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]$ such that $L\,(\mathcal{E}[F|\Sigma_i],\Sigma_i) = F$.

To prove injectivity, note that $L\,((F_i,\Sigma_i)) = L\,((G_i,\Sigma_i))$ implies $\lim_{i\to\infty}F_i = \lim_{i\to\infty}G_i$. An application of Theorem 7.5 yields

$$F_k = \mathcal{E}[\lim_{i\to\infty}F_i \,|\, \Sigma_k\,] = \mathcal{E}[\lim_{i\to\infty}G_i \,|\, \Sigma_k\,] = G_k$$

for each $k\in\mathbb{N}$. Thus, $(F_i,\Sigma_i) = (G_i,\Sigma_i)$, completing the proof that L is injective.

Define A by $A(F) = (\mathcal{E}[F|\Sigma_i],\Sigma_i)$ for each $F \in \overline{\bigcup_{i=1}^{\infty}\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]}$. Then, by Theorem 7.5,

$$A\colon \overline{\bigcup_{i=1}^{\infty}\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]} \to \mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i).$$

Since

$$A\,(L(F_i,\Sigma_i)) = A\left(\lim_{i\to\infty}F_i\right) = \left(\mathcal{E}[\lim_{k\to\infty}F_k\,|\Sigma_i],\Sigma_i)\right) = (F_i,\Sigma_i),$$

and $L\,(A(F)) = L\,((\mathcal{E}[\,F\,|\Sigma_i],\Sigma_i)) = F$, we get that $A = L^{-1}$.

(b) Let $(F_i,\Sigma_i),(G_i,\Sigma_i) \in \mathcal{L}^1[\Sigma,\mathrm{cf}(X)]$. If $(F_i,\Sigma_i) \le (G_i,\Sigma_i)$, then, by Theorem 6.3,

$$L((F_i,\Sigma_i)) = \lim_{i\to\infty}F_i \le \lim_{i\to\infty}G_i = L((G_i,\Sigma_i),$$

showing that L is order preserving. It follows trivially from the definition of L^{-1} in (a) that L^{-1} is order preserving.

(c) Suppose $(F_i,\Sigma_i),(G_i,\Sigma_i) \in \mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$, $F,G \in \mathcal{L}^1[\Sigma,\mathrm{cf}(X)]$ and

$$\lim_{i\to\infty}\Delta(F_i,F) = 0 \text{ and } \lim_{i\to\infty}\Delta(G_i,G) = 0.$$

Then, by Theorem 7.5 and (E6),

$$\begin{aligned}\Delta_{\mathcal{M}}\left((F_i,\Sigma_i),(G_i,\Sigma_i)\right) &= \sup_{i\in\mathbb{N}}\Delta(F_i,G_i)\\ &= \sup_{i\in\mathbb{N}}\Delta\left(\mathcal{E}[F|\Sigma_i],\mathcal{E}[G|\Sigma_i]\right)\\ &\le \Delta(F,G).\end{aligned}$$

But, for each $i\in\mathbb{N}$,

$$\Delta(F_i,G_i) \le \sup_{i\in\mathbb{N}}\Delta(F_i,G_i) = \Delta_{\mathcal{M}}\left((F_i,\Sigma_i),(G_i,\Sigma_i)\right).$$

Since $\lim_{i\to\infty}\Delta(F_i,G_i) = \Delta(F,G)$, we get that

$$\Delta(F,G) \le \Delta_{\mathcal{M}}\left((F_i,\Sigma_i),(G_i,\Sigma_i)\right).$$

Thus

$$\Delta\left(L(F_i,\Sigma_i),L(G_i,\Sigma_i)\right)=\Delta_{\mathcal{M}}\left((F_i,\Sigma_i),(G_i,\Sigma_i)\right),$$

proving that L is an isometry. □

As a consequence of the preceding result, we are now able to establish our main result:

Theorem 8.2. *The complete metric space $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i),\Delta_{\mathcal{M}})$ is a join-semilattice. Moreover, for all $(F_i,\Sigma_i),(G_i,\Sigma_i)\in\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$,*

(a) $(F_i,\Sigma_i)\vee(G_i,\Sigma_i)=(\lim_{j\to\infty}\mathcal{E}[F_j\vee G_j|\Sigma_i],\mathcal{E}_i)$, *and*
(b) $(F_i,\Sigma_i)^+=\left(\lim_{j\to\infty}\mathcal{E}[F_j^+|\Sigma_i],\mathcal{E}_i\right)$.

Proof. Since L is an isomerty and $\overline{\bigcup_{i=1}^\infty\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]}$ is a closed subspace of $\mathcal{L}^1[\Sigma_i,\mathrm{cf}(X)]$, we get that $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$ is complete.

The maps L and L^{-1} are both order preserving and $\mathcal{L}^1[\Sigma,\mathrm{cf}(X)]$ is a join-semilattice; thus, $\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$ is also a join-semilattice: if $(F_i,\Sigma_i),(G_i,\Sigma_i)\in\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$, then it readily follows that $L^{-1}\left(L(F_i,\Sigma_i)\vee L(G_i,\Sigma_i)\right)$ is the least upper bound of $\{L\left((F_i,\Sigma_i)\right),L\left((G_i,\Sigma_i)\right)\}$ in $\mathcal{L}^1[\Sigma,\mathrm{cf}(X)]$.

The formula in (b) is a special case of that in (a). It remains to prove the formula in (a). Let $(F_i,\Sigma_i),(G_i,\Sigma_i)\in\mathcal{M}_\Delta(\mathcal{L}^1[\Sigma,\mathrm{cf}(X)],\Sigma_i)$. Then we have

$$\begin{aligned}(F_i,\Sigma_i)\vee(G_i,\Sigma_i) &= L^{-1}\left(L(F_i,\Sigma_i)\vee L(F_i,\Sigma_i)\right)\\ &= L^{-1}\left(\lim_{k\to\infty}F_k\vee\lim_{k\to\infty}G_k\right)\\ &= L^{-1}\left(\lim_{k\to\infty}(F_k\vee G_k)\right)\\ &= \left(\mathcal{E}[\lim_{k\to\infty}(F_k\vee G_k)|\Sigma_i],\Sigma_i\right)\\ &= \left(\lim_{k\to\infty}\mathcal{E}[F_k\vee G_k|\Sigma_i],\Sigma_i\right),\end{aligned}$$

where the third equality results from the compatibility of $\leq$ and Δ, as in Theorem 6.3. This completes the proof. □

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